Wandering domains and Singularities

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Complex dynamics and Quasiconformal Geometry
To celebrate Tan Lei
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NATO conference, Hillerød 1993

Afterwards Paris (93), MSRI (95), etc, etc.

Bodil Fest, Hollbæk, 2003

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Bob’s Fest, Tossa de Mar, 2008

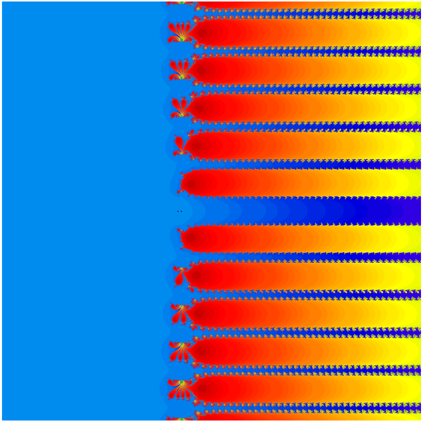
Dynamics around Thurston's Theorem, Roskilde, 2010

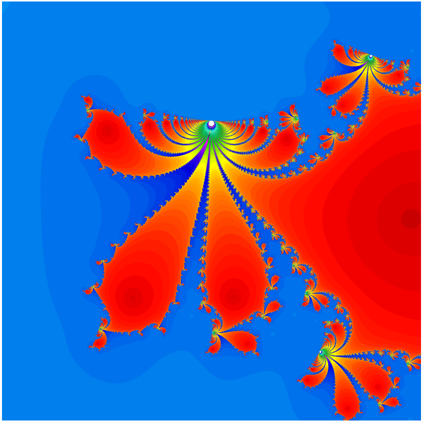
Topics in complex dynamics, Barcelona, 2015

Newton's method of entire transcendental maps

Can these examples have wandering domains?

Newton's method for $F(z) = z + e^z$.





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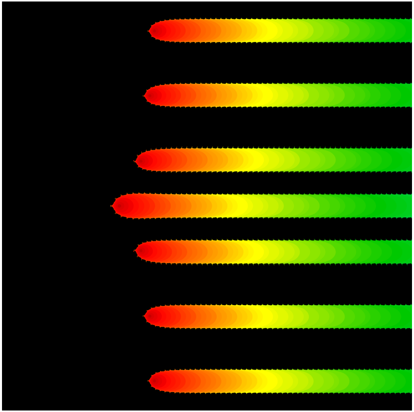
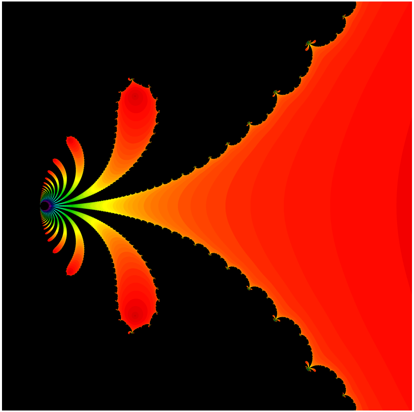

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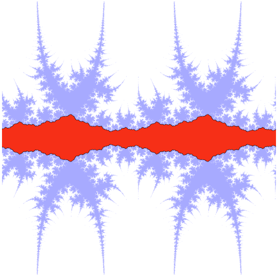
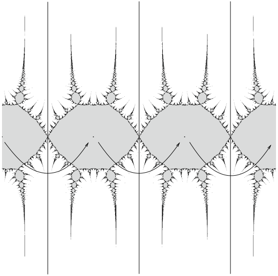
  - U is a Baker domain of period 1 if $f^n|_U \to \infty$ loc. unif.
  - U is a wandering domain if $f^n(U) \cap f^m(U) = \emptyset$ for all $n \neq m$.





$z + a + b\sin(z)$ [Figures: Christian Henriksen] $z + 2\pi + \sin(z)$

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- A point $a \in \widehat{\mathbb{C}}$ is an asymptotic value if there exists a curve $\gamma(t) \to \infty$ such that $f(\gamma(t)) \to a$. (Morally, a has infinitely many preimages collapsed at infinity).

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  Example: $z = 0$ for $f(z) = \lambda e^z$.
- Define the postsingular set of f as

$$P(f) = \cup_{s \in S} \cup_{n \geq 0} f^n(s).$$

Singular values

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- Basins of attraction of attracting or parabolic cycles contain at least one singular value
- Boundaries of Siegel disks or Herman rings, and Cremer cycles belong to $\overline{P(f)}$.
- The relation with Baker domains and wandering domains is not so clear.

Theorem (Bergweiler'95, Mihaljevic-rempe'13, Baranski-F-Jarque-Karpinska'17)

f transcendental meromorphic, U invariant Baker domain, $U \cap S(f) = \emptyset$. Then $\exists p_n \in P(f)$ st

1. $|p_n| \to \infty$
2. $\left|\frac{p_{n+1}}{p_n}\right| \to 1$
3. $\frac{\text{dist}(p_n, U)}{|p_n|} \to 0$

Special classes

Some classes of maps are singled out depending on their singular values.

