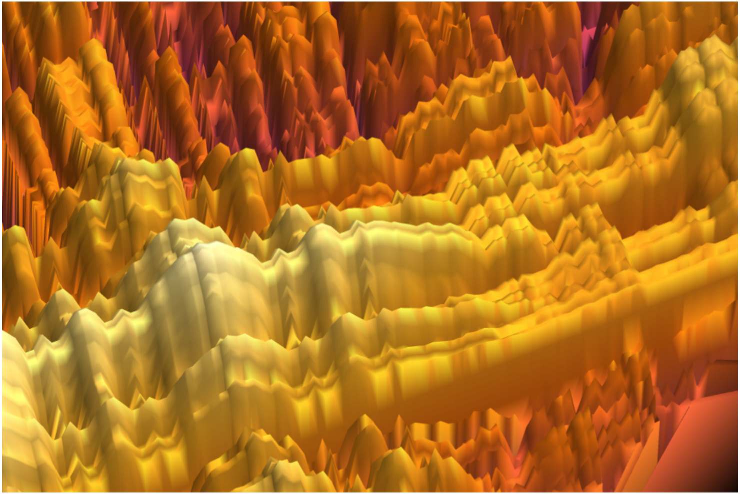
The core entropy of polynomials of higher degree

Giulio Tiozzo
University of Toronto

In memory of Tan Lei

Angers, October 23, 2017



First email: March 4, 2012

Hi Mr. Giulio Tiozzo,
My name is Tan Lei. I am a chinese mathematician working in France in the field of holomorphic dynamics. Curt McMullen suggested me to contact you for the following questions that you might help.
It seems that one can think of the core entropy as a function on the Mandelbrot set itself. And Milnor had a student who proved entropy is monotone on M.
Do you have a copy of this thesis? How to define the core entropy when the Hubbard tree is topologically infinite? Or worse when the critical orbit is dense in J? Is the monotonicity proved using puzzles? Is there a continuity result of the core entropy as a function of the external angle?
Many thanks in advance for your help.
Sincerely yours, Tan Lei

In total: 569 emails

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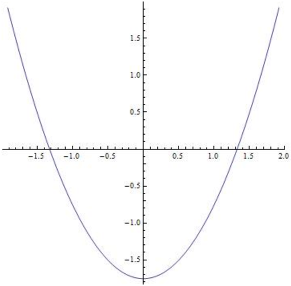

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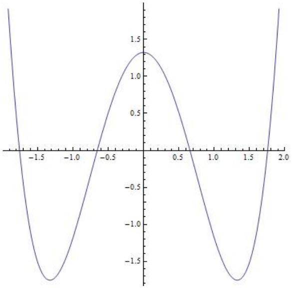

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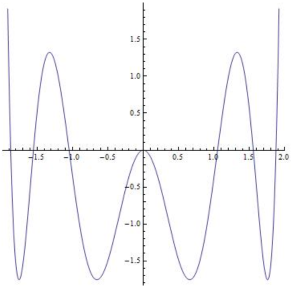

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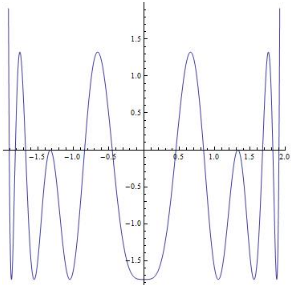

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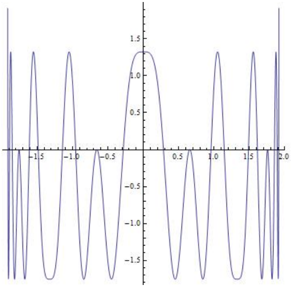

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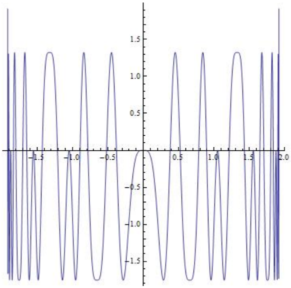

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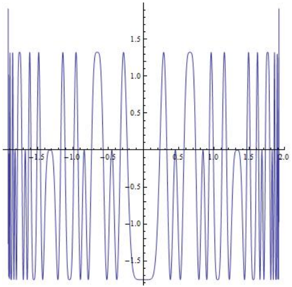

