NONLINEAR PROPAGATION OF WAVE PACKETS

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Ritsumeikan University, 2010 - 10 - 21 and 22

Our aim in this lecture is to explain the proof of a recent Theorem obtained in collaboration with R. Carles (see [?]). It is the opportunity to explain the context, to stress the important results of the theory and give references to the students.

We are concerned with the solutions of a semi-classical nonlinear Schrödinger equation

(0.1)
$$\begin{cases} i\varepsilon\partial_t\psi^\varepsilon + \frac{\varepsilon^2}{2}\Delta\psi^\varepsilon = V(x)\psi^\varepsilon + \Lambda\,\varepsilon^\alpha|\psi^\varepsilon|^{2\sigma}\psi^\varepsilon, \ (t,x)\in\mathbf{R}_+\times\mathbf{R}^d,\\ \psi^\varepsilon_{|t=0} = \psi^\varepsilon_0 \end{cases}$$

with initial data which are wave packets

$$\psi_0^{\varepsilon}(x) = \varepsilon^{-d/4} a\left(\frac{x - x_0}{\sqrt{\varepsilon}}\right) e^{i(x - x_0) \cdot \xi_0 / \varepsilon}, \ a \in \mathcal{S}(\mathbf{R}^d).$$

More precisely, we are interested in finding asymptotics for the solution ψ^{ε} when ε goes to 0. We suppose $\Lambda \in \mathbf{R}^+$, $\sigma < \frac{2}{d-2}$ for $d \ge 3$ so that, by the results of [?], there exists a unique global solution in $L^2(\mathbf{R}^d) \cap \mathcal{F}(L^2(\mathbf{R}^d))$ under the condition that the potential V is at most of quadratic growth (see assumption ?? for precise statement). Note that the reading of reference [?] is a good way to learn technics about existence results for nonlinear Schrödinger equation. We will also explain the relevances of the hypothesis about σ (see Remark ??) and precise choices of $\alpha \in \mathbf{R}^+$ will be made (see Section 2.1).

In Section 1, we describe the class of wave packets that we will consider and study their basic properties. Section 2 is devoted to derive and analyze an ansatz that we will prove in Section 3 to be an approximated solution to (??). There appears a critical index α_c :

- for $\alpha > \alpha_c$, the ansatz is the same than in the linear regime $\Lambda = 0$,
- for $\alpha = \alpha_c$, there appears nonlinear effects.

We focus in Section 3 on the critical regime $\alpha = \alpha_c$ and prove the main Theorem of this talk thanks to Strichartz estimates.

Notation. For two positive numbers a^{ε} and b^{ε} , the notation $a^{\varepsilon} \leq b^{\varepsilon}$ means that there exists C > 0 independent of ε such that for all $\varepsilon \in]0, 1]$, $a^{\varepsilon} \leq C b^{\varepsilon}$.

1. Wave packets

1.1. **Definition.** Let $(x_0, \xi_0) \in \mathbf{R}^{2d}$ and consider the operator

$$\Gamma^{\varepsilon}_{x_0,\xi_0}: \quad L^2(\mathbf{R}^d) \to L^2(\mathbf{R}^d) \\
 a \mapsto \varepsilon^{-d/4} a\left(\frac{x-x_0}{\sqrt{\varepsilon}}\right) e^{\frac{i}{\varepsilon}\xi_0 \cdot (x-x_0)}$$

The operator $T_{x_0,\xi_0}^{\varepsilon}$ is a unitary operator of $L^2(\mathbf{R}^d)$, its adjoint satisfies

$$\forall f \in L^2(\mathbf{R}^d), \ \forall X \in \mathbf{R}^d, \ (T^{\varepsilon}_{x_0,\xi_0})^* f(X) = \varepsilon^{d/4} f(x_0 + \sqrt{\varepsilon}X) \mathrm{e}^{-\frac{i}{\sqrt{\varepsilon}}\xi_0 \cdot X}.$$

Note also that we have

$$\varepsilon^{-d/2} \mathcal{F}\left(T_{x_0,\xi_0}^{\varepsilon}a\right) \left(\frac{\xi}{\varepsilon}\right) = \varepsilon^{-d/4} \mathcal{F}(a) \left(\frac{\xi-\xi_0}{\sqrt{\varepsilon}}\right) e^{-\frac{i}{\varepsilon}x_0\cdot\xi}.$$

We consider the functions $T_{x_0,\xi_0}^{\varepsilon}a$ for $a \in \mathcal{S}(\mathbf{R}^d)$ that we call **wave packets**. These wave packets have similar features in configuration space variables (the variable x) and in momenta variables (the Fourier variable ξ/ε). There exists more refined version of these wave packets introduced by G. Hagedorn in [?].