- The Speisser class or finite type maps:

$$\mathcal{S} = \{f \text{ ETF (or MTF) such that } S(f) \text{ is finite}\}$$

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- The Eremenko-Lyubich class

$$\mathcal{B} = \{f \text{ ETF (or MTF) such that } S(f) \text{ is bounded}\}$$

Example: $z \mapsto \lambda \frac{z}{\sin(z)}$.

Existence of wandering domains

- Maps of finite type are the most similar to rational maps. Indeed,

$$f \in \mathcal{S} \Longrightarrow f \text{ has no wandering domains}$$

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- Maps in class $\mathcal{B}$ also have special properties among which:

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If U is a wandering domain, and $L(U)$ is the set of limit functions of f^n on U, then

$$U \text{ is } \begin{cases} \text{escaping} & \text{if } L(U) = \{\infty\} \\ \text{oscillating} & \text{if } \{\infty, a\} \subset L(U) \text{ for some } a \in \mathbb{C}. \\ \text{“bounded”} & \text{if } \infty \notin L(U). \end{cases}$$

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Open question

Does there exist a map with a "bounded" wandering domain?

The proof is based on quasiconformal folding, a qc surgery construction.

Examples of wandering domains

Examples of wandering domains are not abundant. Usual methods are:

- Lifiting of maps of $\mathbb{C}^*$ [Herman'89, Henriksen-F'09]. The relation with the singularities is limited to the finite type possibilities.

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- Quasiconformal surgery [Kisaka-Shishikura'05, Bishop'15].

Wandering domains and singularities: Motivating examples

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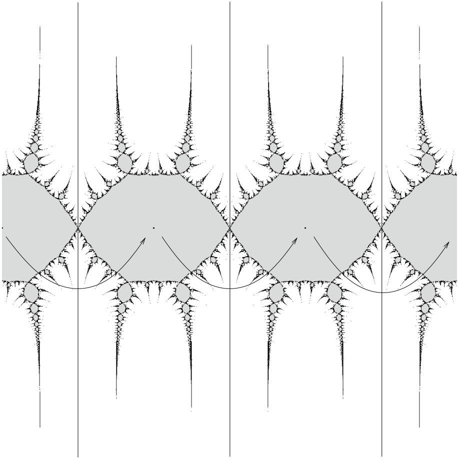
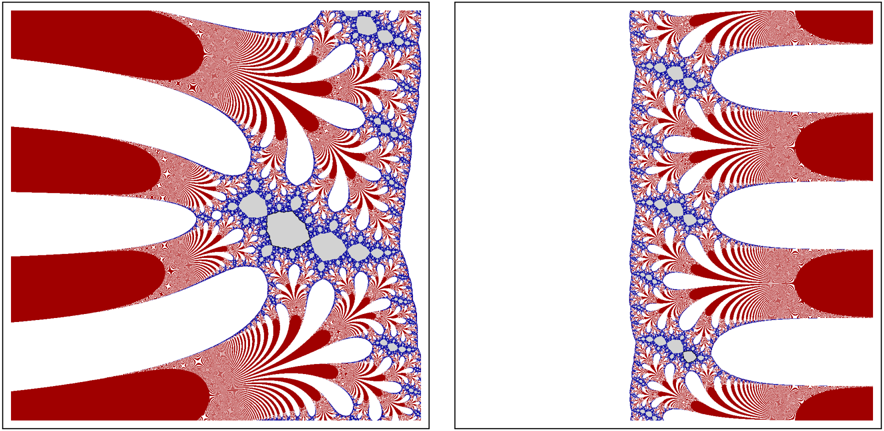
Example 1 (escaping):



$z \mapsto z + 2\pi + \sin(z)$ One critical point in each WD.

Wandering domains and singularities: Examples

Example 2 (escaping and Univalent, $\partial U \subset \overline{P(f)}$):



Left: Siegel disk of $g(w) = \frac{e^{2-\lambda}}{2-\lambda} w^2 e^{-w}$ with $\lambda = e^{2\pi i(1-\sqrt{5})/2)}$, around $w = 2 - \lambda$. Right: Lift to a wandering domain U.

Wandering domains and singularities: Examples

Example 3 [Kisaka-Shishilkura'05, Bergweiler-Rippon-Stallard'13].
Wandering orbit of annuli such that

- $\mathcal{U} \cap P(f) = \emptyset$
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Example 4 [Bishop'15]
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Example 4 [Bishop'15]
The oscillating domain of Bishop in class $\mathcal{B}$ contains critical points of arbitrary high multiplicity, responsible for the high contraction necessary.

Question

Does there exist an oscillating wandering domain in class $\mathcal{B}$ on which f^n is univalent for all $n > 0$?

