SEMI-CLASSICAL ANALYSIS ON GRADED LIE GROUPS

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Abstract. This text consists in the lecture of a Spring Class given in Göttingen in April 2024. The material is taken from works started with Véronique Fischer and extended in collaborations with Cyril Letrouit, Steven Flynn and Lino Benedetto. We explain how the semi-classical approach that has been widely used in the Euclidean setting since the 70s can be extended to the case of nilpotent Lie groups.

Contents

1. Introduction

Semi-classical analysis has its roots in the foundations of quantum mechanics. Simultaneously with this new theory arose the question of understanding the links between classical and quantum mechanics. It turned out that the Planck constant $\hbar$ can be understood as the obstruction to give a classical description of a quantum particule by the simultaneous knowledge of its position and its impulsion. This is expressed by the Heisenberg uncertainty principle that we first discuss.

1.1. **Uncertainty principle.** In quantum mechanics, a particule is described by a probability measure $|\psi(x)|^2dx$, with ψ a normalized square integrable function on the configuration space $\mathbb{R}^d_x$, called its *wave function*. Denoting by x_j the coordinates of $x\in\mathbb{R}^d$, the *average position* of the particule is the expectation value of the observables x_j

$$\langle x_j\rangle_\psi=\int_{\mathbb{R}^d}x_j|\psi(x)|^2dx,\ 1\le j\le d.$$

Similarly, the *average impulsion* is

$$\langle \xi_j\rangle_\psi=\int_{\mathbb{R}^d}\hbar D_{x_j}\psi(x)\,\overline{\psi}(x)dx=\int_{\mathbb{R}^d}\hbar\xi_j|\widehat{\psi}(\xi)|^2\frac{d\xi}{(2\pi)^d},\quad D_{x_j}=\frac{1}{i}\partial_{x_j},$$

where we have used the Plancherel theorem for the Fourier transform. Considering the variance of these expectation values,

$$\begin{aligned}(d_\psi x_j)^2&=\langle(x_j-\langle x_j\rangle_\psi)^2\rangle_\psi=\int_{\mathbb{R}^d}(x_j-\langle x_j\rangle_\psi)^2|\psi(x)|^2dx,\\(d_\psi\xi_j)^2&=\langle(\xi_j-\langle \xi_j\rangle_\psi)^2\rangle_\psi=\int_{\mathbb{R}^d}(\xi_j-\langle \xi_j\rangle_\psi)^2|\widehat{\psi}(\xi)|^2\frac{d\xi}{(2\pi)^d},\end{aligned}$$

the *Heisenberg uncertainty principle* reads

$$d_\psi x_j\,d_\psi\xi_j\ge\frac{\hbar}{2},\ \ 1\le j\le d. \tag{1.1}$$

The Planck constant $\hbar$ reflects the difference between quantum and classical mechanics, since, in the latter, the position and the impulsion are deterministic variables that can be known with precision. The subject of semi-classical analysis is to understand how one can derive classical mechanics from quantum mechanics, by letting the obstruction $\hbar$ go to 0, even though $\hbar$ is a physical constant. Semi-classical analysis has led to the development of asymptotic technics that are now used in various fields of applied mathematics. For this reason, we will skip the notation $\hbar$ and denote ε a small parameter that is present in some problems of interest involving PDEs. Carrying a semi-classical analysis of a problem consists in investigating the properties of a phenomenon of interest in the limit $\varepsilon\to0$, when ε is a small parameter present in the equation.

1.2. **Gaussian wave packets.** The uncertainty principle is optimal in the sense that there exists a unique family among L^2-functions that saturates the uncertainty principle. This family consists in *Gaussian wave packets*. They are wave functions associated with a classical state $z=(q,p)\in\mathbb{R}^{2d}$ according to

$$g^\varepsilon_z(x)=(\pi\varepsilon)^{-d/4}\exp(-\tfrac{1}{2\varepsilon}|x-q|^2+\tfrac{i}{\varepsilon}p\cdot(x-q)),\qquad x\in\mathbb{R}^d.$$

It is normalized, $\|g^\varepsilon_z\|_{L^2}=1$, and centered in z,

$$\langle x_j\rangle_{g^\varepsilon_z}=q_j\ \text{ and }\ \langle \xi_j\rangle_{g^\varepsilon_z}=p_j,\ \ 1\le j\le d,$$

and saturates the uncertainty principle:

$$d_{g^\varepsilon_z}x_j=d_{g^\varepsilon_z}\xi_j=\sqrt{\frac{\varepsilon}{2}},\ \ 1\le j\le d.$$

Gaussian wave packets have the property of being very localized in the sense that if $\phi\in\mathcal{C}^\infty_c(\mathbb{R}^d)$,

$$\int_{\mathbb{R}^d}\phi(x)|g^\varepsilon_z(x)|^2dx=\phi(q)+O(\sqrt{\varepsilon}). \tag{1.2}$$

They also have this property in Fourier variable

$$\int_{\mathbb{R}^d} \phi(\varepsilon D_x) g_z^\varepsilon(x) \overline{g}_z^\varepsilon(x) dx = \phi(p) + O(\sqrt{\varepsilon}), \tag{1.3}$$

where $\phi(\varepsilon D)$ is the *Fourier multiplier* defined by $\widehat{\phi(\varepsilon D) f} = \phi(\varepsilon\xi)\widehat{f}(\xi)$. This results from the fact that, after rescaling , the Fourier transform of g_z^ε has the same Gaussian structure

$$\varepsilon^{-\frac{d}{2}} \,\mathrm{e}^{i\frac{p\cdot q}{2\varepsilon}}\, \widehat{g_z^\varepsilon}\left(\frac{\xi}{\varepsilon}\right) = \mathrm{e}^{i\frac{(-p)\cdot q}{2\varepsilon}} g_{Jz}^\varepsilon, \quad Jz = (-p, q) \in \mathbb{R}^{2d}.$$

The relations (1.2) and (1.3) suggest that the phase-space point $z = (q, p)$ is the only obstruction to the strong convergence to 0 in $L^2(\mathbb{R}^d)$ of the sequence g_z^ε: if $\phi(q) = 0$, then ϕg_z^ε converges strongly to 0, and similarly, if $\phi(p) = 0$, then $\phi(\varepsilon D) g_z^\varepsilon$ too.

We close this short description of Gaussian wave packets by mentioning their additional frame property: any wave function can be written as a superposition of Gaussian wave packets according to the *Bargmann formula*: for all $f \in L^2(\mathbb{R}^d)$

$$f = (2\pi\varepsilon)^{-\frac{d}{2}} \int_{\mathbb{R}^{2d}} \mathcal{B}_\varepsilon[f](z) g_z^\varepsilon dz,$$

where the *Bargmann transform* [3] is the isometry from $L^2(\mathbb{R}^d)$ into $L^2(\mathbb{R}^{2d})$ defined by

$$\mathcal{B}_\varepsilon[f] = (2\pi\varepsilon)^{-\frac{d}{2}} \left(f, g_z^\varepsilon\right)_{L^2}, \quad z \in \mathbb{R}^{2d}.$$

This formula has been used to construct approximated propagators for Schrödinger equations (see [25, 24]). We refer to the book [3] for more about wave packets.

1.3. **Semiclassical pseudodifferential operators.** The previous analysis of the Gaussian wave packets suggests that the description of oscillating families requires a simultaneous analysis in position and in a rescaled Fourier variables. The theory of semi-classical pseudodifferential operators provided a tool for performing such a program. The quantization problem, or how to associate an operator to an energy, also called Hamiltonian, is a question from quantum mechanics. It gives a mathematical setting for exploring the correspondence between classical and quantum mechanics, and analyzing oscillating phenomena.

Let $a(x, \xi)$ be a *semi-classical observable*, i.e. a function of the Schwartz space $\mathcal{S}(\mathbb{R}^{2d})$. The *semi-classical pseudodifferential operator* of symbol a is the operator $\mathrm{op}_\varepsilon(a)$ defined on functions $f \in \mathcal{S}(\mathbb{R}^d)$ by

$$\mathrm{op}_\varepsilon(a) f(x) = (2\pi\varepsilon)^{-d} \int_{\mathbb{R}^{2d}} a\big(a, \xi\big) \mathrm{e}^{\frac{i}{\varepsilon}\xi\cdot(x-y)} f(y) dy\, d\xi. \tag{1.4}$$

This form is called the *classical quantization*, also called *Kohn-Niremberg quantization*, of the symbol a [5, 27]. Other types of quantizations are possible like the, 'left' quantizations, where the symbol appears in the form $a(y, \xi)$, or the Weyl quantization that has the advantage to be a symmetric quantization [17, 27].

The operator $\mathrm{op}_\varepsilon(a)$ maps $\mathcal{S}(\mathbb{R}^d)$ into itself and, by duality, $\mathcal{S}'(\mathbb{R}^d)$ into itself. Its kernel k_ε can be expressed in terms of the inverse Fourier transform of a in the variable ξ

$$\kappa(x, v) = (2\pi)^{-d} \int_{\mathbb{R}^d} a(x, \xi) \mathrm{e}^{i\xi\cdot v} d\xi, \quad (x, v) \in \mathbb{R}^{2d}. \tag{1.5}$$

Indeed, one has

$$k_\varepsilon(x, y) = \frac{1}{\varepsilon^d}\, \kappa\left(x, \frac{x-y}{\varepsilon}\right), \quad (x, y) \in \mathbb{R}^{2d}.$$

As a consequence of the Schur Lemma, the operator $\mathrm{op}_\varepsilon(a)$ maps $L^2(\mathbb{R}^d)$ into itself and

$$\|\mathrm{op}_\varepsilon(a)\|_{\mathcal{L}(L^2(\mathbb{R}^d))} \leq \int_{\mathbb{R}^d} \sup_{x\in\mathbb{R}^d} |\kappa(x,v)| dv$$

for $C>0$ independent of a and ε.

Moreover, the set of pseudos is an algebra that enjoys symbolic calculus: if $a,b\in\mathcal{S}(\mathbb{R}^{2d})$, then in $\mathcal{L}(L^2(\mathbb{R}^d))$,

$$\mathrm{op}_\varepsilon(a)\mathrm{op}_\varepsilon(b) = \mathrm{op}_\varepsilon(ab) + \frac{\varepsilon}{i}\,\mathrm{op}_\varepsilon\left(\nabla_\xi a\cdot\nabla_x b\right) + O\left(\varepsilon^2\right).$$

Regarding the adjoint, one has $\mathrm{op}_\varepsilon(a)^* = \mathrm{op}_\varepsilon(\overline{a}) + O(\varepsilon)$ in $\mathcal{L}(L^2(\mathbb{R}^d))$.