With $p \in \mathcal{C}_0^{\infty}(\mathbf{R}^{2d})$, we associate the pseudodifferential semiclassical operator $p(x, \varepsilon D)$ which is defined with classical quantization by

$$p(x,\varepsilon D)f(x) = (2\pi\varepsilon)^{-d} \int e^{\frac{i}{\varepsilon}\xi \cdot (x-y)} p(x,\xi)f(y)dy\,d\xi, \quad \forall f \in \mathcal{S}(\mathbf{R}^d).$$

The main interest of these wave packets relies on the following proposition.

$$\begin{split} \mathbf{Proposition 1.1.} & \text{If } a \in \mathcal{S}(\mathbf{R}^d) \text{ and } p \in \mathcal{C}_0^{\infty}(\mathbf{R}^{2d}), \text{ then} \\ (1.1)(T_{x_0,\xi_0}^{\varepsilon})^* p(x,\varepsilon D) T_{x_0,\xi_0}^{\varepsilon} a &= p(x_0,\xi_0)a + \sqrt{\varepsilon} \, d_{x,\xi} p(x_0,\xi_0)(X,D)a \\ &\quad + \frac{\varepsilon}{2} \, d_{x,\xi}^2 p(x_0,\xi_0)(X,D) \cdot (X,D)a + \mathcal{O}(\varepsilon^{3/2}) \end{split}$$

in $L^2(\mathbf{R}^d)$.

Proof. The proof relies on Taylor formula. We have

$$\begin{aligned} (T^{\varepsilon}_{x_{0},\xi_{0}})^{*}p(x,\varepsilon D)T^{\varepsilon}_{x_{0},\xi_{0}}a(X) \\ &= (2\pi\varepsilon)^{-d}\mathrm{e}^{-\frac{i}{\sqrt{\varepsilon}}X\cdot\xi_{0}}\int p(x_{0}+\sqrt{\varepsilon}X,\xi)\mathrm{e}^{\frac{i}{\varepsilon}\xi\cdot(x_{0}+\sqrt{\varepsilon}X-y)}\mathrm{e}^{\frac{i}{\varepsilon}\xi_{0}\cdot(y-x_{0})}a\left(\frac{y-x_{0}}{\sqrt{\varepsilon}}\right)dyd\xi \end{aligned}$$

The change of variables $\xi = \xi_0 + \sqrt{\varepsilon}\zeta$ and $y = x_0 + \sqrt{\varepsilon}Y$ gives

$$(T_{x_0,\xi_0}^{\varepsilon})^* p(x,\varepsilon D) T_{x_0,\xi_0}^{\varepsilon} a(X) = (2\pi)^{-d} \int p(x_0 + \sqrt{\varepsilon}X,\xi_0 + \sqrt{\varepsilon}\zeta) e^{i\zeta \cdot (X-Y)} a(Y) dY d\zeta$$

We perform a Taylor expansion of $p(x_0 + \sqrt{\varepsilon}X,\xi_0 + \sqrt{\varepsilon}\zeta)$
 $p(x_0 + \sqrt{\varepsilon}X,\xi_0 + \sqrt{\varepsilon}\zeta) = p(x_0,\xi_0) + \sqrt{\varepsilon} dp_{x,\xi}(x_0,\xi_0)(X,\zeta)$

$$+\varepsilon d_{x,\xi}^2 p(x_0,\xi_0)(X,\zeta) \cdot (X,\zeta) + DT_3(x_0,\xi_0,\sqrt{\varepsilon}X,\sqrt{\varepsilon}\zeta)$$

where

$$DT_3(x_0,\xi_0,\sqrt{\varepsilon}X,\sqrt{\varepsilon}\zeta) = \frac{\varepsilon^{3/2}}{2} \int_0^1 d_{x,\xi}^3 p(x_0+s\sqrt{\varepsilon}X,\xi_0+s\sqrt{\varepsilon}\zeta)(1-s)^2 [X,\zeta]^3 ds.$$

In view of

$$(2\pi)^{-d} \int Xa(Y) e^{i\zeta \cdot (X-Y)} dY d\zeta = Xa(X),$$
$$(2\pi)^{-d} \int \zeta a(Y) e^{i\zeta \cdot (X-Y)} dY d\zeta = Da(X),$$

we obtain the three first terms of (??). It remains to check that the remainder is small in $L^2(\mathbf{R}^d)$. We write

$$\int DT_3(x_0,\xi_0,\sqrt{\varepsilon}X,\sqrt{\varepsilon}\zeta)a(Y)\mathrm{e}^{i\zeta\cdot(X-Y)}dYd\zeta = \int DT_3(x_0,\xi_0,\sqrt{\varepsilon}X,\sqrt{\varepsilon}\zeta)\widehat{a}(\zeta)\mathrm{e}^{i\zeta\cdot X}d\zeta$$
$$=\varepsilon^{3/2}\sum_{|\alpha|+|\beta|=3}c_{\alpha,\beta}\int_0^1 I_{\alpha,\beta}(X,s)ds$$

where $c_{\alpha,\beta}$ are real numbers and

$$I_{\alpha,\beta}(X,s) = \int \partial_x^{\alpha} \partial_{\xi}^{\beta} p(x_0 + \sqrt{\varepsilon}sX, \xi_0 + \sqrt{\varepsilon}s\zeta) X^{\alpha} \zeta^{\beta} \widehat{a}(\zeta) e^{i\zeta \cdot X} d\zeta dX.$$