Known results

Recall, for U a wandering domain, the set of limit functions

$$L(U) = \{a \in \widehat{\mathbb{C}} \mid f^{n_k}|_U \rightrightarrows a \text{ for some } n_k \to \infty\}.$$

Theorem (Bergweiler *et al*'93, Baker'02, Zheng'03)

Let f be a MTF with a wandering domain U. If $a \in L(U)$ then $a \in P(f)' \cap J(f)$.

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Theorem (Mihaljevic-Rempe'13)

If $f \in \mathcal{B}$ and $f^n(S(f)) \rightrightarrows \infty$ uniformly (+ extra geometric assumption), then f has no wandering domains.

Univalent WD in class $\mathcal{B}$

Theorem A (F-Lazebnik-Jarque'17)

There exists an ETF $f \in \mathcal{B}$ such that f has a wandering domain U on which $f^n|_U$ is univalent for all $n \geq 0$.

Univalent WD in class $\mathcal{B}$

Theorem A (F-Lazebnik-Jarque'17)

There exists an ETF $f \in \mathcal{B}$ such that f has a wandering domain U on which $f^n|_U$ is univalent for all $n \geq 0$.

The proof is based on Bishop's quasiconformal folding construction.
We substitute the high degree maps $(z - z_n)^{d_n}$ on the disk components by $(z - z_n)^{d_n} + \delta_n(z - z_n)$, which are univalent near z_n and show that that the critical values can be kept outside (but very close to) the wandering component.

More detail

Wandering domains and singular orbits

Theorem B (Baranski-F-Jarque-Karpinska'17)

Let f be a MTF and U be a wandering domain of f. Let U_n be the Fatou component such that $f^n(U) \subset U_n$. Then for every $z \in U$ there exists a sequence $p_n \in P(f)$ such that

$$\frac{\text{dist}(p_n, U_n)}{\text{dist}(f^n(z), \partial U_n)} \to 0 \quad \textit{as } n \to \infty.$$

In particular, if for some $d > 0$ we have $\text{dist}(f^n(z), \partial U_n) < d$ for all n (for instance if the diameter of U_n is uniformly bounded), then $\text{dist}(p_n, U_n) \to 0$ as n tends to ∞.

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Proof: normal families argument, hyperbolic geometry.... Based on a technical lemma from Bergweiler on Baker domains. Compare also [Mihaljevic-Rempe'13].

Application: Topologically hyperbolic functions

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- Topologically hyperbolic maps do not possess parabolic cycles, rotation domains or wandering domains which do not tend to infinity
- Examples include many Newton's methods of entire functions.

Application: Topologically hyperbolic functions

Corollary C

Let f be a MTF topologically hyperbolic. Let U be a wandering domain s.t. $U_n \cap P(f) = \emptyset$ for $n > 0$. Then for every compact set $K \subset U$ and every $r > 0$ there exists n_0 such that for every $z \in K$ and every $n \geq n_0$,

$$\mathbb{D}(f^n(z), r) \subset U_n.$$

In particular,

$$\operatorname{diam} U_n \to \infty \qquad \textit{and} \qquad \operatorname{dist}(f^n(z), \partial U_n) \to \infty$$

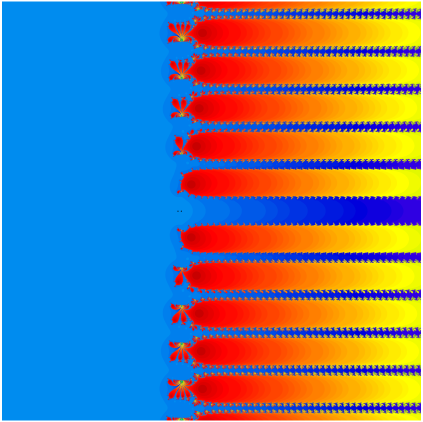
for every $z \in U$, as $n \to \infty$.

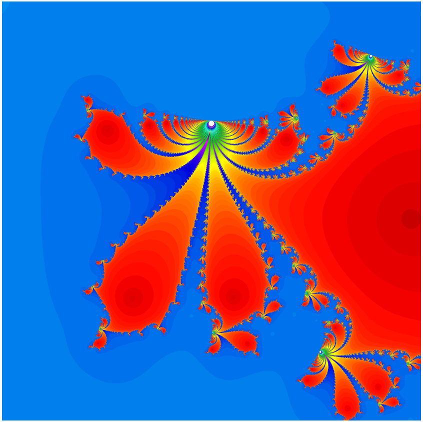
This can be applied to show that many functions, including Newton's method of $h(z) = ae^z + bz + c$ with $a, b, c \in \mathbb{R}$, have no wandering domains

[c.f. Bergweiler-Terglane,Kriete].

No wandering domains

Newton's method for $F(z) = z + e^z$.





Thank you for your attention!