1.4. **Semi-classical measures.** Even though the quantization is not positive, the *Gårding inequality* gives positivity in the semi-classical limit: if $a\geq 0$, there exists a constant $C>0$ such that for all $f\in L^2(\mathbb{R}^d)$,

$$(\mathrm{op}_\varepsilon(a)f,f)_{L^2} \geq -C\varepsilon\|f\|_{L^2}.$$

As a consequence if $(f^\varepsilon)_{\varepsilon>0}$ is a bounded family in $L^2(\mathbb{R}^d)$, there exists subsequences $\varepsilon_k \underset{k\to+\infty}{\longrightarrow} 0$ and a positive measure μ on $\mathbb{R}^{2d}$ such that for all $a\in\mathcal{S}(\mathbb{R}^{2d}$,

$$(\mathrm{op}_\varepsilon(a)f^{\varepsilon_k},f^{\varepsilon_k})_{L^2} \underset{k\to+\infty}{\longrightarrow} \int_{\mathbb{R}^{2d}} a(x,\xi)\mu(dx,d\xi).$$

Such measures are called *semi-classical measures* of the family $(f^\varepsilon)_{\varepsilon>0}$. They characterize the obstruction to the strong convergence of f^ε. For example, when considering Gaussian wave packets, one has

$$(\mathrm{op}_\varepsilon(a)g^\varepsilon_z,g^\varepsilon_z)_{L^2} \underset{\varepsilon\to 0}{\longrightarrow} a(z).$$

The family $(g^\varepsilon_z)_{\varepsilon>0}$ has only one semi-classical measure which is the Dirac mass in $z=(q,p)$.

1.5. **Correspondence principle.** The tools developed above can be combined with evolution problems. Consider a real-valued Hamiltonian H, for example

$$H = \frac{|\xi|^2}{2} + V(x),$$

for some ice function V. At the classical level, one associates with h the Newtonian trajectories $z\mapsto \Phi^t(z) = (q(t),p(t))$ such that

$$\dot{q}(t) = \nabla_\xi H(q(t),p(t)),\ \ \dot{p}(t) = -\nabla_x H(q(t),p(t)),$$

with initial value $(q(0),p(0)) = z = (q,p)\in\mathbb{R}^{2d}$. At the quantum level, one considers the operator $\mathrm{op}_\varepsilon(H)$. The following property connects the classical and the quantum picture.

Proposition 1.1. *Assume $(\psi^\varepsilon)_{\varepsilon>0}$ is a bounded family in $L^2(\mathbb{R}^d)$. Let μ be one of its semi-classical measures. Then,*

(1) *If $\mathrm{op}_\varepsilon(H)\psi^\varepsilon = o(1)$ then μ is supported on $\{H(x,\xi)=0\}$.*
(2) *If $\mathrm{op}_\varepsilon(H)\psi^\varepsilon = o(\varepsilon)$ then μ is invariant by the flow Φ^t.*

Property (1) is refered to as localization of the semiclassical measure and Property (2) as its invariance. The proof comes from the analysis of $(\mathrm{op}_\varepsilon(aH)\psi^\varepsilon,\psi^\varepsilon)$ and of $(\mathrm{op}_\varepsilon(\{a,H\})\psi^\varepsilon,\psi^\varepsilon)$ for $a\in\mathcal{S}(\mathbb{R}^{2d})$. Both quantities are related to the equation satisfied by $(\psi^\varepsilon)_{\varepsilon>0}$.

In this lecture, our objective is to extend the semiclassical approach in the setting of graded Lie groups. We will devote the first section to preliminaries on these groups. Then, we will introduce the semi-classical calculus, discuss wave packets, Gårding inequality, construct semi-classical measures, and state an Egorov theorem in a simple case. We will not discuss geometric invariance of the calculus with respect to filtration preserving diffeomorphisms and we point out the reference [**?**]

on this subject. The reader interested in the semiclassical approach in the Euclidean setting will benefit from the references [27, 24, 5].

2. Preliminaries on graded nilpotent groups

The material of this section is taken from [13], we also refer to [26]. A graded group G is a connected simply connected nilpotent Lie group whose (finite dimensional, real) Lie algebra $\mathfrak{g}$ admits an $\mathbb{N}$-gradation into linear subspaces,

$$\mathfrak{g} = \oplus_{j=1}^{\infty} \mathfrak{g}_j \quad \text{with} \quad [\mathfrak{g}_i, \mathfrak{g}_j] \subseteq \mathfrak{g}_{i+j}, \;\; 1 \leq i \leq j,$$

where all but a finite number of subspaces $\mathfrak{g}_j$ are trivial. We denote by $r = r_G$ the smallest integer j such that all the subspaces $\mathfrak{g}_j$, $j > r$, are trivial. If the first stratum $\mathfrak{g}_1$ generates the whole Lie algebra, then $\mathfrak{g}_{j+1} = [\mathfrak{g}_1, \mathfrak{g}_j]$ for all $j \in \mathbb{N}_0$ and r is the step of the group; the group G is then said to be stratified, and also (after a choice of basis or inner product for $\mathfrak{g}_1$) Carnot.

Example 2.1. The first examples we give below are stratified, not the last one.

(1) *The Heisenberg group* $\mathbb{H}$. Its Lie algebra has two strata: $\mathfrak{h} = \mathfrak{v} \oplus \mathfrak{z}$ with

$$\mathfrak{v} = \text{vect}(X_j, Y_j, \;, 1 \leq j \leq d), \;\; \mathfrak{z} = \text{vect}\, Z, \;\; Z = [X_j, Y_j], \; 1 \leq j \leq d.$$

(2) *H-type groups*. These groups are generalizations of the Heisenberg group. They are step 2 stratified groups with Lie algebra $\mathfrak{h} = \mathfrak{v} \oplus \mathfrak{z}$, characterized by the following property: for any $\lambda \in \mathfrak{z}^*$, the matrix $B(\lambda)$ of the skew-symmetric bilinear defined on $\mathfrak{v} \times \mathfrak{v}$ by $(U, V) \mapsto \lambda([U, V])$ is such that $B(\lambda)^2 = -|\lambda|^2 \text{Id}$.

(3) *The Engel group* $\mathbb{E}$. It is a 3 step group with Lie algebra $\mathfrak{E} = \mathfrak{E}_1 \oplus \mathfrak{E}_2 \oplus \mathfrak{E}_3$,

$$\mathfrak{E}_1 = \text{vect}(X_1, X_2), \;\; \mathfrak{E}_2 = \text{vect}\, X_3, \;\; \mathfrak{E}_3 = \text{vect}\, X_4,$$

with $[X_1, X_2] = X_3$ and $[X_1, X_3] = X_4$.

(4) A *non-stratified graded group* G can be constructed by adding a direction V to the $3d$ Heisenberg Lie algebra of example (1): one sets $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ with

$$\mathfrak{g}_1 = \text{vect}(X, Y), \;\; \mathfrak{g}_2 = \text{vect}(Z, V), \;\; Z = [X, Y], \;\; [V, X] = [V, Y] = [V, Z] = 0.$$

2.1. The exponential map and functional spaces.

The product law on G is derived from the exponential map

$$\exp_G : \mathfrak{g} \to G$$

which is a global diffeomorphism from $\mathfrak{g}$ onto G. Let $n = \dim \mathfrak{g}$. Once a basis $X_1, \ldots, X_n$ for $\mathfrak{g}$ has been chosen, we may identify the point $(x_1, \ldots, x_n) \in \mathbb{R}^n$ with the element $x = \exp(x_1 X_1 + \cdots + x_n X_n) \in G$.

The exponential map allows us to define the (topological vector) spaces $\mathcal{C}^\infty(G)$, $\mathcal{C}_c(G)$ and $\mathcal{S}(G)$ of, respectively, smooth, continuous and compactly supported, and Schwartz functions on G, identified with $\mathbb{R}^n$. The resulting spaces are intrinsically defined as spaces of functions on G and do not depend on a choice of basis.

The exponential map also induces a *Haar measure* dx on G which is invariant under left and right translations and defines Lebesgue spaces on G. The *non-commutative convolution* is given via

$$(f_1 * f_2)(x) := \int_G f_1(y) f_2(y^{-1}x) dy, \quad x \in G \tag{2.1}$$

for $f_1, f_2 \in \mathcal{C}_c(G)$.

2.2. **Adapted basis and dilations.** We now construct a *basis adapted to the gradation*. Set $d_j = \dim \mathfrak{g}_j$ for $1 \leq j \leq r$. We choose a basis $\{X_1, \dots, X_{d_1}\}$ of $\mathfrak{g}_1$ (this basis is possibly reduced to $\emptyset$), then $\{X_{d_1+1}, \dots, X_{d_1+d_2}\}$ a basis of $\mathfrak{g}_2$ (possibly $\{0\}$) and so on. Such a basis $\mathcal{B} = (X_1, \cdots, X_{d_1+\cdots+d_r})$ of $\mathfrak{g}$ is said to be adapted to the gradation; and we have $n = d_1 + \cdots + d_r$. The integer r is the *step of the gradation*.

The Lie algebra $\mathfrak{g}$ is a homogeneous Lie algebra equipped with the family of *dilations* $\{\delta_t, t > 0\}$, $\delta_r : \mathfrak{g} \to \mathfrak{g}$, defined by $\delta_t X = t^\ell X$ for every $X \in \mathfrak{g}_\ell$, $\ell \in \mathbb{N}$ [14, 13]. We re-write the set of integers $\ell \in \mathbb{N}$ such that $\mathfrak{g}_\ell \neq \{0\}$ into the increasing sequence of positive integers $\upsilon_1, \dots, \upsilon_n$, counted with multiplicity, the multiplicity of $\mathfrak{g}_\ell$ being its dimension. The associated group dilations are defined by

$$\delta_t(x) = tx := (t^{\upsilon_1} x_1, t^{\upsilon_2} x_2, \dots, t^{\upsilon_n} x_n), \quad x = (x_1, \dots, x_n) \in G,\ t > 0.$$

In this way, the integers $\upsilon_1, \dots, \upsilon_n$ become the *weights of the dilations* and we have $\delta_t X_j = t^{\upsilon_j} X_j$, $j = 1, \dots, n$, on the chosen basis of $\mathfrak{g}$.

In a canonical way, this leads to the notions of *homogeneity* for functions and operators. For instance, the Haar measure is homogeneous of degree

$$Q := \sum_{1 \leq \ell \leq r} w_\ell \dim \mathfrak{g}_\ell,$$

which is called the homogeneous dimension of the group. Here, the w_ℓ, $1 \leq \ell \leq r$ are the weights of the gradation counted with multiplicity and the vector fields corresponding to an element $X \in \mathfrak{g}_\ell$ are w_ℓ-homogeneous.