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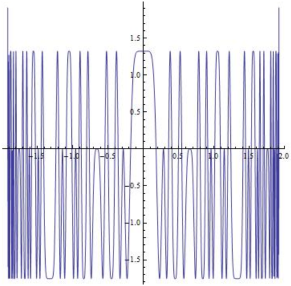

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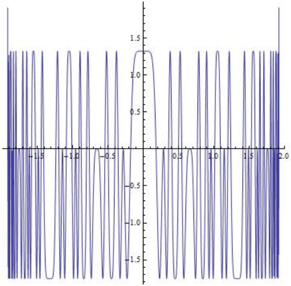

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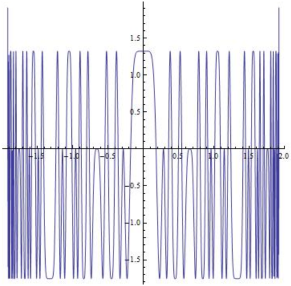

Agrees with general definition for maps on compact spaces using open covers (Misiurewicz-Szlenk)

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How does entropy change with the parameter c?

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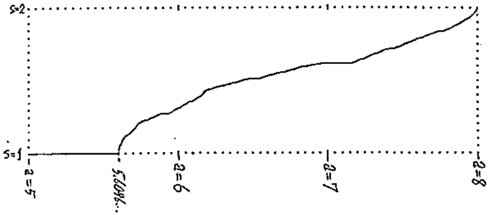

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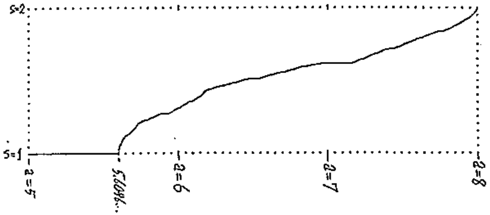

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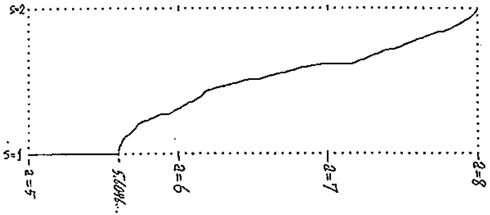

Question : Can we extend this theory to complex polynomials?

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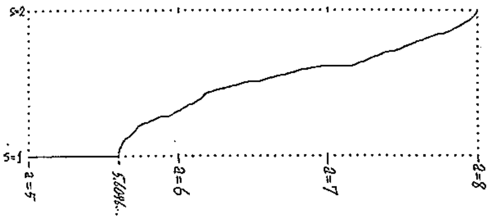
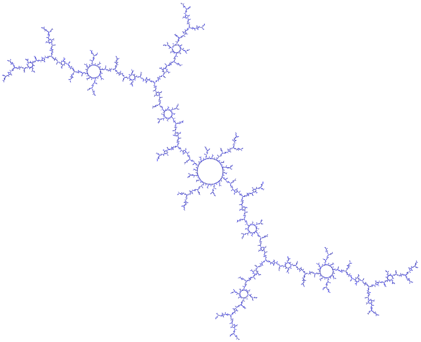


Remark. If we consider $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ entropy is constant $h_{top}(f, \hat{\mathbb{C}}) = \log d$.

The complex case: Hubbard trees

The Hubbard tree T of a postcritically finite polynomial is a forward invariant, connected subset of the filled Julia set which contains the critical orbit.



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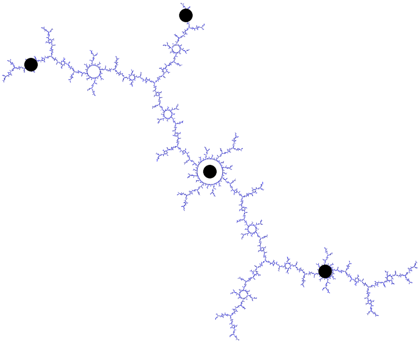