By integration by parts, we obtain for $\gamma \in \mathbf{N}^d$,

$$\begin{aligned} |X^{\gamma}I_{\alpha,\beta}(X)| &= \left| \int \partial_{\zeta}^{\gamma} \left(\zeta^{\beta}\partial_{x}^{\alpha}\partial_{\xi}^{\beta}p(x_{0}+\sqrt{\varepsilon}sX,\xi_{0}+\sqrt{\varepsilon}s\zeta)\widehat{a}(\zeta) \right) \mathrm{e}^{iX\zeta}d\zeta \right| \\ &\lesssim \sup_{|\kappa|\leqslant|\gamma|} \|\partial_{\zeta}^{\beta}\widehat{a}(\zeta)\|_{L^{1}}. \end{aligned}$$

1.2. Wave packets and P.D.E.s. As a consequence, if we consider a semiclassical evolution equation of pseudodifferential type

(1.2)
$$\begin{cases} i\varepsilon\partial_t\psi^\varepsilon = p(x,\varepsilon D)\psi^\varepsilon, \\ \psi^\varepsilon_{|t=0} = T^\varepsilon_{x_0,\xi_0}a, \ a\in\mathcal{S}(\mathbf{R}^d), \end{cases}$$

one can look for solutions of the form

$$\psi^{\varepsilon}(t,x) = T^{\varepsilon}_{x(t),\xi(t)}a^{\varepsilon}(t,\cdot)$$

where $x(t), \xi(t) \in \mathbf{R}^d$ and $a^{\varepsilon}(t, .)$ have to be determined with $x(0) = x_0, \xi(0) = \xi_0$ and $a^{\varepsilon}(0, \cdot) = a(\cdot)$.

A simple computation gives

$$(T_{x(t),\xi(t)}^{\varepsilon})^{*} (i\varepsilon\partial_{t} - p(x,\varepsilon D)) T_{x(t),\xi(t)}^{\varepsilon} = \xi(t) \cdot \dot{x}(t) - p(x(t),\xi(t)) + \sqrt{\varepsilon} \left(\dot{x}(t) \cdot D - \dot{\xi}(t) \cdot X - d_{x,\xi}p(x(t),\xi(t)) \cdot (X,D) \right) + \varepsilon \left(i\partial_{t} - d^{2}p(x(t),\xi(t))(X,D) \cdot (X,D) \right) + O(\varepsilon^{3/2})$$

in $\mathcal{L}(L^2(\mathbf{R}^d))$. As a consequence, it is natural to choose the trajectories $(x(t), \xi(t))$ such that

$$\begin{cases} \dot{x}(t) = \nabla_{\xi} p(x(t), \xi(t)), \ x(0) = x_0, \\ \dot{\xi}(t) = -\nabla_x p(x(t), \xi(t)), \ \xi(0) = \xi_0, \end{cases}$$

and the function $a^{\varepsilon}(t, x)$ of the form

$$a^{\varepsilon}(t,X) = e^{\frac{i}{\varepsilon}S(t,x)}b(t,X) \text{ with } S(t) = \int_0^t \left(\xi(s) \cdot \dot{x}(s) - p(x(s),\xi(s))\right) ds$$

and b solution to

$$\begin{cases} i\partial_t b = d^2 p(x(t), \xi(t))(X, D) \cdot (X, D) b, \\ b_{|t=0} = a. \end{cases}$$

The curves $(x(t), \xi(t))$ are called **the classical trajectories** or **Hamiltonain** curves of the function $p(x, \xi)$ and the function S(t) is the classical action.

Then, the function

(1.3)
$$\psi_{app}^{\varepsilon} = e^{\frac{i}{\varepsilon}S(t)}T_{x(t),\xi(t)}^{\varepsilon}b(t)$$

is a natural ansatz. Of course, there is something to prove according to the assumptions than one makes on the symbol p. In the next subsection, we justify such an asymptotic in the case of the linear semi-classical Schrödinger equation (equation (??) with $\Lambda = 0$). This equation is particularly relevant in quantum mechanics and the fact that ε is small implies that its solutions ψ^{ε} oscillate rapidly. For this reason, it is difficult to perform numerics on the semi-classical equation. However, the computation of ψ^{ε}_{app} given by (??) is made by solving ε -independent ode's or pde's, which is much more easy from a numerical point of view. Therefore, these wave packets allow to construct approximated solutions of the semi-classical Schrödinger equation. Note that there exists more sophisticated families of wave packets, such as Hagedorn's wave packets; the interested reader can refer to [?].

1.3. Application to the linear Schrödinger equation. The equation

$$i\varepsilon\partial_t\psi^\varepsilon + \frac{\varepsilon^2}{2}\Delta\psi^\varepsilon = V(x)\psi^\varepsilon$$

is of the form studied in the previous section with

$$p(x,\xi) = \frac{|\xi|^2}{2} + V(x).$$

Assumption 1.2. From now on, we suppose that V is smooth and at most quadratic:

$$\forall \gamma \in \mathbf{N}^d, \ |\gamma| \ge 2, \ \exists C_{\gamma} > 0, \ \forall x \in \mathbf{R}^d, \ |\partial^{\gamma} V(x)| \leqslant C_{\gamma}$$

The classical trajectories satisfy

$$\left\{ \begin{array}{l} \dot{x}(t) = \xi(t), \\ \dot{\xi}(t) = -\nabla V(x(t)). \end{array} \right. \label{eq:constraint}$$

Because of the assumption on the potential V, these trajectories grow at most exponentially (see the book [?]).