An important class of homogeneous map are the *homogeneneous quasi-norms*, that is, a 1-homogeneous non-negative map $G \ni x \mapsto |x|$ which is symmetric and definite in the sense that $|x^{-1}| = |x|$ and $|x| = 0 \Longleftrightarrow x = 0$. In fact, all the homogeneous quasi-norms are equivalent in the sense that if $|\cdot|_1$ and $|\cdot|_2$ are two of them, then

$$\exists C > 0, \ \ \forall x \in G, \ \ C^{-1}|x|_1 \leq |x|_2 \leq C|x|_1.$$

Examples may be constructed easily, such as

$$|x| = (\sum_{j=1}^{n} |x_j|^{N/\upsilon_j})^{1/N} \text{ for any } N > 0,$$

with the convention above.

In the rest of the paper, we will assume that we have fixed a basis $X_1, \dots, X_n$ of $\mathfrak{g}$ adapted to the gradation. We keep the same notation for the associated left-invariant vector fields on G. For a multi-index $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}_0^n$, we set $X^\alpha = X_1^{\alpha_1} \dots X_n^{\alpha_n}$. The differential operators X^α are homogeneous of degree

$$[\alpha] = \upsilon_1 \alpha_1 + \ldots + \upsilon_n \alpha_n.$$

2.3. **Fourier analysis.** Recall that a (unitary) *representation* $(\mathcal{H}_\pi, \pi)$ of G is a pair consisting in a Hilbert space $\mathcal{H}_\pi$ and a group morphism π from G to the set of unitary operators on $\mathcal{H}_\pi$. In this paper, the representations will always be assumed (unitary) strongly continuous, and their associated Hilbert spaces separable. A representation is said to be *irreducible* if the only closed subspaces of $\mathcal{H}_\pi$ that are stable under π are $\{0\}$ and $\mathcal{H}_\pi$ itself. Two representations π_1 and π_2 are equivalent if there exists a unitary transform $\mathbb{U}$ called an *intertwining map* that sends $\mathcal{H}_{\pi_1}$ on $\mathcal{H}_{\pi_2}$ with

$$\pi_1 = \mathbb{U}^{-1} \circ \pi_2 \circ \mathbb{U}.$$

The *dual set* $\widehat{G}$ is obtained by taking the quotient of the set of irreducible representations by this equivalence relation. We may still denote by π the elements of $\widehat{G}$ and we keep in mind that different representations of the class are equivalent through intertwining operators. The dilations extend to the dual set $\widehat{G}$ via

$$\delta_t\pi(x) = \pi(\delta_t x), \quad x \in G, \quad \pi \in \widehat{G}, \quad t > 0.$$

The *Fourier transform* of an integrable function $f \in L^1(G)$ at a representation π of G is the operator acting on $\mathcal{H}_\pi$ via

$$\widehat{f}(\pi) := \mathcal{F}(f)(\pi) := \int_G f(z)\, (\pi(z))^*\, dz.$$

Note that if $f_1, f_2 \in \mathcal{C}_c(G)$ then

$$\widehat{f_1 * f_2} = \widehat{f}_2 \widehat{f}_1. \tag{2.2}$$

If π_1, π_2 are two equivalent representations of G with $\pi_1 = \mathbb{U}^{-1} \circ \pi_2 \circ \mathbb{U}$ for some intertwining operator $\mathbb{U}$, then

$$\mathcal{F}(f)(\pi_1) = \mathbb{U}^{-1} \circ \mathcal{F}(f)(\pi_2) \circ \mathbb{U}.$$

Hence, this defines the measurable field of operators $\{\mathcal{F}(f)(\pi), \pi \in \widehat{G}\}$ modulo equivalence. The unitary dual $\widehat{G}$ is equipped with its natural Borel structure, and the equivalence comes from quotienting the set of irreducible representations of G together with understanding the resulting fields of operators modulo intertwiners.

We now recall the *Plancherel Theorem* due to Dixmier [6, Ch. 18]. It states the existence and uniqueness of the *Plancherel measure*, that is, the positive Borel measure μ on $\widehat{G}$ such that the Plancherel formula

$$\|f\|^2_{L^2(G)} = \int_G |f(x)|^2 dx = \int_{\widehat{G}} \|\widehat{f}(\pi)\|^2_{HS(\mathcal{H}_\pi)}\, d\mu(\pi), \tag{2.3}$$

or equivalently

$$(f_1, f_2)_{L^2(G)} = \int_G f_1(x)\overline{f_2(x)} dx dx = \int_{\widehat{G}} \mathrm{Tr}_{\mathcal{H}_\pi} \left(\widehat{f_1}(\pi) \widehat{f_2}(\pi)^* \right)\, d\mu(\pi)$$

holds for any $f \in \mathcal{C}_c(G)$. Here $\| \cdot \|_{HS(\mathcal{H}_\pi)}$ denotes the Hilbert-Schmidt norm on $\mathcal{H}_\pi$. This implies that the group Fourier transform is a unitary map from $L^1(G) \cap L^2(G)$ equipped with the norm of $L^2(G)$ to the Hilbert space

$$L^2(\widehat{G}) := \int_{\widehat{G}} \mathcal{H}_\pi \otimes \mathcal{H}_\pi^*\, d\mu(\pi).$$

We identify $L^2(\widehat{G})$ with the space of μ-square integrable Hilbert-Schmidt fields on $\widehat{G}$; its Hilbert norm and scalar products are then given by

$$\|\tau\|^2_{L^2(\widehat{G})} = \int_{\widehat{G}} \|\tau(\pi)\|^2_{HS(\mathcal{H}_\pi)}\, d\mu(\pi), \quad \tau \in L^2(\widehat{G}),$$

$$(\tau_1, \tau_2)_{L^2(\widehat{G})} = \int_{\widehat{G}} \mathrm{Tr}_{\mathcal{H}_\pi}(\tau_1(\pi)\ \tau_2(\pi)^*)\, d\mu(\pi), \quad \tau_1, \tau_2 \in L^2(\widehat{G}).$$

Here $\mathrm{Tr}_{\mathcal{H}_\pi}$ denotes the trace of operators on the Hilbert space $\mathcal{H}_\pi$. The group Fourier transform $\mathcal{F}$ extends unitarily from $L^2(G)$ onto $L^2(\widehat{G})$. The dual set $\widehat{G}$ and the Plancherel measure μ can be explicitly described via Kirillov's orbit method [4].

Finally, we denote by $L^\infty(\widehat{G})$ the space of measurable fields (modulo equivalence) of bounded operators $\sigma = \{\sigma(\pi) \in \mathscr{L}(\mathcal{H}_\pi) : \pi \in \widehat{G}\}$ on $\widehat{G}$ such that

$$\|\sigma\|_{L^\infty(\widehat{G})} := \sup_{\pi \in \widehat{G}} \|\sigma(\pi)\|_{\mathscr{L}(\mathcal{H}_\pi)}$$

is finite; here the supremum refers to the essential supremum with respect to the Plancherel measure μ of $\widehat{G}$. In fact, $L^\infty(\widehat{G})$ is naturally a Banach space and moreover a von Neumann algebra, sometimes called the *von Neumann algebra of the group* G.

The set $L^\infty(\widehat{G})$ acts naturally on $L^2(\widehat{G})$ by composition on the left:

$$(\sigma\tau)(\pi) = \sigma(\pi)\ \tau(\pi), \quad \pi \in \widehat{G},\ \sigma \in L^\infty(\widehat{G}) \text{ and } \tau \in L^2(\widehat{G}),$$

(it also acts on the right) and this action is continuous

$$\|\sigma\tau\|_{L^2(\widehat{G})} \leq \|\sigma\|_{L^\infty(\widehat{G})} \|\tau\|_{L^2(\widehat{G})}.$$

Dixmier's Plancherel theorem implies that $L^\infty(\widehat{G})$ is isomorphic to the von Neumann algebra $\mathscr{L}(L^2(G))^G$ of linear bounded operators on G that are invariant under left translations. The isomorphism is given by the fact that the Fourier multiplier with symbol σ, i.e. the operator $f \mapsto \mathcal{F}^{-1}(\sigma\widehat{f})$, is an operator in $\mathscr{L}(L^2(G))^G$.

Note that $\mathcal{F}L^1(G) \subseteq L^\infty(\widehat{G})$ with $\|\widehat{f}\|_{L^\infty(\widehat{G})} \leq \|f\|_{L^1(G)}$ for all $f \in L^1(G)$.

2.4. **Rockland operators.** We associate with the left-invariant vector fields X_j, $1 \leq j \leq n$, their Fourier symbol $\pi(X_j) = d\pi(X_j)$. They satisfy

$$\mathcal{F}(X_j f) = \pi(X_j)\mathcal{F}(f),\ \ f \in \mathcal{S}(G),\ \ 1 \leq j \leq n. \tag{2.4}$$

this definition extends to any left-invariant differential operator.

A *Rockland operator* $\mathcal{R}$ on G is a left-invariant differential operator which is homogeneous of positive degree and satisfies the Rockland condition:

(R) for each unitary irreducible representation π on G, except for the trivial representation, the operator $\pi(\mathcal{R})$ is injective on $\mathcal{H}_\pi^\infty$, that is,

$$\forall v \in \mathcal{H}_\pi^\infty, \qquad \pi(\mathcal{R})v = 0 \implies v = 0.$$

In the stratified case, any (left-invariant negative) *sub-Laplacian*, that is

$$\mathcal{L} = Z_1^2 + \ldots + Z_{n'}^2 \quad \text{with } Z_1, \ldots, Z_{n'} \text{ forming any basis of the first stratum } \mathfrak{g}_1,$$

is a positive Rockland operator.

More generally, on any graded group G, the operator

$$\mathcal{R} = \sum_{j=1}^{n} (-1)^{\frac{\nu_o}{\upsilon_j}} c_j X_j^{2\frac{\nu_o}{\upsilon_j}} \quad \text{with} \quad c_j > 0, \tag{2.5}$$

is a positive Rockland operator of homogeneous degree $2\nu_o$ if ν_o is any common multiple of $\upsilon_1, \ldots, \upsilon_n$.

3. Semi-classical pseudodifferential calculus

In this section, we introduce a semi-classical pseudodifferential calculus on graded Lie groups. Main ideas are taken from [10, 11, 12, 1].