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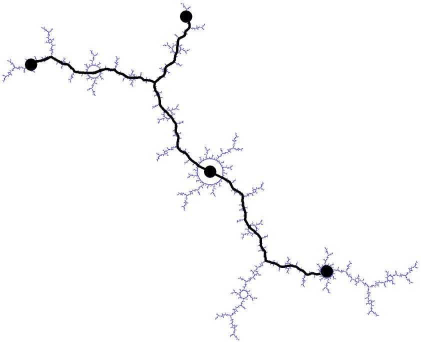

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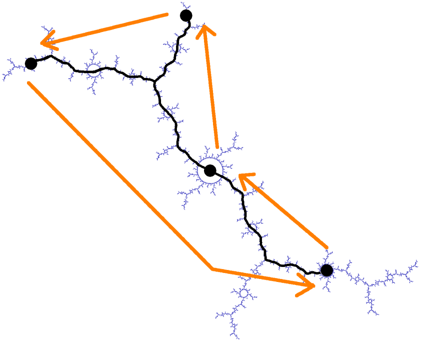

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Question: How does $h(f)$ vary with the polynomial f?

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A critical portrait m is said to be a primitive major if moreover the elements of m are pairwise disjoint.

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Define the distance between primitive majors as

$$d(m_1, m_2) := \sup_{x,y} |d(\pi_{m_1}(x), \pi_{m_1}(y)) - d(\pi_{m_2}(x), \pi_{m_2}(y))|$$

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Then

$$\Theta := \{\Theta(c_1), \ldots, \Theta(c_k)\}$$

is a critical marking (Poirier).

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$$PM(2) \cong \partial\mathbb{D}$$

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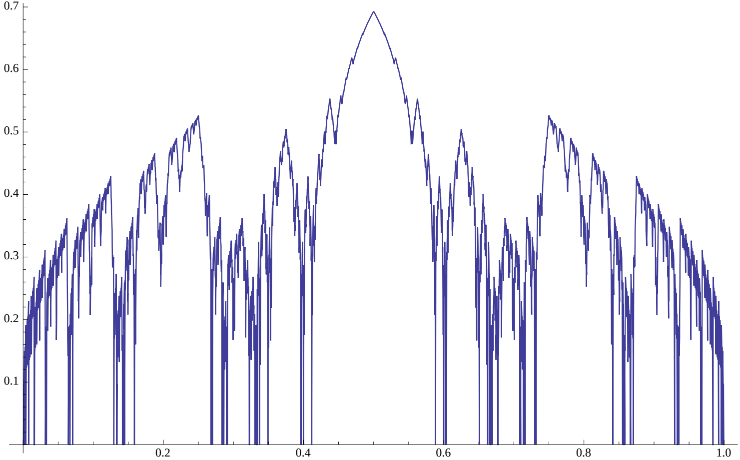

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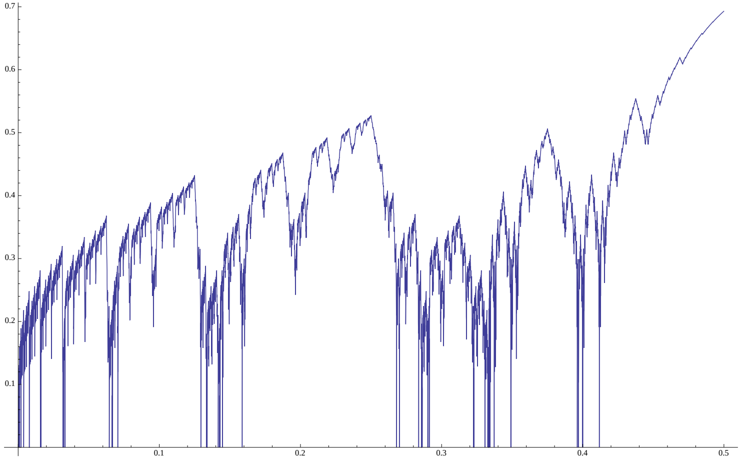