Lemma 1.3. There exist $C_0, C_1 > 0$ such that

$$\forall t \in \mathbf{R}^+, \ |x(t)| + |\xi(t)| \leqslant C_0 \,\mathrm{e}^{C_1 t}.$$

Proof. We observe that

$$\ddot{x}(t) + \nabla V\left(x(t)\right) = 0.$$

Multiply this equation by $\dot{x}(t)$,

$$\frac{d}{dt}\left(\left(\dot{x}\right)^{2}+V\left(x(t)\right)\right)=0,$$

and notice that in view of Assumption ??, $V(x) \lesssim (1+|x|^2)^{1/2}$:

$$\dot{x}(t) \lesssim \langle x(t) \rangle$$

and the estimate follows.

The classical action is

$$S(t) = \int_0^t \left(\frac{1}{2}|\xi(s)|^2 - V(x(s))\right) ds.$$

The profile equation becomes

$$\begin{cases} \partial_t b + \frac{1}{2}\Delta b = \frac{1}{2}V''(x(t))X \cdot Xb \\ b_{|t=0} = a. \end{cases}$$

We refer to [?] for a proof of the existence of solutions to the profile equation. Besides, by multiplying the equation by x and by differentiating it, one obtains a closed system on Xb and $\nabla_X b$ and by a recursive argument, one can prove the following result

Lemma 1.4. For all $k \in \mathbf{N}$, there exists constants c_k and C_k such that

$$\forall \alpha, \beta \in \mathbf{N}^d, \ |\alpha| + |\beta| = k, \ \left\| X^{\alpha} \partial_X^{\beta} b(t, \cdot) \right\|_{L^2} \leqslant C_k \mathrm{e}^{c_k t}.$$

The estimates of these two Lemma are sharp: the special case V''(x(t)) = -Id shows it. For references on the properties of classical trajectories, the reader can refer to the book [?] and for the existence of solution of the profile equation to [?].

We can now state the result in the linear case. Note that many authors have worked on this subject and we will note refer to any special one.

Theorem 1.5. There exists two constants c and C such that

$$\forall t \in \mathbf{R}, \ \left\| \psi^{\varepsilon}(t) - \mathrm{e}^{\frac{i}{\varepsilon}S(t)} T^{\varepsilon}_{x(t),\xi(t)} b(t, \cdot) \right\|_{L^{2}(\mathbf{R}^{d})} \leqslant \sqrt{\varepsilon} C \, \mathrm{e}^{ct}.$$

The approximation of ψ^{ε} by

(1.4)
$$\phi_{lin}^{\varepsilon} = e^{\frac{i}{\varepsilon}S(t)}T_{x(t),\xi(t)}^{\varepsilon}b(t,\cdot)$$

holds until times of order Log $\left(\frac{1}{\varepsilon}\right)$ which is called the **Ehrenfest time**.

Proof. The proof of the Theorem relies on an energy estimate. The function

$$w^{\varepsilon} = \psi^{\varepsilon} - \phi^{\varepsilon}_{lin}$$

satisfies $w^{\varepsilon}(0) = 0$ and

$$\begin{split} i\varepsilon\partial_t w^{\varepsilon} &+ \frac{\varepsilon^2}{2}\Delta w^{\varepsilon} - V(x)w^{\varepsilon} \\ &= -\varepsilon^{3/2}\mathrm{e}^{\frac{i}{\varepsilon}S(t)}T^{\varepsilon}_{x(t),\xi(t)}\int_0^1 V^{(3)}(x(t) + \sqrt{\varepsilon}sX)[X,X,X]b(t,X)(1-s)^2ds. \end{split}$$

This implies

$$\frac{d}{dt} \| w^{\varepsilon}(t) \|_{L^2} \lesssim \sqrt{\varepsilon} \| X^3 b(t, \cdot) \|_{L^2} \lesssim \sqrt{\varepsilon} e^{Ct}.$$

5

2. Nonlinear Schrödinger equation

2.1. The ansatz and the critical exponent. We now consider the nonlinear Schrödinger equation (??) ($\Lambda > 0$) and we argue similarly than in the linear case. We observe that for $a \in \mathcal{S}(\mathbf{R}^d)$,

$$\left|T_{x(t),\xi(t)}^{\varepsilon}a\right|^{2\sigma}T_{x(t),\xi(t)}^{\varepsilon}a=\varepsilon^{-d\sigma/2}T_{x(t),\xi(t)}^{\varepsilon}\left(|a|^{2\sigma}a\right).$$