3.1. **The algebra of semi-classical symbols.** We define $\mathcal{A}_0$, the space of *regularizing symbols* $\sigma = \{\sigma(x,\pi) : (x,\pi) \in G \times \widehat{G}\}$ of the form

$$\sigma(x,\pi) = \mathcal{F}\kappa_x(\pi) = \int_G \kappa_x(y)(\pi(y))^* dy,$$

where $(x,y) \mapsto \kappa_x(y)$ is a function of the topological vector space $\mathcal{S}(G \times G)$ of Schwartz functions on $G \times G$. The function $x \mapsto \kappa_x$ is called the *convolution kernel* of σ. The notation $\mathcal{A}_0$ is reminiscent to the set of symbols $\mathcal{A}$ introduced in [21] in the Euclidean setting.

We endow $\mathcal{A}_0$ with the norm

$$\|\sigma\|_{\mathcal{A}_0} = \|\sup_{x\in G} |\kappa_x(\cdot)|\|_{L^1(G)}. \tag{3.1}$$

In view of (2.2), the set $\mathcal{A}_0$ is an algebra. Moreever, it is stable via the action of the left-invariant vector fields X_j, and via the *difference operator* Δ_q associated to $q \in \mathcal{C}^\infty(G)$ and defined by:

$$\Delta_q \widehat{\kappa} = \mathcal{F}(q\kappa), \qquad \kappa \in \mathcal{C}_c(G). \tag{3.2}$$

We set $\Delta_j := \Delta_{x_j}$.

3.2. **Semi-classical quantization.** With the symbol $\sigma \in \mathcal{A}_0$, we associate the (family of) *semi-classical pseudodifferential operators*

$$\mathrm{Op}_\varepsilon(\sigma) = \int_{\pi\in\widehat{G}} \mathrm{Tr}_{\mathcal{H}_\pi} \left(\pi(x)\sigma(x,\delta_\varepsilon\pi)\mathcal{F}f(\pi)\right) d\mu(\pi).$$

In other words, we have

$$\mathrm{Op}_\varepsilon(\sigma)f(x) = \int_{\pi\in\widehat{G}} \mathrm{Tr}_{\mathcal{H}_\pi} \left(\pi(x)\sigma(x,\delta_\varepsilon\pi)\mathcal{F}f(\pi)\right) d\mu(\pi), \quad f\in\mathcal{S}(G),\, x\in G.$$

In terms of the convolution kernel $\kappa_x = \mathcal{F}^{-1}\sigma(x,\cdot)$, we have

$$\mathrm{Op}_\varepsilon(\sigma)f(x) = f * \kappa_x^{(\varepsilon)}(x), \quad f\in\mathcal{S}(G),\, x\in G.$$

Above, $\kappa_x^{(\varepsilon)}$ is the convolution kernel of $\sigma(\cdot,\delta_\varepsilon\,\cdot)$ and is given by a rescaling of the convolution kernel of σ:

$$\kappa_x^{(\varepsilon)}(z) := \varepsilon^{-Q}\kappa_x(\delta_\varepsilon^{-1}z), \quad x,z\in G.$$

Proposition 3.1. *Let $\sigma \in \mathcal{A}_0$, then $\mathrm{Op}_\varepsilon(\sigma)$ is bounded in $L^2(G)$. Moreover, there exists a constant $C>0$ such that for all*

$$\forall \sigma\in\mathcal{A}_0, \;\; \forall\varepsilon>0, \;\; \|\mathrm{Op}_\varepsilon(\sigma)\|_{\mathcal{L}(L^2(G))} \leq C\,\|\sigma\|_{\mathcal{A}_0}.$$

We point out that other estimates can be useful, in particular those implying the symbol norms introduced in [13].

Proof. We observe that if $f\in\mathcal{S}(G)$ then

$$|\mathrm{Op}_\varepsilon(\sigma)f(x)| = |\int_G f(y)\kappa_x^\varepsilon(y^{-1}x)dy| \leq \int_G |f(y)| \sup_{x_1\in G} |\kappa_{x_1}^\varepsilon(y^{-1}x)|\, dy = |f| * \sup_{x_1\in G}|\kappa_{x_1}^\varepsilon(\cdot)|(x),$$

so the Young convolution inequality implies

$$\|\mathrm{Op}_\varepsilon(\sigma)f\|_{L^2(G)} \leq \|f\|_{L^2(G)} \|\sup_{x_1\in G}|\kappa_{x_1}^\varepsilon(\cdot)|\|_{L^1(G)}.$$

We recognise this L^1-norm as $\|\sigma\|_{\mathcal{A}_0}$:

$$\|\sup_{x_1\in G}|\kappa_{x_1}^\varepsilon(\cdot)|\|_{L^1(G)} = \|\sup_{x_1\in G}|\kappa_{x_1}(\cdot)|\|_{L^1(G)} = \|\sigma\|_{\mathcal{A}_0}.$$

□

3.3. **Symbolic calculus.** The semi-classical pseudodifferential operators enjoy a symbolic calculus. In the next statement, if $\alpha = (\alpha_1, \cdots, \alpha_n) \in \mathbb{N}^n$, we denote by Δ_α the difference operator Δ_{q_α} associated with the function $q_\alpha(x) = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$. We recall that d_1 denotes the dimension of the first strata of he group G. The homogeneous length of the multi-index α is the integer

$$[\alpha] = \alpha_1 \upsilon_1 + \cdots + \alpha_n \upsilon_n.$$

Theorem 3.2. *Let $\sigma \in \mathcal{A}_0$. Then, in $\mathcal{L}(L^2(G))$, for $\varepsilon \in (0,1]$,*

$$\mathrm{Op}_\varepsilon(\sigma)^* = \mathrm{Op}_\varepsilon(\sigma^*) - \varepsilon^{\upsilon_1} \sum_{j=1}^{d_1} \mathrm{Op}_\varepsilon(X_j \Delta_j \sigma^*) + O(\varepsilon^{\upsilon_1+1}).$$

Let $\sigma_1, \sigma_2 \in \mathcal{A}_0$. Then in $\mathcal{L}(L^2(G))$,

$$\mathrm{Op}_\varepsilon(\sigma_1) \circ \mathrm{Op}_\varepsilon(\sigma_2) = \mathrm{Op}_\varepsilon(\sigma_1\, \sigma_2) - \varepsilon^{\upsilon_1} \sum_{j=1}^{d_1} \mathrm{Op}_\varepsilon\left(\Delta_j \sigma_1 \cdot X_j\, \sigma_2\right) + O(\varepsilon^{\upsilon_1+1}).$$

The proof of this result requires Taylor estimates adapted to graded groups (see [14] and its extension to graded groups in Theorem 3.1.51 in [13]). We associate with a function $f \in \mathcal{C}^\infty(G)$ its Taylor polynomial of order N at x, $\mathbb{P}_{f,x,N}$, i.e. the unique linear combination of monomials of homogeneous degree $\leq N$ satisfying

$$X^\beta \mathbb{P}_{f,x,N}(0) = X^\beta f(x)$$

for any $\beta \in \mathbb{N}_0^n$ with $[\beta] \leq N$. With these notations in hands, we have the following result.

Theorem 3.3 (Folland & Stein's Taylor formula)**.** *We fix a quasi-norm $|\cdot|$ on G and take $\varepsilon \in (0,1]$. There exists $\eta > 1$ such that for any $N \in \mathbb{N}_0$, there exists $C_N > 0$ such that for any $f \in C^{\lceil N \rfloor}(G)$ we have*

$$\forall x, y \in G, \qquad |f(xy) - \mathbb{P}_{f,x,N}(y)| \leq C_N \sum_{\substack{[\alpha] \leq \lceil N \rfloor + 1 \\ [\alpha] > N}} |y|^{[\alpha]} \sup_{|z| \leq \eta^{\lceil N \rfloor + 1}|y|} |(X^\alpha f)(xz)|,$$

where $\lceil N \rfloor = \max\{[\alpha] : \alpha \in \mathbb{N}_0^n$ with $[\alpha] \leq N\}$.

Proof of Theorem 3.2. Let us start with the adjoint. Let $\sigma \in \mathcal{A}_0$ with convolution kernel κ_x. The kernel of $\mathrm{Op}_\varepsilon(\sigma)$ is the function $(x,y) \mapsto \varepsilon^{-Q} \kappa^*_{\varepsilon,x}(\delta_\varepsilon^{-1}(y^{-1}x))$ with

$$\kappa^*_{\varepsilon,x}(z) = \overline{\kappa}_{x\delta_\varepsilon(-z)}(-z).$$

Therefore $\mathrm{Op}_\varepsilon(\sigma)^* = \mathrm{Op}^\varepsilon(\sigma^*_\varepsilon)$ with $\sigma^*_\varepsilon = \mathcal{F}\kappa^*_{\varepsilon,x}$. We thus focus on the asymptotics in ε of the symbol σ^*_ε for the norm $\|\cdot\|_{\mathcal{A}_0}$. By Theorem 3.3 applied with $N = 1$, there exists $C, C' > 0$ such that for $x, z \in G$,

$$\begin{aligned}\left| \kappa^\varepsilon_{x,\sigma^*}(z) - \overline{\kappa}_x(-z) + \varepsilon^{\upsilon_1} \sum_{j=1}^{d_1} z_j X_{j,x} \overline{\kappa}_x(-z) \right| &\leq C \varepsilon^{\upsilon_1+1} \sum_{\substack{[\alpha] \leq \lceil 2 \rfloor + 1 \\ [\alpha] \geq 2}} \sup_{|z'| \leq \eta^{\lceil N \rfloor + 1}|z|} |z|^{[\alpha]} |X_x^\alpha \overline{\kappa}_{x\delta_\varepsilon z'}(-z)| \\ &\leq C' \varepsilon^{\upsilon_1+1} \sum_{[\alpha] \leq \lceil 2 \rfloor + 1} |z|^{[\alpha]} \sup_{x' \in G} |X_x^\alpha \overline{\kappa}_{x'}(-z)|.\end{aligned}$$

Using the properties of κ_x, we deduce that for all $N \in \mathbb{N}$ there exists a constant $c_N > 0$ such that for all $x, z \in G$

$$(1 + |z|^N) \left| \kappa^\varepsilon_{x,\sigma^*}(z) - \overline{\kappa}_x(-z) + \varepsilon^{\upsilon_1} \sum_{j=1}^{d_1} z_j X_{j,x} \overline{\kappa}_x(-z) \right| \leq c_N \varepsilon^{\upsilon_1+1}.$$

We then recognize in $\overline{\kappa}_x(-z)$ the convolution kernel of the the symbol σ^* and in $\sum_{j=1}^{d_1} z_j X_{j,x}\overline{\kappa}_x(-z)$ the convolution kernel of the symbol $\sum_{j=1}^{d_1}\Delta_j X_j\sigma$ and we conclude to the boundedness of the quantity $\varepsilon^{-\upsilon_1-1}\|\sigma_\varepsilon^*-\sigma^*+\varepsilon^{\upsilon_1}\sum_{j=1}^{d_1}\Delta_j X_j\sigma\|_{\mathcal{A}_0}$.