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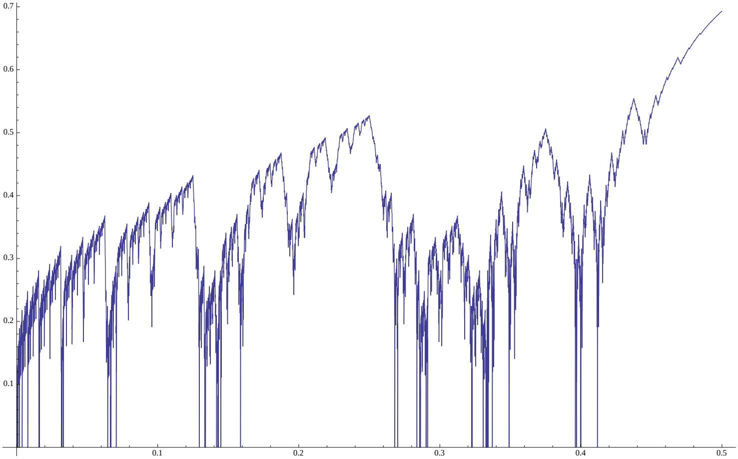

Question Can you see the Mandelbrot set in this picture?

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- Core entropy also proportional to Hausdorff dimension of angles landing on the corresponding vein (T., Jung)

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Any pair of distinct angles $\theta^{\pm}$ defines four partitions of the circle: $L(\theta^{\pm})$ is the circle minus the four points $\theta^{\pm}/2$, and $\theta^{\pm}/2 + 1/2$ and Full($\theta^{\pm}$) is S^1 minus the two intervals $[\frac{\theta^-}{2}, \frac{\theta^+}{2}]$ and $[\frac{\theta^-}{2} + \frac{1}{2}, \frac{\theta^+}{2} + \frac{1}{2}]$.

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Now, rather than, as Douady and Tao Li, looking at angles landing as the Hubbard tree, we look at pairs of angles landing together and pairs of angles landing at the tree.

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With pictures the idea would be a lot easer to explain.
All the best, Tan Lei

Continuity in the quadratic case

Question (Thurston, Hubbard):
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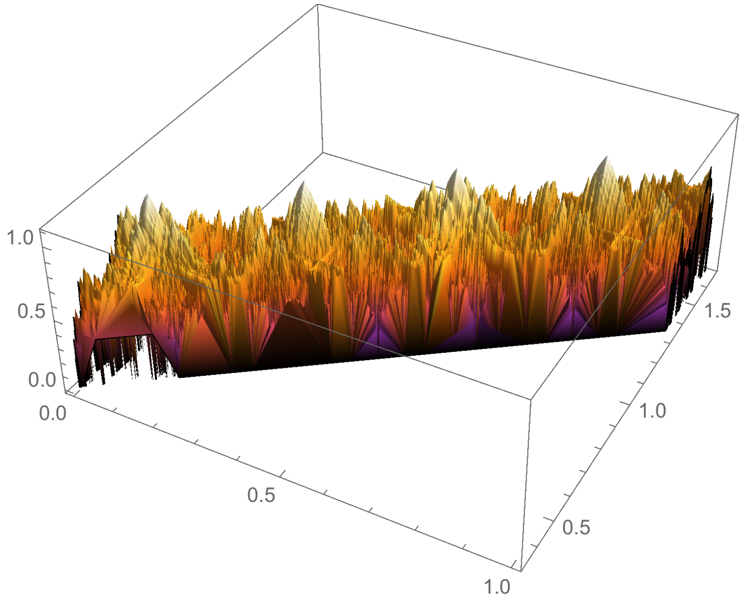
Theorem (T., Dudko-Schleicher)

The core entropy function $h(\theta)$ extends to a continuous function from $\mathbb{R}/\mathbb{Z}$ to $\mathbb{R}$.

The core entropy for cubic polynomials



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A critical portrait m is said to be a primitive major if moreover the elements of m are pairwise disjoint.

