Julia sets with a wandering branching point

Xavier Buff

Université de Toulouse

joint work with Jordi Canela and Pascale Roesch

Dendrites and branching points

- A Julia set $\mathcal{J}_P \subset \mathbb{C}$ is a dendrite if $\mathcal{J}_P$ is connected, locally connected, with connected complement.
- If $z \in \mathcal{J}_P$, then $\nu(z)$ is the number of components of $\mathcal{J}_P \setminus \{z\}$.
- A point $z \in \mathcal{J}_P$ is a branching point if $\nu(z) \geq 3$.

Theorem (Thurston 1985)

If P has degree 2, every branching point is (pre)periodic or (pre)critical.

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Theorem (Thurston 1985)

If P has degree 2, every branching point is (pre)periodic or (pre)critical.

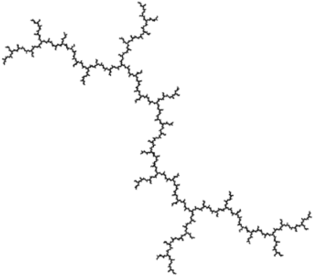
Theorem (Kiwi 2002)

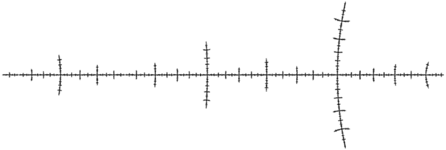
If P has degree d and z is neither (pre)periodic nor (pre)critical, then $\nu(z) \leq d$.

Examples of dendrites

Definition

A polynomial $P : \mathbb{C} \to \mathbb{C}$ is *strictly postcritically finite* if every (finite) critical point is preperiodic to a repelling cycle.





The Julia sets of $z^2 + i$ and $z^3 + 3\sqrt{3}z^2 + 27z/4$

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Communications in Mathematical Physics

Similarity Between the Mandelbrot Set and Julia Sets

TAN Lei*

Institut für Dynamische Systeme, Universität Bremen, W-2800 Bremen 33, Federal Republic of Germany

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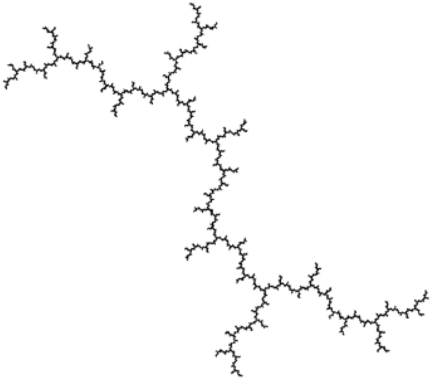
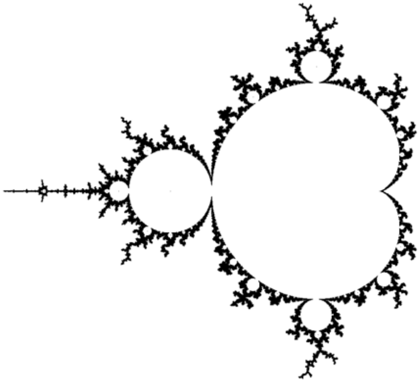
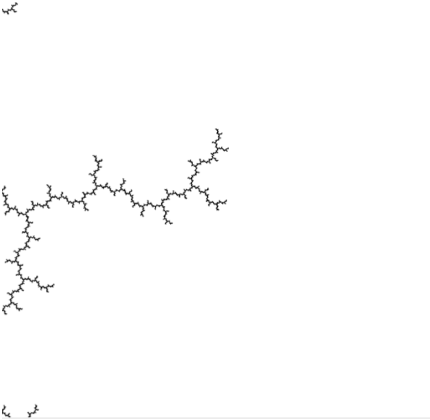
Abstract. The Mandelbrot set M is "self-similar" about any Misiurewicz point c in the sense that if we examine a neighborhood of c in M with a very powerful microscope, and then increase the magnification by a carefully chosen factor, the picture will be unchanged except for a rotation. The corresponding Julia set J_c is also "self-similar" in the same sense, with the same magnification factor. Moreover, the two sets M and J_c are "similar" in the sense that if we use a very powerful microscope to look at M and J_c, both focused at c, the structures we see look like very much the same.