We argue in a similar manner for the composition. Denote by κ_1 and κ_2 the convolution kernels of the symbols σ_1 and σ_2. The kernel of the operator $\mathrm{Op}_\varepsilon(\sigma_1)\circ\mathrm{Op}_\varepsilon(\sigma_2)$ is the function

$$(x,y)\mapsto\varepsilon^{-2Q}\int_G \kappa_{1,x}(\delta_\varepsilon^{-1}(u^{-1}x))\kappa_{2,u}(\delta_\varepsilon^{-1}(y^{-1}u))du.$$

Therefore, $\mathrm{Op}_\varepsilon(\sigma_1)\circ\mathrm{Op}_\varepsilon(\sigma_2)=\mathrm{Op}_\varepsilon(\sigma_\varepsilon)$ with $\sigma_\varepsilon=\mathcal{F}\kappa_{\varepsilon,x}$,

$$\begin{aligned}\kappa_{\varepsilon,x}(z)&=\varepsilon^{-Q}\int_G \kappa_{1,x}(\delta_\varepsilon^{-1}(u^{-1}x))\kappa_{2,u}(z\delta_\varepsilon^{-1}(x^{-1}u))du\\&=\int_G \kappa_{1,x}(v)\kappa_{2,x\delta_\varepsilon(-v)}(zv^{-1})dv\end{aligned}$$

We now use the Taylor formula of Theorem 3.3 and write

$$\kappa_{2,x\delta_\varepsilon(-v)}(zv^{-1})=\kappa_{2,x}(zv^{-1})-\varepsilon^{\upsilon_1}\sum_{j=1}^{d_1}v_jX_{j,x}\kappa_{2,x}(zv^{-1})+r^\varepsilon(x,zv^{-1})$$

with

$$|r^\varepsilon(x,zv^{-1})|\leq C\varepsilon^{\upsilon_1+1}\sum_{\substack{[\alpha]\leq\lceil 2\rfloor+1\\ [\alpha]\geq 2}}\ \sup_{|v'|\leq\eta^{\lceil 2\rfloor+1}|v|}|v|^{[\alpha]}|X_x^\alpha\kappa_{2,x\delta_\varepsilon v'}(zv^{-1})|,\quad z,v,x\in G,$$

for some constant $C>0$. Using the properties of $\kappa_{2,\sigma}$, we obtain the existence of $C'>0$, and then of c_N associated to $N\in\mathbb{N}$, such that for all $z,v,x\in G$, we have

$$\begin{aligned}|r^\varepsilon(x,zv^{-1})|&\leq C'\varepsilon^{\upsilon_1+1}\sum_{[\alpha]\leq\lceil 2\rfloor+1}\sup_{x'\in G}|v|^{[\alpha]}|X_x^\alpha\kappa_{2,x'}(zv^{-1})|\\&\leq c_N\varepsilon^{\upsilon_1+1}(1+|zv^{-1}|^2)^{-N}|v|^{\lceil 2\rfloor+1}\end{aligned}$$

We now identify $\int_G\kappa_{1,x}(v)\kappa_{2,x}(zv^{-1})dv$ as the convolution kernel of the symbol $\sigma_1\sigma_2$, and in the function $\int_G v_j\kappa_{1,x}(v)X_{j,x}\kappa_{2,x}(zv^{-1})dv$ as the convolution kernel of the symbol $\Delta_j\sigma_1X_j\sigma_2$. We deduce

$$\begin{aligned}\|\sigma_\varepsilon-\sigma_1\sigma_2+\varepsilon^{\upsilon_1}\sum_{j=1}^{d_1}\Delta_j\sigma_1X_j\sigma_2\|_{\mathcal{A}_0}&=\int_{z\in G}\sup_{x\in G}|\int_G\kappa_{1,x}(v)r^\varepsilon(x,v)dv|dz\\&\leq c_N\varepsilon^{\upsilon_1+1}\sup_{x,v'}|(1+|v'|^2)^N\kappa_{1,x}(v')|\int_{z\in G}\int_{v\in G}(1+|v|^2)^{-N}(1+|zv^{-1}|^2)^{-N}|v|^{\lceil 2\rfloor+1}dvdz,\end{aligned}$$

whence the conclusion. □

Remark 3.4. The reader will have noticed that one could have used Taylor formula at higher order and obtain asymptotics at any order for both the adjoint or the composition of semiclassical pseudodifferential operators.

3.4. **Differential operators and the symbolic calculus.** The symbolic calculus of the preceding section extends to the product of differential operators and pseudodifferential ones. Indeed, the notation Op_ε allows to write for all $\alpha\in\mathbb{N}_0^n$,

$$\varepsilon^{[\alpha]}X^\alpha=\mathrm{Op}_\varepsilon(\pi(X^\alpha)),$$

where $\pi(X^\alpha) = \pi(X_1)^{\alpha_1} \cdots \pi(X_n)^{\alpha_n}$ and the operators $\pi(X_j)$ are defined in (2.4). Moreover, differential operators of the form

$$P^\varepsilon = \sum_{[\alpha]\leq N} c_\alpha(x)\varepsilon^{[\alpha]}X^\alpha$$

can be written in a symbolic way:

$$P^\varepsilon = \sum_{[\alpha]\leq N} \varepsilon^{[\alpha]}\mathrm{Op}_\varepsilon\left(c_\alpha(x)\pi(X^\alpha)\right).$$

The formula of Theorem 3.2 then also hold.

Proposition 3.5. *Consider the operator H^ε given by*

$$H^\varepsilon = \varepsilon^2 \sum_{[\alpha]=[\beta]=1} X_\alpha\left(b_{\alpha,\beta}(x)X_\beta .\right) + i\varepsilon^2 \sum_{[\alpha]=2,\ |\alpha|=1} \left(c_\alpha(x)X_\alpha - \frac{1}{2}X_\alpha c_\alpha(x)\right),$$

where the functions c_α and $b_{\alpha,\beta}$ are smooth, real-valued and bounded, have bounded derivatives, and satisfy $b_{\alpha,\beta}(x) = b_{\beta,\alpha}(x)$ for all $x \in G$. Then, the operator H^ε is formally self-adjoint and

$$H^\varepsilon = \mathrm{Op}_\varepsilon(h_0 + \varepsilon h_1 + \varepsilon^2 h_2)$$

with

$$\begin{aligned} h_0(x,\pi) &= \sum_{[\alpha]=[\beta]=1} b_{\alpha,\beta}(x)\pi(X^\alpha)\pi(X^\beta) + i \sum_{[\alpha]=2,\ |\alpha|=1} c_\alpha(x)\pi(X^\alpha), \\ h_1(x,\pi) &= \sum_{[\alpha]=[\beta]=1} X^\alpha b_{\alpha,\beta}(x)\pi(X^\beta), \\ h_2(x,\pi) &= -\frac{i}{2}\sum_{[\alpha]=2,\ |\alpha|=1} X^\alpha c_\alpha(x). \end{aligned}$$

Proof. We take α and β of homogeneous length 2 and of length 1. Then, for $f \in \mathcal{S}(G)$,

$$\begin{aligned} \varepsilon^2 X_\alpha(b_{\alpha,\beta}X_\beta f) &= \varepsilon^2 b_{\alpha,\beta}X_\alpha X_\beta f + \varepsilon^2 X_\alpha b_{\alpha,\beta}X_\beta f \\ &\quad \mathrm{Op}_\varepsilon(b_{\alpha,\beta}\pi(X_\alpha)\pi(X_\beta))f + \varepsilon\mathrm{Op}_\varepsilon(X^\alpha b_{\alpha,\beta}\pi(X_\beta))f. \end{aligned}$$

The result follows. ∎

Let us now consider a Rockland operator $\mathcal{R}$ of homogeneous degree $2\nu_0$ as in (2.5). Then

$$\varepsilon^{2\nu_0}\mathcal{R} = \mathrm{Op}_\varepsilon(\pi(\mathcal{R})).$$

Lemma 3.6. *Let $\chi \in \mathcal{C}_c^\infty(\mathbb{R})$ and consider the operator $\chi(\mathcal{R})$ defined by the functional calculus. Therefore, for any $\phi \in \mathcal{S}(G)$, the symbol $\phi(x)\chi(\pi(\mathcal{R}))$ is in $\mathcal{A}_0$ and one has*

$$\mathrm{Op}_\varepsilon(\phi(x)\chi(\pi(\mathcal{R}))) = \phi(x)\chi(\varepsilon^{2\nu_0}\pi(\mathcal{R})).$$

Proof. It is proved in [19] that the operator $\chi(\mathcal{R})$ has a Schwartz convolution kernel symbol $\chi(\mathcal{R})\delta_0$. Moreover, one has $\mathrm{Op}_\varepsilon(\phi(x)\chi(\pi(\mathcal{R}))) = \phi(x)\mathrm{Op}_\varepsilon(\chi(\pi(\mathcal{R})))$ and, by homogeneity of $\mathcal{R}$, $\delta_\varepsilon\pi(\mathcal{R}) = \varepsilon^{2\nu_0}\pi(\mathcal{R})$, whence $\mathrm{Op}_\varepsilon(\chi(\pi(\mathcal{R}))) = \chi(\varepsilon^{2\nu_0}\pi(\mathcal{R}))$. ∎

4. Bargman transform and applications

In this section, following [1], we introduce a Bargmann transform on the group G and derive consequences of the introduction of this concept: a notion of Wick quantization, a Gårding inequality, and a families of wave packets that realize a frame of $L^2(G)$.

4.1. **The Bargman transform.** Let $a \in \mathcal{S}(G)$ such that $\|a\|_{L^2(G)} = 1$. We set

$$a_\varepsilon := \varepsilon^{-\frac{Q}{4}} a \circ \delta_{\varepsilon^{-\frac{1}{2}}}, \qquad \varepsilon > 0,$$

so that $a_\varepsilon \in \mathcal{S}(G)$ with $\|a_\varepsilon\|_{L^2(G)} = 1$. All the quantities we are going to define in the following will depend on the choice of the function a. In the Euclidean case, one chooses the normalized centered Gaussian function: $a(x) = \frac{1}{\sqrt{\pi}} e^{-\frac{|x|^2}{2}}$, $x \in \mathbb{R}^n$. To mimic this case, we could ask that

$$a(x) = a(x^{-1}) \geq 0, \ \ \forall x \in G.$$

We will mention it whenever we make additional assumptions on a.