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Theorem (Thurston)

For each $r > 0$, we have a homeomorphism

$$Y_d(r) \cong PM(d)$$

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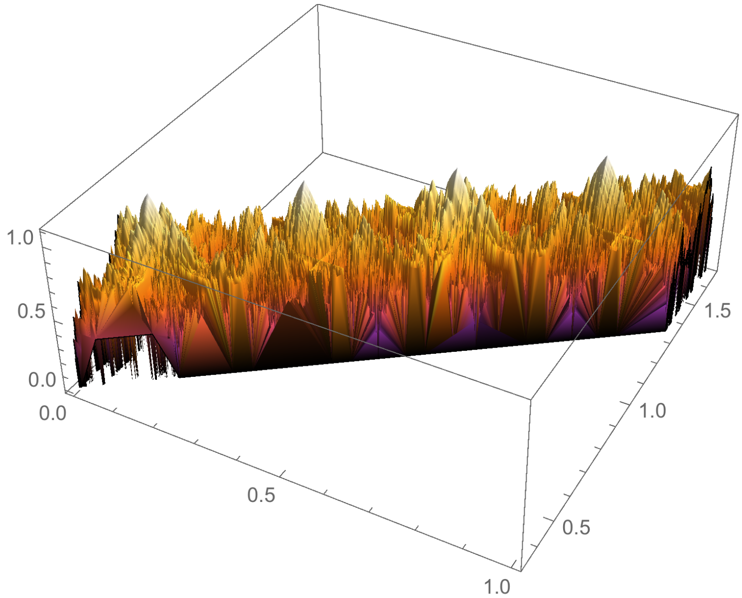

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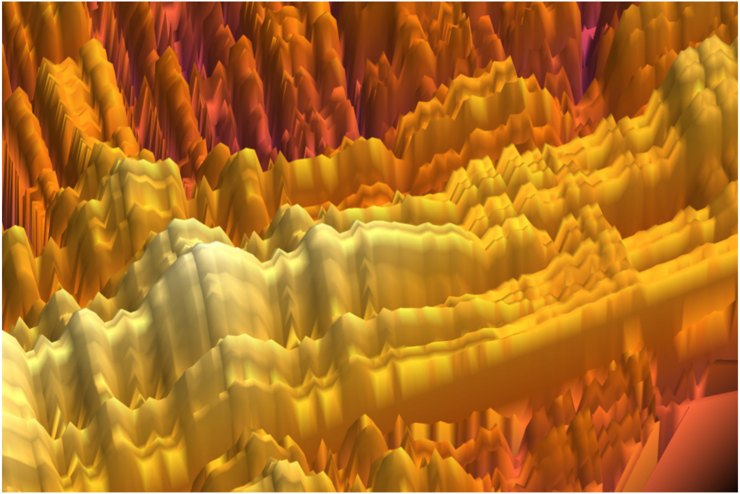

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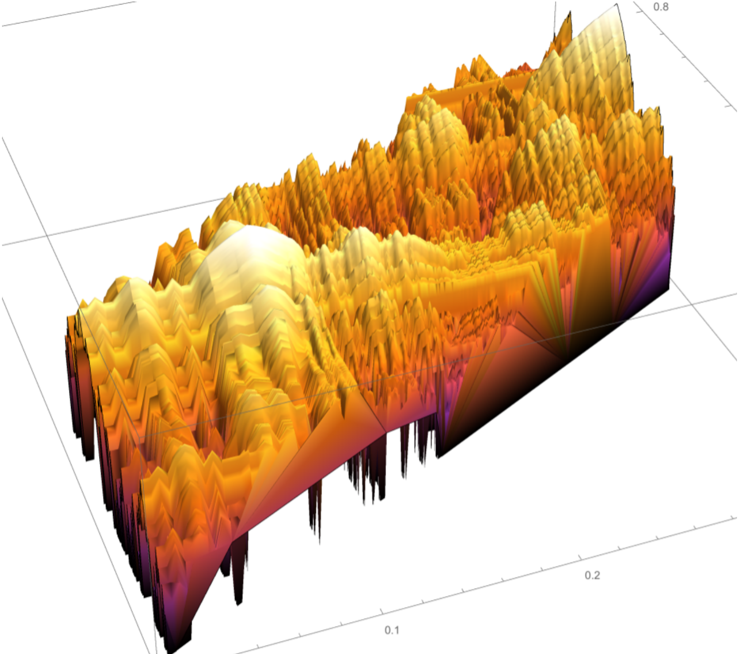