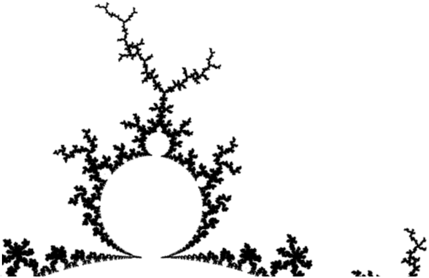
Similarity between the Mandelbrot set and Julia sets





Similarity between Julia sets and the Mandelbrot set





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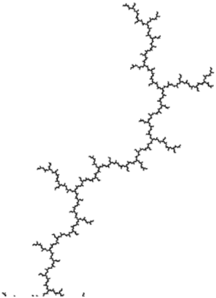
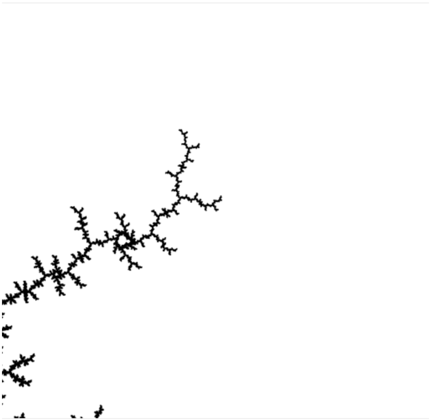




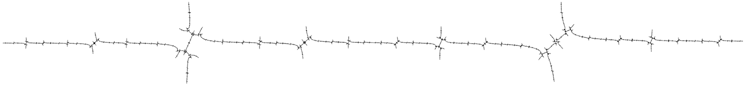
Existence of wandering branching points

Theorem (Blokh 2005)

If a cubic polynomial has a branching point which is neither (pre)periodic nor (pre)critical, then the two critical points are recurrent one to each other.

Theorem (Blokh-Oversteegen 2008)

There exist cubic polynomials with a branching point which is neither (pre)periodic nor (pre)critical.



Admissible polynomials

Definition

A cubic polynomial Q is *admissible* if it has two distinct critical points ω and ω', two distinct repelling fixed points α and β and a branching point γ, such that

1. γ is precritical to ω, ω is precritical to ω', ω' is prefixed to α,
2. γ separates β, β' and β'' in $\mathcal{J}_Q$, with $Q^{-1}(\beta) = \{\beta, \beta', \beta''\}$;
3. other technical assumptions.

Key Proposition

- For an admissible polynomial Q, we denote by γ_Q the branching point separating β, β' and β'' and by $j_Q \geq 0$ the least integer j such that $Q^{\circ j}(\gamma_Q)$ is a critical point.

Proposition

Assume Q is an admissible polynomial. Then, there is a sequence $\{P_m\}$ of admissible polynomials which converges to Q, such that $\{\gamma_{P_m}\}$ converges to γ_Q and $j_{P_m} > j_Q$ for all m.

Constructing the sequence $\{P_m\}$

- An admissible polynomial Q has (k, ℓ)-configuration if $Q^{\circ k}(\omega) = \omega'$ and $Q^{\circ \ell}(\omega') = \alpha$.
- Assume $(\xi_{-m})_{m \geq 0}$ is a backward orbit for Q satisfying
  - $\ldots \mapsto \xi_{-m-1} \mapsto \xi_{-m} \mapsto \ldots \mapsto \xi_{-1} \mapsto \xi_0 = \omega$ and
  - $\xi_{-m} \to \alpha$ as $m \to +\infty$.

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- Let $\boldsymbol{\xi}_{-m} : \mathcal{U} \to \mathbb{C}$ be maps following holomorphically the precritical orbit on a neighborhood of Q.
- The polynomials P_m have $(m + \ell, k + \ell)$-configuration with distinct critical points ω_m and ω'_m satisfying:

$$\omega_m \xrightarrow{P_m^{\circ \ell}} \zeta_m = \boldsymbol{\xi}_{-m}(P_m) \xrightarrow{P_m^{\circ m}} \omega'_m = \boldsymbol{\xi}_0(P_m) \xrightarrow{P_m^{\circ k+\ell}} \alpha_m = P_m(\alpha_m).$$

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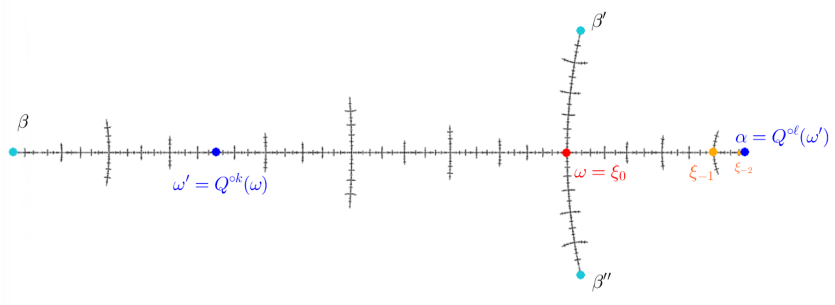
- If ζ_m separates α_m and β_m, then the branching point γ_m is precritical.