For each $(x, \pi) \in G \times \widehat{G}$, we define an operator on $\mathcal{H}_\pi$ depending on the point $y \in G$,

$$\mathrm{WP}^\varepsilon_{x,\pi}[a](y) = a_\varepsilon(x^{-1}y)\delta_{\varepsilon^{-1}}\pi(y). \tag{4.1}$$

The function $y \mapsto \mathrm{WP}^\varepsilon_{x,\pi}[a](y)$ is a smooth map from G to the set of fields of operators on $G \times \widehat{G}$. Moreover,

$$\int_G \|\mathrm{WP}^\varepsilon_{x,\pi}[a]\|^2_{\mathcal{L}(\mathcal{H}_\pi)} dx d\mu(\pi) = \|a\|^2_{L^2(G)} = 1. \tag{4.2}$$

The *Bargmann transform* on G is the operator $\mathcal{B}^\varepsilon$ defined on $\mathcal{S}(G)$ via

$$\mathcal{B}^\varepsilon[f](x,\pi) = \varepsilon^{-\frac{Q}{2}} \int_G f(y) \mathrm{WP}^\varepsilon_{x,\pi}[a](y)^* dy, \quad f \in \mathcal{S}(G), \ (x,\pi) \in G \times \widehat{G}. \tag{4.3}$$

The Bargmann transform associates with a function $f \in \mathcal{S}(G)$ a field of operators on the set $G \times \widehat{G}$.

In the following, it will be convenient to consider the ε-Fourier transform that we write

$$\mathcal{F}^\varepsilon \kappa(\pi) = \varepsilon^{-\frac{Q}{2}} \mathcal{F}\kappa(\delta_{\varepsilon^{-1}}\pi) = \varepsilon^{-\frac{Q}{2}} \int_G f(y) \delta_{\varepsilon^{-1}}\pi(y)^*, \ \ \kappa \in \mathcal{S}(G), \ \ \pi \in \widehat{G}.$$

Then, for all $f \in \mathcal{S}(G)$, we have the Plancherel relation

$$\|f\|^2_{L^2(G)} = \varepsilon^{-Q} \int_{\widehat{G}} \|\mathcal{F}^\varepsilon f(\pi\|^2_{HS} d\mu(\pi),$$

and $\mathcal{B}^\varepsilon[f]$ is the field of operators on $G \times \widehat{G}$ given by

$$\mathcal{B}^\varepsilon[f](x,\pi) = \mathcal{F}^\varepsilon \left(f\, a_\varepsilon(x^{-1} \cdot) \right)(\pi), \qquad (x,\pi) \in G \times \widehat{G}. \tag{4.4}$$

The Bargmann transform on the group G enjoys properties similar to those of the Bargmann transform in the Euclidean setting.

Proposition 4.1. (1) *The map $\mathcal{B}^\varepsilon$ extends uniquely to an isometry from $L^2(G)$ to $L^2(G \times \widehat{G})$ for which we keep the same notation.*

(2) *The adjoint map $\mathcal{B}^{\varepsilon,*} : L^2(G \times \widehat{G}) \to L^2(G)$ is given by*

$$\mathcal{B}^{\varepsilon,*}[\tau](y) = \varepsilon^{-\frac{Q}{2}} \int_{G \times \widehat{G}} \mathrm{Tr}_{\mathcal{H}_\pi} \left(\tau(x,\pi) \mathrm{WP}^\varepsilon_{x,\pi}[a](y) \right) dx d\mu(\pi), \qquad \tau \in L^2(G \times \widehat{G}), \ y \in G.$$

(3) *Moreover $\mathcal{B}^{\varepsilon,*}\mathcal{B}^\varepsilon = \mathrm{id}_{L^2(G)}$ while $\mathcal{B}^\varepsilon \mathcal{B}^{\varepsilon,*}$ is a projection on a closed subspace of $L^2(G \times \widehat{G})$.*

Proof. Point 1. From (4.4) and the Plancherel formula (2.3), we obtain

$$\int_{\widehat{G}} \|\mathcal{B}^\varepsilon[f](x,\pi)\|^2_{HS(\mathcal{H}_\pi)} d\mu(\pi) = \|f\, a_\varepsilon(x^{-1} \cdot)\|^2_{L^2(G)}, \qquad x \in G.$$

Integrating against dx yields Point (1). Point (2) follows from

$$\int_{G\times\widehat{G}} \mathrm{Tr}_{\mathcal{H}_\pi}\left(\tau(x,\pi)\left(\mathcal{B}^\varepsilon[f](x,\pi)\right)^*\right)dxd\mu(\pi) = \int_{G\times\widehat{G}} \mathrm{Tr}_{\mathcal{H}_\pi}\left(\tau(x,\pi)\left(\mathcal{F}^\varepsilon\left(fa_\varepsilon(x^{-1}\cdot)\right)(\pi)\right)^*\right)dxd\mu(\pi),$$

by (4.4). By Plancherel Theorem, if $\tau(x,\pi)=\mathcal{F}\kappa_x(\pi)$,

$$\begin{aligned}
&\int_{G\times\widehat{G}} \mathrm{Tr}_{\mathcal{H}_\pi}\left(\tau(x,\pi)\left(\mathcal{F}^\varepsilon\left(fa_\varepsilon(x^{-1}\cdot)\right)(\pi)\right)^*\right)dxd\mu(\pi)\\
&\qquad = \varepsilon^{-\frac{Q}{2}}\int_{G\times G}\overline{f}(y)\overline{a}_\varepsilon(x^{-1}y)\kappa_x(\delta_{\varepsilon^{-1}}y)dydx\\
&\qquad = \varepsilon^{-\frac{Q}{2}}\int_{G\times G}\overline{f}(y)\mathrm{Tr}_{\mathcal{H}_\pi}(\pi(\varepsilon^{-1}y)\tau(x,\pi))\overline{a}_\varepsilon(x^{-1}y)dydx,
\end{aligned}$$

and one gets Point (2).
Finally, Part (3) follows from Parts (1) and (2) since they imply for any $f,g\in L^2(G)$

$$(f,g)_{L^2(G)} = (\mathcal{B}^\varepsilon[f],\mathcal{B}^\varepsilon[g])_{L^2(G\times\widehat{G})} = (\mathcal{B}^{\varepsilon,*}\mathcal{B}^\varepsilon[f],g)_{L^2(G)}.$$

□

In the next sections, we use the Bargmann transform for defining a positive quantization that we will compare to the semi-classical quantization introduced in the previous Section 3. That will allow us to prove positivity in the limit $\varepsilon\to 0$ for semi-classical pseudodifferential operators with non-negative symbols. Finally, we will construct wave packets that consists in ε-dependent families microlocalized in the phase space.

4.2. **Wick quantization.** We define the semi-classical Wick quantization for $\sigma\in L^\infty(G\times\widehat{G})$

$$\mathrm{Op}_\varepsilon^{\mathrm{Wick}}(\sigma) := \mathcal{B}^{\varepsilon,*}\sigma\,\mathcal{B}^\varepsilon.$$

Proposition 4.2. *The symbolic quantization* $\mathrm{Op}_\varepsilon^{\mathrm{Wick}}$ *is well defined on* $L^\infty(G\times\widehat{G})$ *and satisfies*

$$\forall\sigma\in L^\infty(G\times\widehat{G}),\qquad \|\mathrm{Op}_\varepsilon^{\mathrm{Wick}}(\sigma)\|_{\mathscr{L}(L^2(G))}\leq\|\sigma\|_{L^\infty(G\times\widehat{G})}.$$

Moreover, it is a positive quantization in the sense that if $\sigma\geq 0$*, then* $\mathrm{Op}_\varepsilon^{\mathrm{Wick}}(\sigma)\geq 0$.

Proof. We have for any $f\in L^2(G)$:

$$\begin{aligned}
\|\mathrm{Op}_\varepsilon^{\mathrm{Wick}}(\sigma)f\|_{L^2(G)} &= \|\mathcal{B}^{\varepsilon,*}\sigma\mathcal{B}[f]\|_{L^2(G)}\\
&\leq \|\mathcal{B}^{\varepsilon,*}\|_{\mathscr{L}(L^2(G\times\widehat{G}),L^2(G))}\|\sigma\|_{L^\infty(G\times\widehat{G})}\|\mathcal{B}^\varepsilon\|_{\mathscr{L}(L^2(G),L^2(G\times\widehat{G}))}\|f\|_{L^2(G)}.
\end{aligned}$$

Since $\mathcal{B}^\varepsilon$ is an isometry, the operator norms of $\mathcal{B}^\varepsilon$ and $\mathcal{B}^{\varepsilon,*}$ are equal to 1.

Let us now suppose that $\sigma(x,\pi)\geq 0$ for all $(x,\pi)\in G\times\widehat{G}$. Then

$$\left(\mathrm{Op}_\varepsilon^{\mathrm{Wick}}(\sigma)f,f\right)_{L^2(G)} = (\sigma\mathcal{B}^\varepsilon[f],\mathcal{B}^\varepsilon[f])_{L^2(G\times\widehat{G})}\geq 0,\ \ \forall f\in L^2(G).$$

□

We can also compute the convolution kernel of $\mathrm{Op}_\varepsilon^{\mathrm{Wick}}(\sigma)$ in order to compare it with a semi-classical pseudodifferential operator.

Lemma 4.3. *If* $\sigma\in\mathcal{A}_0$*, then*

$$\mathrm{Op}_\varepsilon^{\mathrm{Wick}}(\sigma) = \mathrm{Op}_\varepsilon(\sigma^{\varepsilon,\mathrm{Wick}}),$$

where $\sigma^{\varepsilon,\mathrm{Wick}}\in\mathcal{A}_0$ *has the convolution kernel*

$$\kappa_x^{\varepsilon,\mathrm{Wick}}(w) = \int_G \bar{a}(z'\delta_{\sqrt{\varepsilon}}w^{-1})a(z')\kappa_{x\delta_{\sqrt{\varepsilon}}z'^{-1}}(w)dz'.$$

Moreover, for all $\sigma \in \mathcal{A}_0$, $\|\sigma^{\varepsilon,\mathrm{Wick}}\|_{\mathcal{A}_0} \leq \|\sigma\|_{\mathcal{A}_0}$.