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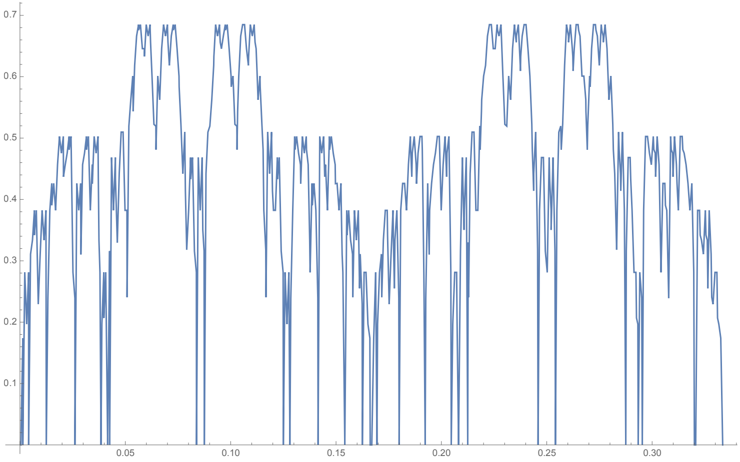

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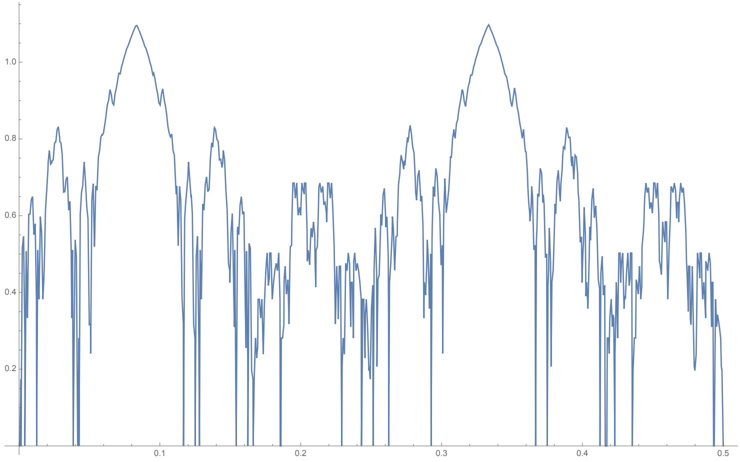

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Fix $d \geq 2$. Then the core entropy extends to a continuous function on the space $\mathrm{PM}(d)$ of primitive majors.

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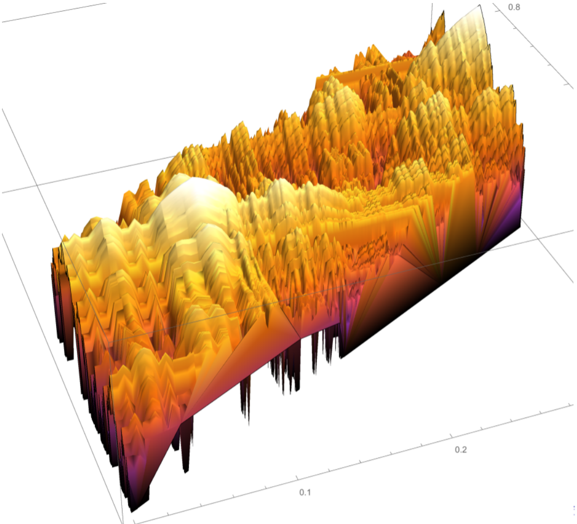

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"I hope you two will make a great paper together!" (January 28, 2015)

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Denote $i_\alpha := f^i(c_\alpha)$ the i^{th} iterate of the critical point c_α, let

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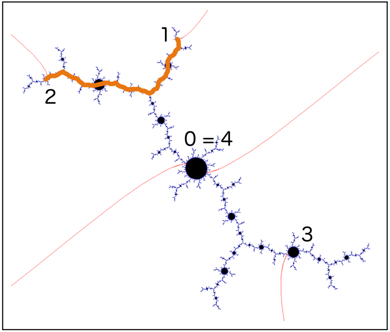