Constructing the sequence $\{P_m\}$



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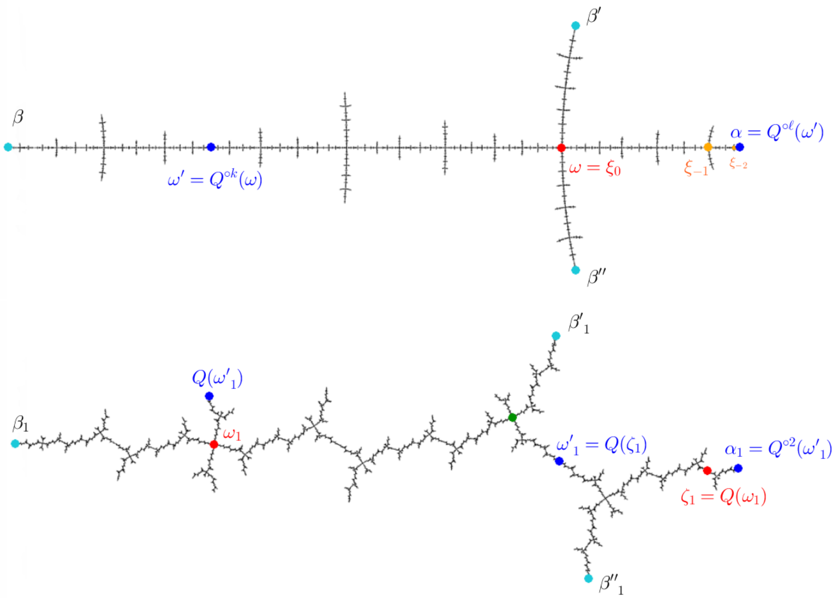

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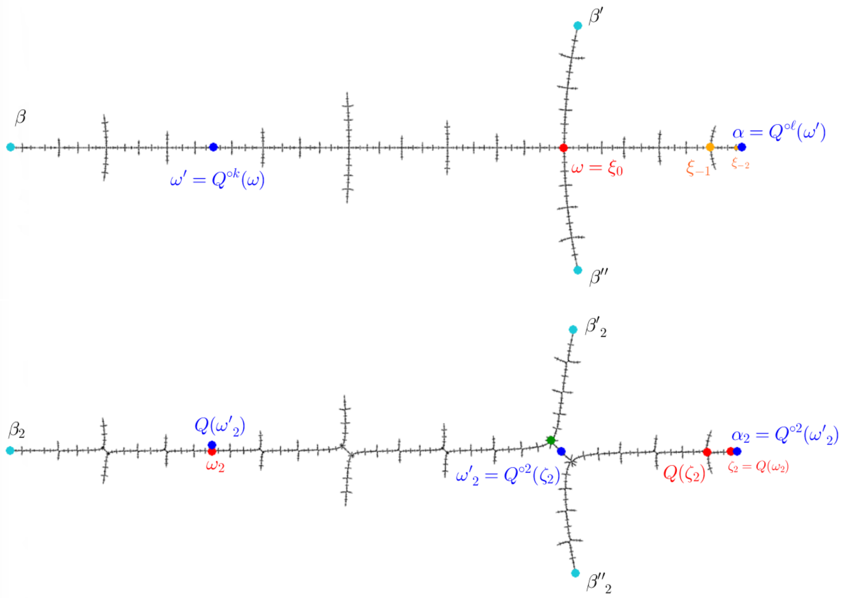

Convergence of Carathéodory loops

Lemma

Let $\{P_m\}$ be a sequence of admissible polynomials which converges to an admissible polynomial Q. Then, the sequence of Carathéodory loops of $\mathcal{J}_{P_m}$ converges uniformly to the Carathéodory loop of $\mathcal{J}_Q$.