Proof. Let $f \in \mathcal{S}(G)$. By the definition of the Bargmann operator and Properties (2.2) and by Part (2) of Proposition 4.1, we obtain

$$\begin{aligned}\mathrm{Op}_\varepsilon^{\mathrm{Wick}}(\sigma)f(x) &= \varepsilon^{-\frac{Q}{2}} \int_{G\times\widehat{G}} \mathrm{Tr}_{\mathcal{H}_\pi}\left(\sigma(z,\pi)\mathcal{B}^\varepsilon[f](z,\pi)\mathrm{WP}^\varepsilon_{z,\pi}[a](x)\right) dzd\mu(\pi)\\ &= \varepsilon^{-Q}\int_G f(y)\overline{a}_\varepsilon(z^{-1}y)a_\varepsilon(z^{-1}x)\left(\int_{\widehat{G}} \mathrm{Tr}_{\mathcal{H}_\pi}\left(\sigma(z,\pi)\pi(\delta_{\varepsilon^{-1}}(y^{-1}x)\right)d\mu(\pi)\right)dx\\ &= \varepsilon^{-Q}\int_G f(y)\overline{a}_\varepsilon(z^{-1}y)a_\varepsilon(z^{-1}x)\kappa_x(\delta_{\varepsilon^{-1}}(y^{-1}x))dx.\end{aligned}$$

We recognize $f * \left(\varepsilon^{-Q}\kappa_x^{\varepsilon,\mathrm{Wick}}(\delta_{\varepsilon^{-1}}\cdot)\right)$ for the function

$$\begin{aligned}\kappa_x^{\varepsilon,\mathrm{Wick}}(w) &= \int_G \overline{a}_\varepsilon(z^{-1}x\delta_\varepsilon w^{-1})a_\varepsilon(z^{-1}x)\kappa_z(w)dz\\ &= \int_G \overline{a}_\varepsilon(z'\delta_\varepsilon w^{-1})a_\varepsilon(z')\kappa_{xz'^{-1}}(w)dz'\\ &= \int_G a(z'\delta_{\sqrt{\varepsilon}}w^{-1})\bar{a}(z')\kappa_{x\delta_{\sqrt{\varepsilon}}z'^{-1}}(w)dz'.\end{aligned} \tag{4.5}$$

For concluding the proof, we check that $\kappa^{\varepsilon,\mathrm{Wick}} \in \mathcal{S}(G\times G)$ and we observe, using (4.5),

$$\begin{aligned}\int_G \sup_{x\in G}|\kappa_x^{\varepsilon,\mathrm{Wick}}(w)|dw &\leq \int_G \sup_{x'\in G}|\kappa_{x'}(w)|\left(\int_G a_\varepsilon(z'w^{-1})a_\varepsilon(z')dz'\right)dw\\ &\leq \int_G \sup_{x'\in G}|\kappa_{x'}(w)|dw,\end{aligned}$$

by the Cauchy-Schwartz inequality (we have also used $a_\varepsilon \geq 0$ and $\|a_\varepsilon\|_{L^2(G)} = \|a\|_{L^2(G)} = 1$). □

As a corollary, we obtain that the semi-classical Wick quantization coincides at leading order in ε with the standard quantization.

Corollary 4.4. *We choose a function $a \in \mathcal{D}(G)$ that is even, i.e. $a(x^{-1}) = a(x)$. Then for any $\sigma \in \mathcal{A}_0$, there exists $C > 0$ such that for all $\varepsilon \in (0,1]$,*

$$\|\mathrm{Op}_\varepsilon(\sigma) - \mathrm{Op}_\varepsilon^{\mathrm{Wick}}(\sigma)\|_{\mathscr{L}(L^2(G))} \leq C\varepsilon.$$

Proof. By Proposition 3.1, using the $\mathcal{A}_0$-norm defined in (3.1), we have

$$\|\mathrm{Op}_\varepsilon(\sigma) - \mathrm{Op}_\varepsilon^{\mathrm{Wick}}(\sigma)\|_{\mathscr{L}(L^2(G))} \leq \|\sigma - \sigma^{\varepsilon,\mathrm{Wick}}\|_{\mathcal{A}_0} \leq I_1(\varepsilon) + I_2(\varepsilon),$$

where

$$\begin{aligned}I_1(\varepsilon) &:= \int_G \sup_{x\in G}\left|\int_G |a(z)|^2\left(\kappa_x(w) - \kappa_{x\delta_{\sqrt{\varepsilon}}z^{-1}}(w)\right)dz\right|dw,\\ I_2(\varepsilon) &:= \int_G \sup_{x\in G}\left|\int_G (\bar{a}(z) - \bar{a}(z\delta_{\sqrt{\varepsilon}}w^{-1}))a(z)\kappa_{x\delta_{\sqrt{\varepsilon}}z^{-1}}(w)dz\right|dw.\end{aligned}$$

By the Taylor estimates due to Folland and Stein (see Theorem 3.3), we have:

$$I_1(\varepsilon) = \varepsilon^{\frac{\upsilon_1}{2}} \int_G \sup_{x\in G} \left| \sum_{j=1}^{n_1} \int_G (-z_j)|a(z)|^2 dz \; X_{j,x}\kappa_x(w) \right| dw + O(\varepsilon),$$

$$\begin{aligned} I_2(\varepsilon) &= \varepsilon^{\frac{\upsilon_1}{2}} \int_G \sup_{x\in G} \left| \sum_{j=1}^{n_1} \int_G (-w_j) a(z) X_j \bar{a}(z) \kappa_x(w) dz \right| dw + O(\varepsilon) \\ &\leq \varepsilon^{\frac{\upsilon_1}{2}} \sum_{j=1}^{n_1} \left| \int_G a(z) X_j \bar{a}(z) dz \right| \int_G |w_j| \sup_{x'\in G} |\kappa_{x'}(w)| \, dw + O(\varepsilon). \end{aligned}$$

We recall that n_1 denotes the dimension of the first strata (see Section 2.2 where the basis $(X_j)_{1\leq j\leq n}$ has been introduced), and υ_1 the associated weight. We have used that $\upsilon_1 \geq 1$ to estimate the rest as $O(\varepsilon)$. If $\upsilon_1 > 1$, then the result holds. If $\upsilon_1 = 1$, we have to analyze the first term given by the Taylor formula. We then observe the following facts

(i) As a is even, for any polynomial q satisfying $q(z^{-1}) = -q(z)$ we have $\int_G |a(z)|^2 q(z) dz = 0$. This holds in particular for the coordinate polynomials z_j.

(ii) As a is real valued, for any left or right invariant vector field X, an integration by parts shows $\int_G X_j a(z) \bar{a}(z) dz = 0$.

Consequently, $I_1(\varepsilon) = O(\varepsilon)$ and $I_2(\varepsilon) = O(\varepsilon)$ if $\upsilon_1 = 1$. □

4.3. **Gårding inequality.** Let us now state Gårding inequality.

Theorem 4.5. *Let $\sigma \in \mathcal{A}_0$. If σ is non-negative, then there exists a constant $C > 0$ such that*

$$\forall f \in L^2(G), \quad \forall \varepsilon \in (0,1], \qquad \Re \left(\mathrm{Op}_\varepsilon(\sigma) f, f\right)_{L^2(G)} \geq -C\varepsilon \|f\|^2_{L^2(G)}. \tag{4.6}$$

This inequality is typical from the semiclassical setting. For Gårding inequality on groups in a non semi-classical setting, the reader can refer to [2].

Proof. We choose a even and consider the associated Wick quantization. We write

$$\begin{aligned} \Re \left(\mathrm{Op}_\varepsilon(\sigma) f, f\right)_{L^2(G)} &\geq \left(\mathrm{Op}_\varepsilon^{\mathrm{Wick}}(\sigma) f, f\right)_{L^2(G)} - \|\mathrm{Op}_\varepsilon(\sigma) - \mathrm{Op}_\varepsilon^{\mathrm{Wick}}(\sigma)\|_{\mathscr{L}(L^2(G))} \|f\|^2_{L^2(G)} \\ &\geq -\|\mathrm{Op}_\varepsilon(\sigma) - \mathrm{Op}_\varepsilon^{\mathrm{Wick}}(\sigma)\|_{\mathscr{L}(L^2(G))} \|f\|^2_{L^2(G)}. \end{aligned}$$

By Corollary 4.4, $\|\mathrm{Op}_\varepsilon^{KN}(\sigma) - \mathrm{Op}_\varepsilon^{\mathrm{Wick}}(\sigma)\|_{\mathscr{L}(L^2(G))} = O(\varepsilon)$. This concludes the proof of Theorem 4.5. □

4.4. **Wave packets.** In this section, we use the operator $\mathrm{WP}^\varepsilon_{x,\pi}[a](y)$ in order to define some families of Wave packets satisfying a frame relation analogue to the Bargmann formula (1.2). We will not always need strong hypothesis on a, which was supposed to satisfy $a \geq 0$, a is symmetric and $\|a\|_{L^2(G)} = 1$ in the preceding section. We now only assume $a \in \mathcal{S}(G)$ with $\|a\|_{L^2(G)} = 1$.

We set for $(x,\pi) \in G \times \widehat{G}$,

$$g^\varepsilon_{x,\pi,k,\ell}(y) := \left(\mathrm{WP}^\varepsilon_{x,\pi}[a](y)\varphi_k(\pi), \varphi_\ell(\pi)\right)_{\mathcal{H}_\pi}, \; y \in G, \tag{4.7}$$

where $(\cdot,\cdot)_{\mathcal{H}_\pi}$ denotes the inner product of $\mathcal{H}_\pi$ and $(\varphi_k(\pi))_{k\in I_\pi}$, $I_\pi \subset \mathbb{N}$ is an orthonormal basis of $\mathcal{H}_\pi$.

We shall call such a family a *wave packet on G with core (x_0,λ_0), profile a and harmonics (φ_k,φ_ℓ).*

The frame properties in Proposition 4.1 (3) implies the following decomposition.

Corollary 4.6. *Assume $\|a\|_{L^2(G)} = 1$. Then any function $f \in L^2(G)$ decomposes in $L^2(G)$ as*

$$f = \int_{G\times\widehat{G}} \sum_{k,\ell\in I_\pi} \left(f, g^\varepsilon_{x,\pi,k,\ell}\right)_{L^2(G)}\ g^\varepsilon_{x,\pi,k,\ell}\, dxd\mu(\pi),$$

in the sense that

$$\|f\|^2_{L^2(G)} = \int_{G\times\widehat{G}} \sum_{k,\ell\in I_\pi} |\left(f, g^\varepsilon_{x,\pi,k,\ell}\right)_{L^2(G)}|^2 dxd\mu(\pi),$$

or equivalently for any $f_1, f_2 \in L^2(G)$

$$(f_1, f_2)_{L^2(G)} = \int_{G\times\widehat{G}} \sum_{k,\ell\in I_\pi} \left(f_1, g^\varepsilon_{x,\pi,k,\ell}\right)_{L^2(G)}\ \overline{\left(f_2, g^\varepsilon_{x,\pi,k,\ell}\right)}_{L^2(G)}\ dxd\mu(\pi).$$

Note that wavelets have been constructed on stratified groups in [15] (see also [20]).

