When Hyperbolic Maps are Matings

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- Matings were introduced by Douady and Hubbard, following their amazingly detailed and informative description of (in particular) quadratic polynomials $z^2 + c$ for c in the Mandelbrot set.
- Their description came from results they proved about dynamical rays in the basin of infinity of a quadratic polynomial, and parameter rays in the complement of the Mandelbrot set (in parameter space).

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- The Douady-Hubbard results can be summarised more simply for hyperbolic polynomials, and are so complete for these that there seems to be nothing to add.
- For polynomials, every bounded Fatou component is a topological disc, and every periodic bounded Fatou component contains a unique attractive periodic point.
- If the Julia set is connected then it is locally connected and the filled Julia set is a quotient of the closed unit disc, with the dynamics on the Julia set being a quotient of $z \mapsto z^d$ on the unit circle, where d is the degree of the polynomial.

The quadratic case

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- For a hyperbolic quadratic polynomial $f(z) = z^2 + c$ which is not in the hyperbolic component of $z \mapsto z^2$, the quotient of the unit disc and the dynamics on it is completely described by the points on the unit circle which collapse to the point on the boundary of the immediate attractive basin containing the critical value.

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- These points map forward to the common endpoint of finitely many, and at least two, dynamical rays for f, of which two, together with the common endpoint separate any other rays from the critical value (and the Fatou component containing it).

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- These points map forward to the common endpoint of finitely many, and at least two, dynamical rays for f, of which two, together with the common endpoint separate any other rays from the critical value (and the Fatou component containing it).
- The preimages of these two dynamical rays in the exterior of the unit disc are radial lines ending at points of rational argument, that is, at points $e^{2\pi i p_1}$ and $e^{2\pi i p_2}$ where p_1 and p_2 are odd denominator rationals. In addition the points $e^{2\pi i p_1}$ and $e^{2\pi i p_2}$ are of the same period under the map $z \mapsto z^2$.

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- These two radial lines are then also preimages of parameter rays ending at a parabolic parameter value in the Mandelbrot set.
- This parabolic value is in the boundary of the hyperbolic component of f, for which the periodic parabolic basin has the same period as the immediate attractive basin of f containing the critical value.

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- It is also in the boundary of a unique hyperbolic component with a periodic cycle of Fatou components of the same period as the periodic cycle of parabolic basins for the parabolic parameter value.

Conversely...

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- Conversely each odd denominator rational is the preimage of the common endpoint of exactly two parameter rays.
- This common endpoint is a parabolic parameter value.
- The two parameter rays separate $c = 0$ from the hyperbolic component adjacent to the parabolic parameter value with attractive basin of the same period as the parabolic basin.

The critically periodic centre of a hyperbolic component

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- Any hyperbolic component of quadratic polynomials in the Mandelbrot set contains a unique critically periodic polynomial $z^2 + c_0$ called the centre.
- The dynamics on the Julia set is constant, up to topological conjugacy, throughout the hyperbolic component, and also the same up to topological conjugacy for the parabolic parameter value.

S_p

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- For $c_0 \neq 0$, the dynamics of the critically periodic map $z^2 + c_0$ on the Julia set is the quotient of a critically periodic branched covering of $\overline{\mathbb{C}}$ which preserves the unit circle, interior and exterior of the unit disc and is equal to $z \mapsto z^2$ on $\{z : |z| \geq 1\}$.

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- Here p is the argument of one of the dynamical rays landing at the lowest period point on the boundary of the Fatou component containing the critical value, and adjacent to this Fatou component.
- Equivalently p is the argument of one of the parameter rays landing at the parabolic value on the boundary of the hyperbolic component of $z^2 + c_0$ and separating it from 0.

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- s_p chosen to preserve a quadratic invariant lamination on the disc, called L_p, and is also chosen suitably up to topological conjugacy on the complement of the lamination
- The quotient map $[s_p]$ is uniquely determined up to topological conjugacy - and of course is topologically conjugate to the corresponding critically periodic quadratic polynomial.

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$$s_p \amalg s_q(z) = \begin{array}{lll} s_p(z) & \text{if} & |z| \leq 1 \\ (s_q(z^{-1}))^{-1} & \text{if} & |z| \geq 1. \end{array}$$

Which matings are rational maps?