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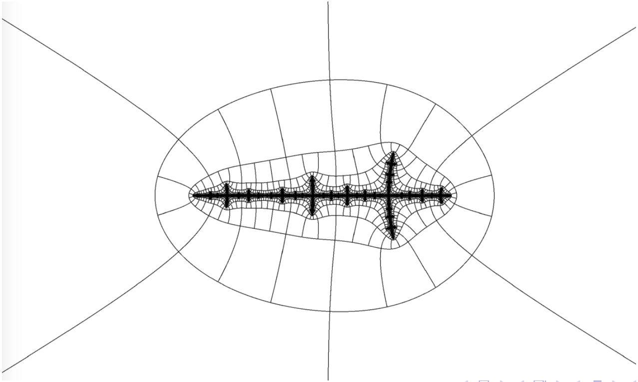

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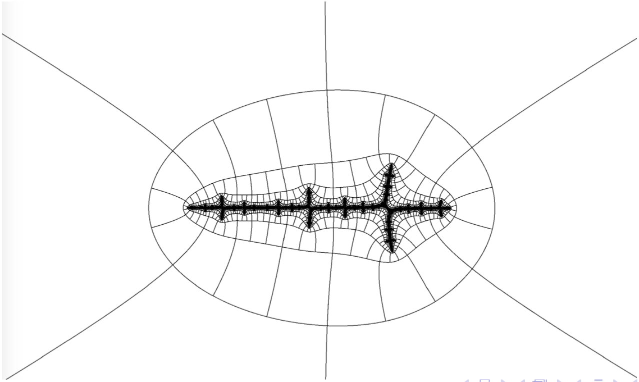
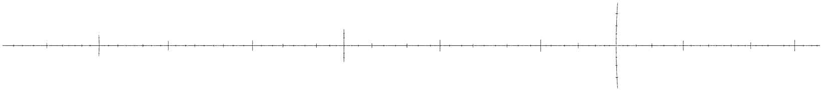


The sequence $\{Q_n\}$



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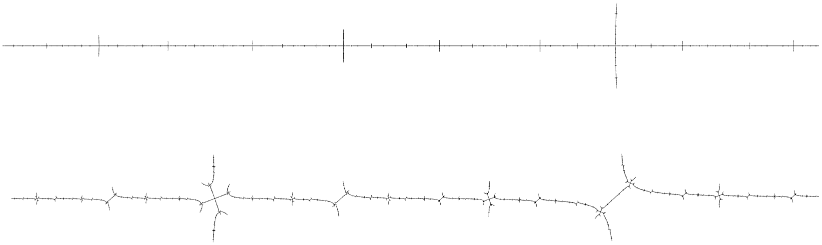

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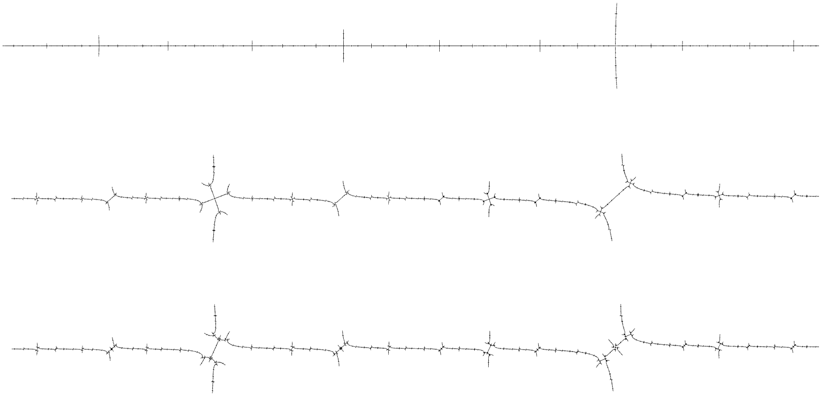

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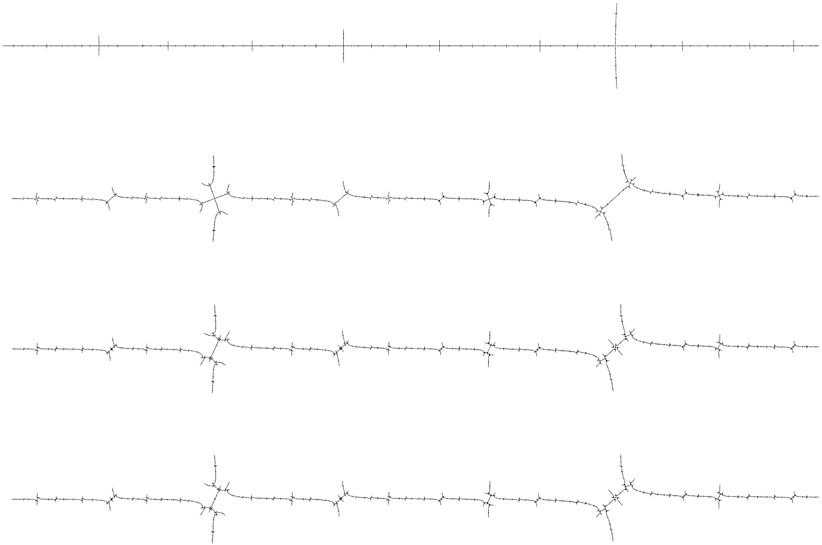