Therefore, if we look for a solution of (??) of the form

$$\psi^{\varepsilon}(t) = \mathrm{e}^{\frac{i}{\varepsilon}S(t)}T^{\varepsilon}_{x(t),\xi(t)}u^{\varepsilon}(t,\cdot),$$

the function $u^{\varepsilon}(t, X)$ must satisfy

$$\begin{cases} i\partial_t u^{\varepsilon} + \frac{1}{2}\Delta u^{\varepsilon} = \frac{1}{2}V''(x(t)X \cdot Xu^{\varepsilon} + \varepsilon^{\alpha - \frac{d\sigma}{2} - 1}|u^{\varepsilon}|^{2\sigma}u^{\varepsilon}, \\ u^{\varepsilon}_{|t=0} = a. \end{cases}$$

There appears a critical exponant

$$\alpha_c = 1 + \frac{d\sigma}{2}$$

and one can prove the following:

If α > α_c, the function ψ^ε(t) is asymptotic to the function φ^ε_{lin} defined in (??). The function ψ^ε(t) is said to be linearizable since it is asymptotic to the solution of the associated linear equation with the same initial data.
If α = α_c, the function ψ^ε(t) is asymptotic to

(2.1)
$$\phi^{\varepsilon} = e^{\frac{i}{\varepsilon}S(t)}T^{\varepsilon}_{x(t),\xi(t)}u(t,\cdot),$$

where u solves

(2.2)
$$\begin{cases} \partial_t u + \frac{1}{2}\Delta u = \frac{1}{2}V''(x(t))X \cdot Xu \\ u_{|t=0} = a. \end{cases}$$

The first question which arises is the existence of solutions to (??) and the control of its momenta (the quantities $X^{\alpha}\partial_X^{\beta}u(t,X)$ for multiindices $\alpha, \beta \in \mathbf{N}^d$). The proof of the linearizable case is similar to the one performed in the critical case, thus we will focus on the situation where $\alpha = \alpha_c$. Before stating the main theorem and proving it, we begin by stating results about the profile equation (??).

2.2. The envelope equation in the critical case. Equations of the form of (??) have been studied by R. Carles in [?]. Before stating his result, let us introduce a notation. We define the assumption $(\text{Exp})_k$.

Definition 2.1. Let $k \in \mathbf{N}$. We say that the function u satisfies $(Exp)_k$ if there exist constants c_k and C_k such that

$$\forall \alpha, \beta \in \mathbf{N}^d, \ |\alpha| + |\beta| \leqslant k, \ \left\| X^{\alpha} \partial_X^{\beta} u(t, \cdot) \right\|_{L^2} \leqslant C_k \, \mathrm{e}^{c_k t}.$$

We also introduce the energy space

$$\Sigma = \{ f \in L^2(\mathbf{R}^d), \ Xf, \nabla_X f \in L^2 \}.$$

We have the following result.

Theorem 2.2 (Carles 2009, [?]).

• If $\sigma < \frac{2}{d-2}$ for $d \ge 3$ and $a \in \Sigma$, then there exists a unique global solution to (??), $u \in C(\mathbf{R}, \Sigma)$.

• If $\sigma = d = 1$ and $a \in \mathcal{S}(\mathbf{R})$, then for all $k \in \mathbf{N}$, $(\mathrm{Exp})_k$ is satisfied.

We point out that, according to [?], the second part of the Theorem is also true if V''(x(t)) is diagonal with negative eigenvalues or if $t \mapsto V''(x(t))$ is compactly supported. For the moment, the question whether it is true or not in other cases is an open problem. Such assumptions are not pertinent in our context. Since in the linear case, we have used $(\text{Exp})_3$, we infer that we will need such an assumption in the nonlinear case. Therefore, our results will be more pertinent when $\sigma = d = 1$ (the one dimensional cubic case); in higher dimension, we will work under the assumption that we have $(\text{Exp})_k$ for k large enough.

3. Analysis of the critical case

We consider the scaled energy space

$$\Sigma_1^{\varepsilon} = \{ f \in L^2(\mathbf{R}^d), \ \|xf\|_{L^2} + \|\varepsilon \nabla f\|_{L^2} < +\infty \}$$

equipped with the norm

$$||f||_{\mathcal{H}} = ||f||_{L^2} + ||A_{\varepsilon}(t)f||_{L^2} + ||B_{\varepsilon}(t)f||_{L^2}$$

where

$$A_{\varepsilon}(t) = i rac{D_x - \xi(t)}{\sqrt{\varepsilon}} ext{ and } B_{\varepsilon}(t) = rac{x - x(t)}{\sqrt{\varepsilon}}.$$

Our aim is to prove the following Theorem

Theorem 3.1 (Carles - Fermanian 2010, [?]). Let $d \ge 1$, $\sigma \in \mathbf{N}$ with $\sigma < \frac{2}{d-2}$ if $d \ge 3$.