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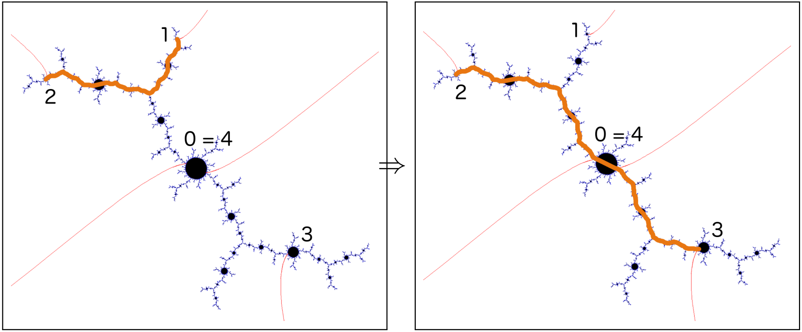

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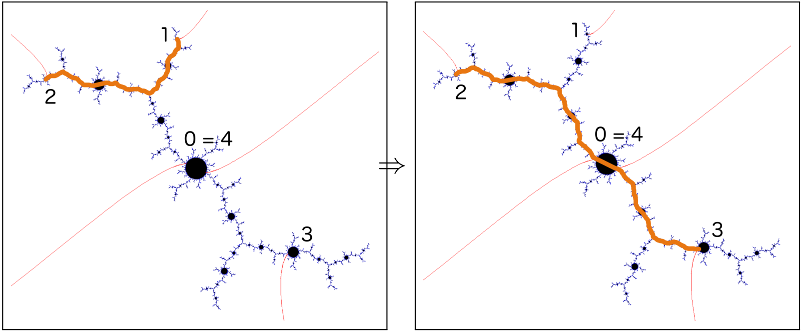

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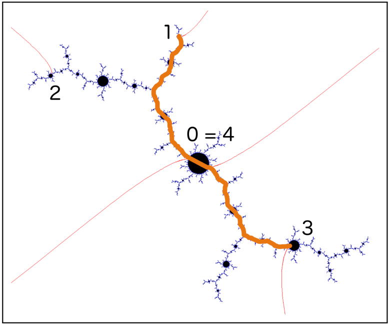

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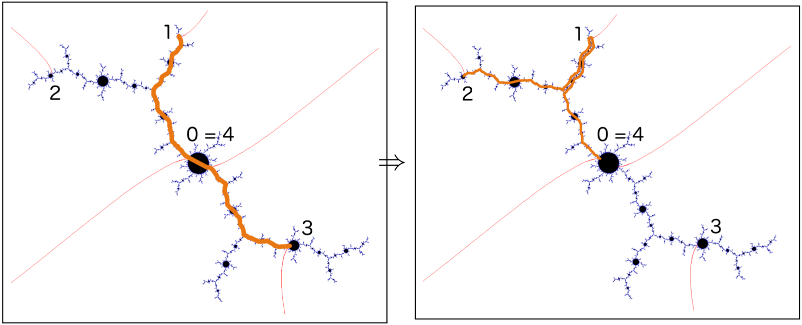

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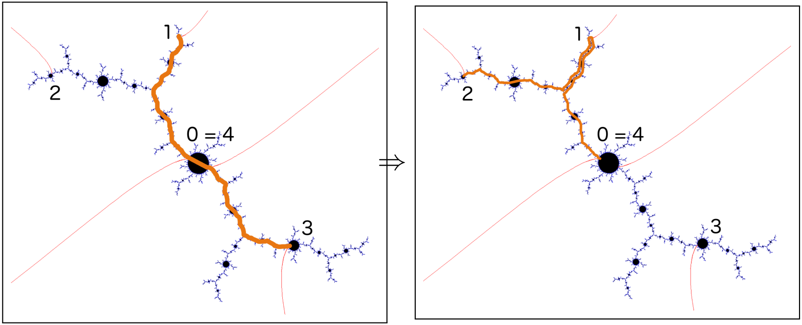

$$(1, 3) \quad \Rightarrow \quad (1, 2) + (1, 4)$$

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Theorem (Thurston; Tan Lei; Gao Yan)

The core entropy of f is given by

$$h(f) = \log \lambda$$

where λ is the leading eigenvalue of A.