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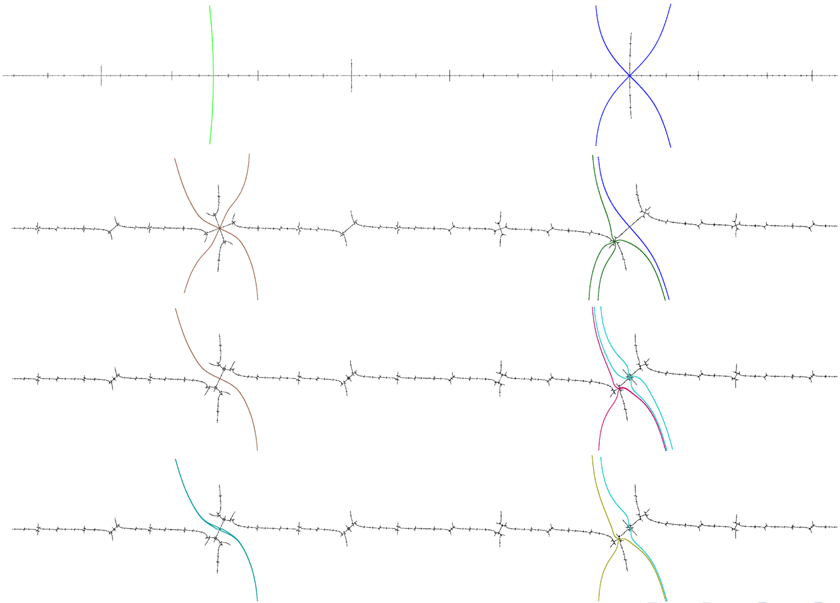
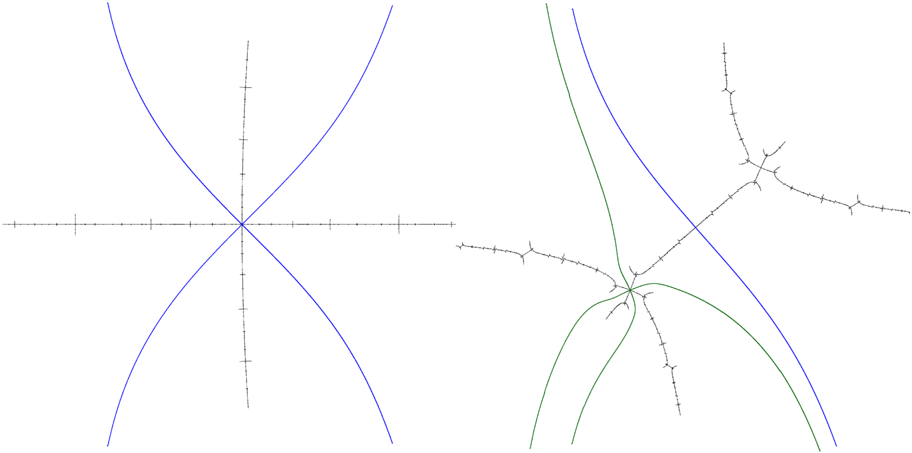


The sequence $\{Q_n\}$; zoom near the branching point



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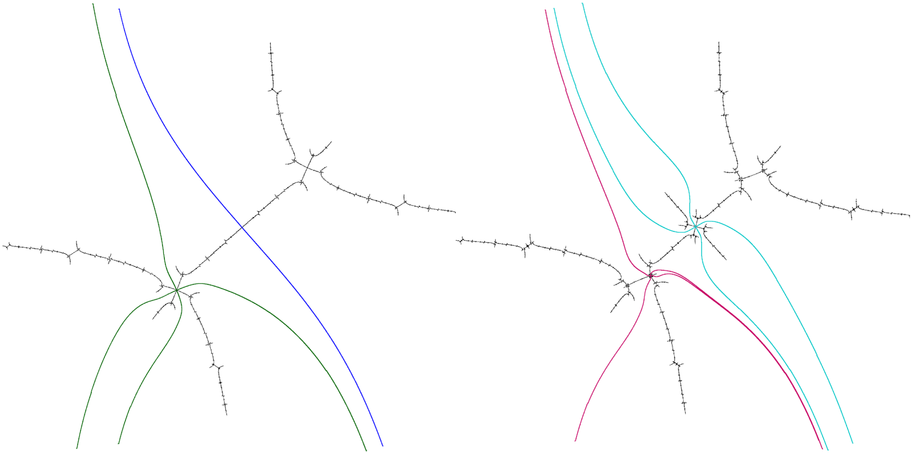