• If $(Exp)_4$ is satisfied, there exists C > 0 such that

$$\sup_{0 \leq t \leq C \operatorname{Log} \operatorname{Log}\left(\frac{1}{\varepsilon}\right)} \|\psi^{\varepsilon}(t) - \phi^{\varepsilon}(t)\|_{\mathcal{H}} \underset{\varepsilon \to 0}{\longrightarrow} 0.$$

• If $d = \sigma = 1$,

$$\sup_{0 \leqslant t \leqslant C \operatorname{Log}\left(\frac{1}{\varepsilon}\right)} \|\psi^{\varepsilon}(t) - \phi^{\varepsilon}(t)\|_{\mathcal{H}} \underset{\varepsilon \to 0}{\longrightarrow} 0.$$

In the case $d = \sigma = 1$, one can prove the approximation on time of order $\text{LogLog}\left(\frac{1}{\varepsilon}\right)$ only by use of energy estimates. In order to gain the result for Ehrenfest time and to treat the other cases, we use Strichartz estimates. Note that the situation $d = \sigma = 1$ is the only situation which is L^2 subcritical ($\sigma < 2/d$) in the nonlinear terminology; this explains why the result is better in that case.

Note also that a nonlinear superposition principle is proved in [?] for initial data which are the sum of wave packets. This is again a manifestation of the fact that the critical regime $\alpha = \alpha_c$ is weakly nonlinear: the nonlinear effects only modify the profile of the wave packets and not its trajectory.

3.1. Strichartz estimates. We denote by H^{ε} the semi-classical Schrödinger operator and by $U^{\varepsilon}(t)$ its associated propagator:

$$H^{\varepsilon} = -\frac{\varepsilon^2}{2}\Delta + V(x), \ U^{\varepsilon}(t) = e^{-itH^{\varepsilon}}.$$

Theorem 3.2 (Fujiwara 1979 and 1980, [?] and [?]). There exists $\delta_0 > 0$ independent of ε and C such that

$$\forall |t| < \delta_0, \ \|U^{\varepsilon}(t)\|_{\mathcal{L}(L^1(\mathbf{R}^d), L^{\infty}(\mathbf{R}^d))} \leqslant C \, (\varepsilon|t|)^{-d/2}.$$

This dispersive estimate implies Strichartz estimates. One says that an exponant (q, r) is admissible if it satisfies

(3.1)
$$\frac{2}{q} = d\left(\frac{1}{2} - \frac{1}{r}\right) := \delta(r)$$

for $2 \leq r < \infty$ if $d = 2, 2 \leq r \leq \infty$ if d = 1 and $2 \leq r \leq \frac{2d}{d-2}$ if d > 2, where \overline{p} is the conjugated exponant to p

$$\frac{1}{p} + \frac{1}{\overline{p}} = 1.$$

Corollary 3.3. Consider I an interval of \mathbf{R}^+ , one has the following scaled Strichartz estimates

$$(3.2) \|U^{\varepsilon}(t)f\|_{L^{q}(I,L^{r}(\mathbf{R}^{d}))} \leq C(r,I) \varepsilon^{-\frac{1}{q}} \|f\|_{L^{2}(\mathbf{R}^{d})} \\ \left\|\int_{0}^{t} U^{\varepsilon}(t-s)f(s)ds\right\|_{L^{q_{1}}(I,L^{r_{1}}(\mathbf{R}^{d}))} \leq C(r_{1},r_{2},I) \varepsilon^{-\frac{1}{q_{1}}-\frac{1}{q_{2}}} \|f\|_{L^{\overline{q_{2}}}(I,L^{\overline{r_{2}}}(\mathbf{R}^{d}))}$$

where (q, r), (q_1, r_1) and (q_2, r_2) are pairs of admissible exponents.

Proof. These Strichartz estimates are derived from the dispersive estimates by the so-called ' TT^* argument' (see [?]). We shortly explain this argument. By the dispersive estimate and by the fact that $U^{\varepsilon}(t)$ is a unitary group, we are left with two estimates

$$\begin{aligned} \|U^{\varepsilon}(t-s)\|_{L^{1}(\mathbf{R}^{d})\mapsto L^{\infty}(\mathbf{R}^{d})} &\leqslant \quad C(\varepsilon|t-s|)^{-d/2}, \\ \|U^{\varepsilon}(t-s)\|_{L^{2}(\mathbf{R}^{d})\mapsto L^{2}(\mathbf{R}^{d})} &= \quad 1 \end{aligned}$$

for $|t - \tau| < \delta_0$. Complex interpolation for $2 \leq r \leq +\infty$ gives for $|t - s| < \delta_0$

$$\|U^{\varepsilon}(t-s)\|_{L^{\overline{r}}(\mathbf{R}^d)\mapsto L^q(\mathbf{R}^d)} \leqslant C(\varepsilon|t-s|)^{-\delta(r)}.$$

We now consider $f \in L^2(\mathbf{R}^d)$ and $g \in L^2(-]\delta_0, \delta_0[\times \mathbf{R}^d)$ and we have

$$\left|\int_{\mathbf{R}} \left(U^{\varepsilon}(t)f,g(t)\right)_{L^{2}(\mathbf{R}^{d})} dt\right| \leq \|f\|_{L^{2}(\mathbf{R}^{d})} \left\|\int_{\mathbf{R}} U^{\varepsilon}(-t)g(t)dt\right\|_{L^{2}(\mathbf{R}^{d})}$$