It was immediately clear that $s_p \amalg s_{1-p}$ could not be Thurston equivalent to a rational map and also that $s_p \amalg s_q$ could not be Thurston equivalent to a rational map if $[s_p]$ and $[s_q]$ are in conjugate limbs.

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In the language of minor leaves the conjugate limbs condition becomes $\mu_r \leq \mu_p$ and $\mu_{1-r} \leq \mu_q$, where μ_p, μ_q , μ_r are the chords in the unit disc joining the preimages of common ray endpoints

$$e^{2\pi ip} \text{ and } e^{2\pi ip_2}, \quad e^{2\pi iq} \text{ and } e^{2\pi iq_2}, \quad e^{2\pi ir} \text{ and } e^{2\pi ir_2}.$$

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However. . .

Theorem

(1986-7) $[s_p]$ *and* $[s_q]$ *are not in conjugate limbs* $\Leftrightarrow$ $s_p \amalg s_q$ *is Thurston equivalent (and* $[s_p \amalg s_q]$ *topologically equivalent) to a rational map.*

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- So a rich class of examples of quadratic rational maps. But
- How many ((sub)hyperbolic) rational maps are matings?

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- A similar result can be proved in some regions of parameter space where one critical point is constrained to have degree k, and the Julia set is star-like

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Theorem

(2016-17)Let $f \simeq s_p \amalg s_q$ be a hyperbolic quadratic rational map with critical points $c_1(f)$ and $c_2(f)$ corresponding to the critical points ∞ and 0 of $s_p \amalg s_q$, such that all Fatou components have disjoint closures. Suppose also that any point on S^1 in the boundary of the gap of L_p containing thr critical value of s_p is not *an endpoint of a leaf of L_q. Let $c_1(f)$ have period m*

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Then, in the variety of rational maps with c_1 of period m, all type IV hyperbolic components sufficiently near the Mandelbrot set copy containing f, are matings.

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- A nearby hyperbolic component is created when a critical point moves to a nearby periodic point.
- If there are no connections between different leaves of $L_p \cup L_q^{-1}$ then the critical point does not cross S^1 in any essential way and moves to another mating.

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- Leaves of L_q^1 ending on the boundary of the *minor gap* of L_p look likely to lead to non-matings.

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- Let σ_β be a homeomorphism which is the identity outside a small neighbourhood of β and moves the start-point to the finish-point.
- Then $f \simeq s_p \simeq \sigma_\zeta^{-1} \circ \sigma_\beta \circ s$.

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- Suppose also that the closures of Fatou components are all disjoint.
- Then type IV hyperbolic rational maps near f with c_1 of fixed period are in two-to-one correspondence with periodic points of f near the closure of the Fatou component of $f(c_2)$.

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- The description of nearby hyperbolic components can again be done in terms of paths in the dynamical plane.
- Let g be hyperbolic type IV in a neighbourhood of f. Let x be the corresponding periodic point in the corresponding neighbourhood of the Fatou component of c_2.
- Then there is a path β from $f(c_2)$ to x in this neighbourhood such that $g \simeq \sigma_\zeta^{-1} \circ \sigma_\beta \circ f$ where $f \circ \zeta = \beta$ and ζ is a path from $c_2(f)$ to the periodic point in $f^{-1}(x)$.

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Is the path β the union of a path in the Fatou component of $f(c_2)$ and the image of a path along S^1?

By induction

As so often in mathematics, the strategy involves an induction.

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- If the path β is in a disc U which intersects the forward orbit of x only in x itself then U determines the homotopy class of β uniquely.

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- If the path β is in a disc U which intersects the forward orbit of x only in x itself then U determines the homotopy class of β uniquely.
- So this is one thing we want to arrange, by an inductive process.

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- Then for g to be a mating we need β to be a path in the union of the Fatou component of $f(c_2)$ and the image of an arc of S^1, up to homotopy.
- This will be true if every component of $S^1 \cap \varphi^{-1}(U)$ intersects the preimage of the closure of the Fatou component of $f(c_2)$.

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- This is a big subject in dynamics in general, and especially in complex dynamics, principally through Yoccoz puzzles and their generalisations.
- This topic is connected to Tan Lei, perhaps rather indirectly, through the people she is associated with, and in particular, former students.