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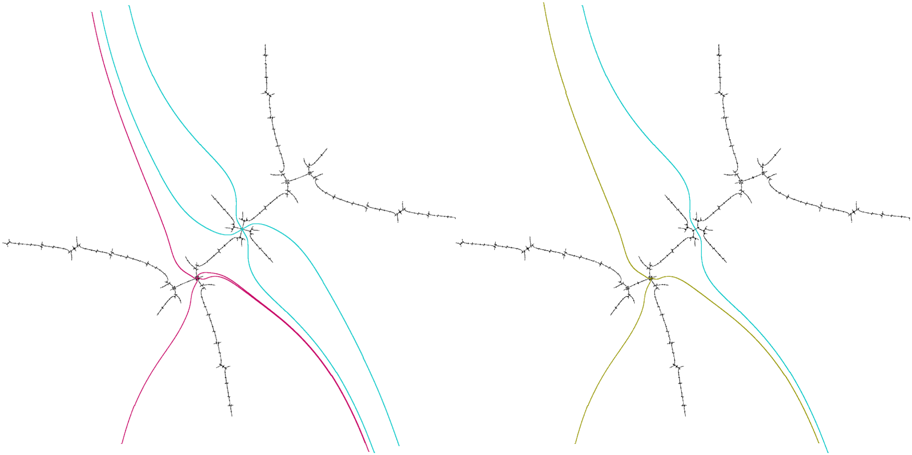

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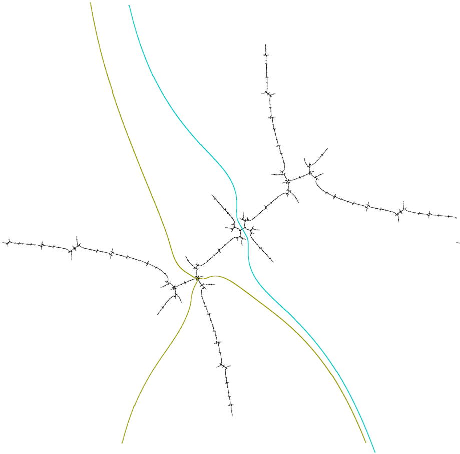

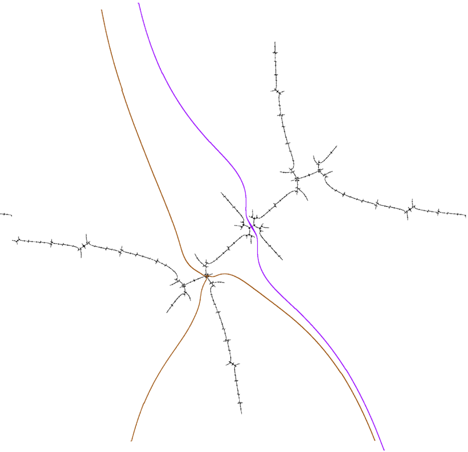

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Renormalizations and Wandering Jordan Curves of Rational Maps

Guizhen Cui[1], **Wenjuan Peng**[1], **Lei Tan**[2]

[1] Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing 100190, People's Republic of China. E-mail: gzcui@math.ac.cn; wenjpeng@amss.ac.cn
[2] LAREMA, UMR 6093 CNRS, Université d'Angers, 2 bd Lavoisier, 49045 Angers, France.
E-mail: Lei.Tan@univ-angers.fr

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Abstract: We realize a dynamical decomposition for a post-critically finite rational map which admits a combinatorial decomposition. We split the Riemann sphere into two completely invariant subsets. One is a subset of the Julia set consisting of uncountably many Jordan curve components. Most of them are wandering. The other consists of components that are pullbacks of finitely many renormalizations, together with possibly uncountably many points. The quotient action on the decomposed pieces is encoded by a dendrite dynamical system. We also introduce a surgery procedure to produce post-critically finite rational maps with wandering Jordan curves and prescribed renormalizations.