We observe that

$$\begin{split} \left\| \int U^{\varepsilon}(-t)g(t)dt \right\|_{L^{2}(\mathbf{R}^{d})}^{2} &= \int_{\mathbf{R}\times\mathbf{R}} \left(U^{\varepsilon}(-t)g(t), U^{\varepsilon}(-s)g(s) \right)_{L^{2}(\mathbf{R}^{d})} dt ds \\ &= \int_{\mathbf{R}\times\mathbf{R}} \left(g(t), U^{\varepsilon}(t-s)g(s) \right)_{L^{2}(\mathbf{R}^{d})} dt ds \\ &\leqslant C \int_{\mathbf{R}\times\mathbf{R}} \left(\varepsilon |t-s| \right)^{-\delta(r)} \|g(t)\|_{L^{\overline{r}}(\mathbf{R}^{d})} \|g(s)\|_{L^{\overline{r}}(\mathbf{R}^{d})} dt ds \\ &\leqslant C \varepsilon^{-2/q} \|g\|_{L^{\overline{q}}([-\delta_{0},\delta_{0}], L^{\overline{r}}(\mathbf{R}^{d}))} \end{split}$$

since $\delta(r) = 2/q$ and where we have used Hardy-Littlewood inequality (for q > 2). In the critical case (q = 2), the above analysis breaks down and we refer to [?]. Therefore, we have obtained

$$\left| \int_{\mathbf{R}} \left(U^{\varepsilon}(t) f, g(t) \right)_{L^{2}(\mathbf{R}^{d})} dt \right| \leq C \varepsilon^{-1/q} \| f \|_{L^{2}(\mathbf{R})} \| g \|_{L^{\overline{q}}([-\delta_{0}, \delta_{0}], L^{\overline{r}}(\mathbf{R}^{d}))},$$

whence

$$\|U^{\varepsilon}(t)f\|_{L^{q}([-\delta_{0},\delta_{0}]),L^{r}(\mathbf{R}^{d})} \leqslant C\varepsilon^{-1/q}\|f\|_{L^{2}(\mathbf{R}^{d})}.$$

In order to conclude to the proof of (??), we decompose I into small intervals of length 2δ .

3.2. Schedule of the proof of Theorem ??. Set $w^{\varepsilon} = \psi^{\varepsilon} - \phi^{\varepsilon}$, then w^{ε} satisfies

$$i\varepsilon\partial_t w^{\varepsilon} + \frac{\varepsilon^2}{2}\Delta w^{\varepsilon} - V(x)w^{\varepsilon} = L^{\varepsilon}(t,x) + NL^{\varepsilon}(t,x)$$

where L^{ε} is the linear contribution to the remainder (the same one than in the linear case):

$$||L^{\varepsilon}(t)||_{L^{2}} \lesssim \varepsilon^{3/2} ||x^{3}u(t)||_{L^{2}}$$

and NL^{ε} contains the nonlinear terms

$$(3.3) NL^{\varepsilon}(t,x) = \varepsilon^{\alpha} \left(|\phi^{\varepsilon} + w^{\varepsilon}|^{2\sigma} (\phi^{\varepsilon} + w^{\varepsilon}) - |\phi^{\varepsilon}|^{2\sigma} \phi^{\varepsilon} \right), \\ \|NL^{\varepsilon}(t,x)\| \lesssim \varepsilon^{\alpha} \left(|\phi^{\varepsilon}|^{2\sigma} + |w^{\varepsilon}|^{2\sigma} \right) w^{\varepsilon}.$$

Using Duhamel formula, we have on step time intervals of length τ

$$w^{\varepsilon}(t+\tau) = U^{\varepsilon}(\tau)w^{\varepsilon}(t) + \frac{i}{\varepsilon}\int_{0}^{\tau}U^{\varepsilon}(\tau-s)L^{\varepsilon}(t+s)ds + \frac{i}{\varepsilon}\int_{0}^{\tau}U^{\varepsilon}(\tau-s)NL^{\varepsilon}(t+s)ds + \frac{i}{\varepsilon}\int_{0}^{\tau}U^{\varepsilon}(\tau-s)NL^{\varepsilon}(\tau-s)NL^{\varepsilon}(t+s)ds + \frac{i}{\varepsilon}\int_{0}^{\tau}U^{\varepsilon}(\tau-s)NL^{\varepsilon$$

First step: Choosing a Strichartz admissible pair. In order to use a Strichartz estimate, we choose the pair (q, r)

$$q = \frac{4\sigma + 4}{d\sigma}$$
 and $r = 2\sigma + 2$

This pair has the advantage that it will fit with the use of Hölder estimate in the nonlinear contribution (??): there exists θ such that

$$\frac{1}{\overline{q}} = \frac{2\sigma}{\theta} + \frac{1}{q}$$
 and $\frac{1}{\overline{r}} = \frac{2\sigma}{r} + 1\overline{r}$

with

$$\theta = \frac{2\sigma(2\sigma+2)}{2-(d-2)\sigma}.$$

Remark 3.4. Note that $\theta > 0$ because $\sigma < \frac{2}{d-2}$.