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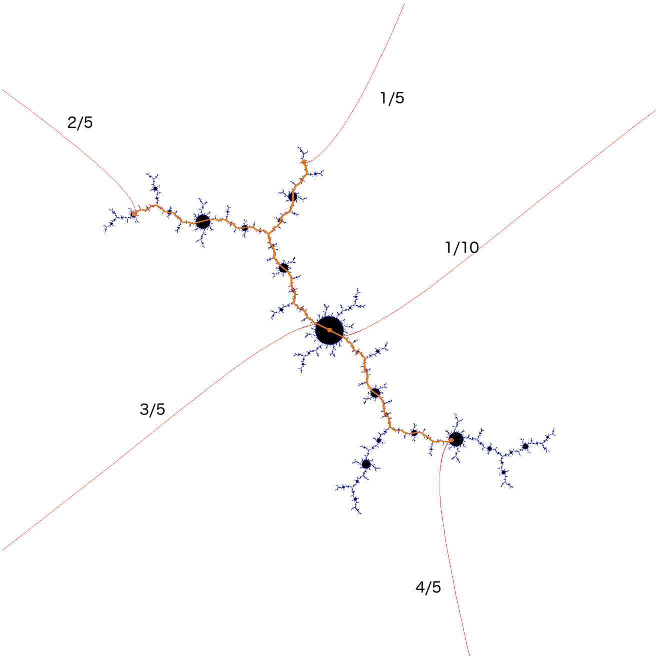

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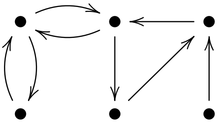

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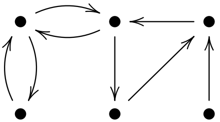

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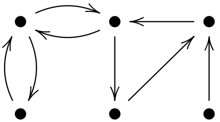

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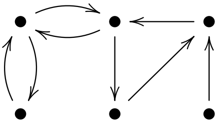

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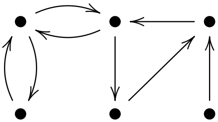

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$$P(t) = 1 - 2t^2 - t^3 + t^5$$

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Then we define the growth rate of Γ as :

$$r(\Gamma) := \limsup \sqrt[n]{C(\Gamma, n)}$$

where $C(\Gamma, n)$ is the number of closed paths of length n.

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Theorem

Let $\sigma \leq 1$. Then $P(t)$ defines a holomorphic function in the unit disk, and its root of minimum modulus is r^{-1}.

Wedges

				$\cdots$
			$(4,5)$	$\cdots$
		$(3,4)$	$(3,5)$	$\cdots$
	$(2,3)$	$(2,4)$	$(2,5)$	$\cdots$
$(1,2)$	$(1,3)$	$(1,4)$	$(1,5)$	$\cdots$

Labeled wedges

Label all pairs as either separated or non-separated

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	$\boxed{(2,3)}$	$(2,4)$	$\cdots$
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(The boxed pairs are the separated ones.)

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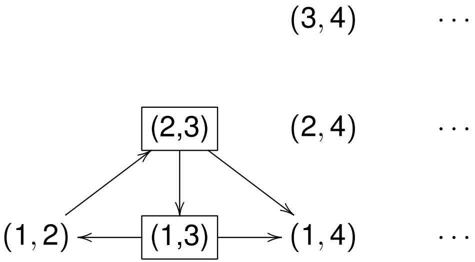

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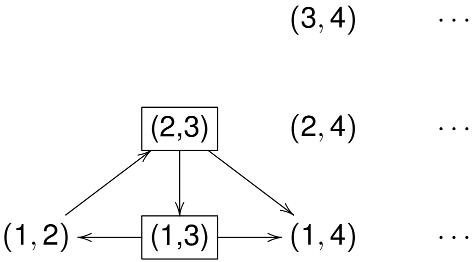

"This sounds like climbing a mountain; you go up step by step, but you chute all the way to the bottom, and in two broken pieces" (August 25, 2014)

Continuity: sketch of proof

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Then	$W_{\theta_n} \to W_\theta$	(wedges)
so	$P_{\theta_n}(t) \to P_\theta(t)$	(spectral determinants)
and	$r(\theta_n) \to r(\theta)$	(growth rates)

Further directions / questions

1. **Conjecture:** In each stratum the maximum of the core entropy equals

$$\max_{m \in \Pi} h(m) = \log(\text{Depth}(\Pi) + 1)$$

where the Depth of a stratum is the maximum length of a chain of nested leaves in the primitive major.

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Merci!