The advantage of this pair is that all the Lebesgue norm on \mathbf{R}^d will be associated with the same exponant r. Using Strichartz estimate, we obtain for $I = [t, t + \tau]$

$$\begin{aligned} \|w^{\varepsilon}\|_{L^{q}(I,L^{r})} &\lesssim \quad \varepsilon^{-1/q} \|w^{\varepsilon}(t)\|_{L^{2}} + \varepsilon^{-1-1/q} \|L^{\varepsilon}\|_{L^{1}(I,L^{2})} \\ &+ \varepsilon^{-1-2/q+\alpha} \left(\|\phi^{\varepsilon}\|_{L^{\theta}(I,L^{r})}^{2\sigma} + \|w^{\varepsilon}\|_{L^{\theta}(I,L^{r})}^{2\sigma} \right) \|w^{\varepsilon}\|_{L^{q}(I,L^{r})}. \end{aligned}$$

One sees that it will be important to know something about the L^r norm of $\phi^{\varepsilon}(t)$ (and $w^{\varepsilon}(t)$).

Second step: Estimation of $\|\phi^{\varepsilon}(t)\|_{L^r}$. We observe that scaled Gagliardo-Nirenberg inequality

(3.4)
$$\|f\|_{L^r} \leqslant C(r)\varepsilon^{-\delta(r)/2} \|f\|_{L^2}^{1-\delta(r)} \|A_{\varepsilon}(t)f\|_{L^2}^{\delta(r)}$$

gives

(3.5)
$$\|\phi^{\varepsilon}(t)\|_{L^r} \lesssim \varepsilon^{-\delta(r)/2} \mathrm{e}^{Ct}.$$

Third step: A bootstrap argument. It is reasonable to suppose that w^{ε} is not worse than ϕ^{ε} . For this reason, we suppose that $w^{\varepsilon}(t)$ satisfies the same type of inequality than ϕ^{ε} and we prove a priori estimates under this assumption.

Lemma 3.5. As long as $||w^{\varepsilon}(t)||_{L^r} \leq C_0 \varepsilon^{-\delta(r)/2} e^{C_1 t}$, we have for some constant C,

(3.6)
$$\|w^{\varepsilon}\|_{L^{q}([0,t],L^{r})} \lesssim \varepsilon^{-1/q} \|w^{\varepsilon}\|_{L^{1}(L^{2})} e^{Ct} + \varepsilon^{1/2 - 1/q} e^{Ct},$$

(3.7) $\|w^{\varepsilon}\|_{L^{\infty}([0,t],\mathcal{H})} \lesssim e^{Ct} \|w^{\varepsilon}\|_{L^{1}([0,t],\mathcal{H})} + \sqrt{\varepsilon} e^{Ct}.$

We refer to Sections 3.2 and Section 4 of [?] for the proof of these estimates.

Fourth step: concluding the proof. Equation (??) and the Gronwall Lemma gives the result as long as the bootstrap assumption is satisfied:

$$\|w^{\varepsilon}\|_{L^{\infty}([0,t],\mathcal{H})} \lesssim \sqrt{\varepsilon} e^{e^{Ct}}.$$

To verify this point, one uses again Gagliardo-Nirenberg inequality (??) which gives in view of (??)

$$\|w^{\varepsilon}(t)\|_{L^{r}} \lesssim \varepsilon^{-\delta(r)/2} \sqrt{\varepsilon} e^{e^{Ct}} \lesssim \varepsilon^{-\delta(r)/2} e^{Ct}$$

for $t < \text{LogLog}\left(\frac{1}{c}\right)$.

The case $d = \sigma = 1$: In this situation, we have q = 8, r = 4 and $\theta = 8/3 < 8$ and it is possible to find bounds on the $L^8(L^4)$ norm of the profile u. WE infer the estimate

$$\|\phi^{\varepsilon}(t)\|_{L^4} \lesssim \varepsilon^{-\delta(4)}$$

and Lemma ?? becomes

$$\begin{aligned} \|w^{\varepsilon}\|_{L^{8}([0,t],L^{4})} &\lesssim \quad \varepsilon^{-1/8} \|w^{\varepsilon}\|_{L^{1}(L^{2})} + \varepsilon^{1/2-1/8} \operatorname{e}^{Ct}, \\ \|w^{\varepsilon}\|_{L^{\infty}([0,t],\mathcal{H})} &\lesssim \quad \|w^{\varepsilon}\|_{L^{1}([0,t],\mathcal{H})} + \sqrt{\varepsilon} \operatorname{e}^{Ct}. \end{aligned}$$

Therefore, Gronwall lemma gives $||w^{\varepsilon}||_{L^{\infty}([0,t],\mathcal{H})} \lesssim e^{Ct}$. We emphasize that if $\sigma \in \mathbf{N}$, the only case where $\sigma < 2/d$ is when $d = \sigma = 1$ and one can check that $\theta < q$ if and only if $\sigma < 2/d$.

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