Proof. By Proposition 4.1, we have for any $f \in L^2(G)$

$$\|f\|^2_{L^2(G)} = \|\mathcal{B}[f]\|^2_{L^2(G\times\widehat{G})} = \int_{G\times\widehat{G}} \|\mathcal{B}[f](x,\pi)\|^2_{HS(\mathcal{H}_\pi)} dxd\mu(\pi).$$

The Hilbert-Schmidt norms may be written in the basis (φ_k) as

$$\|\mathcal{B}^\varepsilon[f](x,\pi)\|^2_{HS(\mathcal{H}_\pi)} = \sum_{k,\ell} |(\mathcal{B}^\varepsilon[f](x,\pi)\varphi_k, \varphi_\ell)_{\mathcal{H}_\pi}|^2$$

$$\begin{aligned}\text{with}\qquad (\mathcal{B}^\varepsilon[f](x,\pi)\varphi_k,\varphi_\ell)_{\mathcal{H}_\pi} &= \int_G f(y)(\mathrm{WP}^\varepsilon_{x,\pi}(y)^*[a]\varphi_k,\varphi_\ell)_{\mathcal{H}_\pi} dy\\ &= \int_G f(y)\overline{(\varphi_\ell, \mathrm{WP}^\varepsilon_{x,\pi}[a](y)^*\varphi_k)}_{\mathcal{H}_\pi} dy\\ &= \int_G f(y)\overline{(\mathrm{WP}^\varepsilon_{x,\pi}[a](y)\varphi_\ell, \varphi_k)}_{\mathcal{H}_\pi} dy = (f, g_{x,\pi,\ell,k})_{L^2(G)}.\end{aligned}$$

We then conclude on $(f_1, f_2)_{L^2(G)}$ by considering $\|f_1 \pm f_2\|^2$ and $\|f_1 \pm i f_2\|^2$. ☐

The frame of wave packets provides with a family of functions that enjoy smoothness properties that we will analyze in the next section. These properties also depend on the properties of the functions $(\varphi_\ell)_{\ell\in\mathbb{N}}$.

5. Semi-classical measures

The semi-classical pseudodifferential theory developed in Section 3 may be used to analyse the oscillations of families $(u^\varepsilon)_{\varepsilon>0}$ that are bounded in $L^2(G)$. Following ideas developed in the Euclidean setting [16] and adapted in [10] to the group setting, we develop in this section a manner to analyze the obstruction to strong convergence of families of square integrable functions on G that have oscillations no larger than some fixed scale $\frac{1}{\varepsilon}$ in a sense that we will make precise. The semi-classical pseudodifferential calculus is then particularly adapted and provided with a picture of the obstructions in the phase space $G \times \widehat{G}$.

5.1. **Definitions.** Let $(u^\varepsilon)_{\varepsilon>0}$ be a bounded family in $L^2(G)$. One considers the functionals ℓ_ε defined on $\mathcal{A}_0$ by

$$\ell_\varepsilon(\sigma) = (\mathrm{Op}(\sigma)u^\varepsilon, u^\varepsilon)_{L^2(G)}, \quad \sigma \in \mathcal{A}_0.$$

The limit points of ℓ_ε as ε goes to 0 have some structures. When the family $(u^\varepsilon)_{\varepsilon>0}$ is L^2-normalized, and after possibly further extraction of subsequences, the limit points of ℓ_ε define states of the C^*-algebra $\mathcal{A}$ obtained by completion of $\mathcal{A}_0$ for the norm

$$\|\sigma\|_{L^\infty(G\times\widehat{G})} := \sup_{(x,\pi)\in G\times\widehat{G}} \|\sigma(x,\pi)\|_{\mathcal{L}(\mathcal{H}_\pi)}.$$

For describing the structure of these limit points, we consider the set of pairs (γ, Γ) where γ is a positive Radon measure on $G \times \widehat{G}$ and

$$\Gamma = \{\Gamma(x,\pi) \in \mathcal{L}(\mathcal{H}_\pi) : \ (x,\pi) \in G \times \widehat{G}\}$$

is a positive γ-measurable field of trace-class operators satisfying

$$\int_{G\times\widehat{G}} \mathrm{Tr}_{\mathcal{H}_\pi}\left(\Gamma(x,\pi)\right) d\gamma(x,\pi) < +\infty.$$

This set is then equipped with the equivalence relation: $(\gamma,\Gamma) \sim (\gamma',\Gamma')$ if there exists a measurable function $f : G \times \widehat{G} \to \mathbb{C} \setminus \{0\}$ such that

$$d\gamma' = f d\gamma \ \text{ and } \ \Gamma' = f^{-1}\Gamma$$

for γ-almost every $(x,\pi) \in U \times \widehat{G}$. The equivalence class of (γ,Γ) is denoted by $\Gamma d\gamma$, it is called a positive vector-valued measure. We denote by $\mathcal{M}_{ov}^+(G\times\widehat{G})$ the set of these equivalence classes. The positive continuous linear functionals of the C^*-algebra $\mathcal{A}$ is naturally identified with $\mathcal{M}_{ov}^+(G\times\widehat{G})$. We will choose representants (Γ,γ) of the class $\Gamma d\gamma$ with $\mathrm{Tr}_{\mathcal{H}_\pi}\Gamma(x,\pi) = 1$, $d\gamma(x,\pi)$ almost everywhere.

Theorem 5.1. *If $(u^\varepsilon)_{\varepsilon>0}$ is a bounded family of $L^2(G)$, there exist a sequence $(\varepsilon_k)_{k\in\mathbb{N}}$ in $(0,+\infty)$ with $\varepsilon_k \underset{k\to+\infty}{\longrightarrow} 0$ and a pair $\Gamma d\gamma \in \mathcal{M}_{ov}^+(G\times\widehat{G})$ such that we have*

$$\forall \sigma \in \mathcal{A}_0, \ \ \left(\mathrm{Op}_{\varepsilon_k}(\sigma)u^{\varepsilon_k}, u^{\varepsilon_k}\right)_{L^2(G)} \underset{k\to+\infty}{\longrightarrow} \int_{G\times\widehat{G}} \mathrm{Tr}_{\mathcal{H}_\pi}\left(\sigma(x,\pi)\Gamma(x,\pi)\right) d\gamma(x,\pi).$$

Moreover, given the sequence $(\varepsilon_k)_{k\in\mathbb{N}}$, one has

$$\int_{U\times\widehat{G}} \mathrm{Tr}_{\mathcal{H}_\pi}\left(\Gamma(x,\pi)\right) d\gamma(x,\pi) \leq \limsup_{\varepsilon>0} \|u^\varepsilon\|^2_{L^2(G)}.$$

The positive vector-valued measure $\Gamma d\gamma$ is called a semi-classical measure of the family $(u^\varepsilon)_{\varepsilon>0}$ for the sequence ε_k.

5.2. **Link with the weak limits of the energy density.** We want to link here the weak limits of the measure $|u^\varepsilon(x)|^2dx$ and the semi-classical measures of the family $(u^\varepsilon)_{\varepsilon>0}$. For this, we introduce the definition of an ε-oscillating family of $L^2(G)$. We consider a positive Rockland operator $\mathcal{R}$ of homogeneous degree $2\nu_0$ (see for example in (2.5)).

Definition 5.2. Let $(u^\varepsilon)_{\varepsilon>0}$ be a bounded family in $L^2(G)$. We say that $(u^\varepsilon)_{\varepsilon>0}$ is *ε-oscillating* if

$$\limsup_{\varepsilon\to 0} \|\mathbf{1}_{\varepsilon^{2\nu_0}\mathcal{R}>M} u^\varepsilon\|_{L^2(G)} \underset{M\to+\infty}{\longrightarrow} 0.$$

Let $\chi \in \mathcal{C}^\infty(\mathbb{R})$ such that

$$0 \le \chi \le 1, \qquad \chi = 0 \text{ on }]-\infty, 1], \qquad \text{and } \chi = 1 \text{ on } [2, +\infty[. \tag{5.1}$$

Equivalently, $(u^\varepsilon)_{\varepsilon>0}$ is ε-oscillating if and only if

$$\limsup_{\varepsilon\to 0} \left\| \chi\left(\frac{\varepsilon^{2\nu_0}\mathcal{R}}{M}\right) u^\varepsilon \right\|_{L^2(G)} \underset{M\to+\infty}{\longrightarrow} 0.$$

Intuitively, a family is ε-oscillating it its scale of oscillations are not larger than $\frac{1}{\varepsilon}$. This can be understood thanks to the following Lemma that links the size of the derivatives of a family with its scale of oscillation.

Proposition 5.3. *Let $s > 0$. Assume that there exists $C > 0$ such that*

$$\forall \varepsilon > 0, \quad \|(\varepsilon^{2\nu_0}\mathcal{R})^s \psi^\varepsilon\|_{L^2(G)} \le C.$$

Then $(u^\varepsilon)_{\varepsilon>0}$ is ε-oscillating.

Proof. We use the Plancherel formula and the fact that for $s > 0$, we have

$$\chi\left(\frac{\varepsilon^{2\nu_0}\mathcal{R}}{M}\right) \le \frac{(\varepsilon^{2\nu_0}\mathcal{R})^s}{M^s}\chi\left(\frac{\varepsilon^{2\nu_0}}{M}\mathcal{R}\right) \le \frac{(\varepsilon^{2\nu_0}\mathcal{R})^s}{M^s}$$

as soon as the function χ is supported in $[1, +\infty)$ and satisfies $0 \le \chi \le 1$. □

The interest of the notion of ε-oscillation relies in the fact that it gives an indication of the size of the oscillations that have to be taken into account. It legitimates the use of semi-classical pseudodifferential operators and semi-classical measures. In particular, we have the following result.

Proposition 5.4. *Let $(u^\varepsilon)_{\varepsilon>0}$ be an ε-oscillating family admitting a semi-classical measure $\Gamma d\gamma$ for the sequence $(\varepsilon_k)_{k\in\mathbb{N}}$, then for all $\phi \in \mathcal{S}(G)$*

$$\lim_{k\to+\infty} \int_G \phi(x)|u^{\varepsilon_k}(x)|^2 dx = \int_{G\times\widehat{G}} \phi(x) d\gamma(x,\pi).$$

Proof. Note that it is enough to prove the result for smooth compactly supported functions ϕ. Consider the function χ defined in (5.1) We write for $M > 0$

$$\int_G \phi(x)|u^{\varepsilon_k}(x)|^2 dx = I_0^{k,M} + I_1^{k,M},$$

where

$$I_0^{k,M} := \int_G \phi(x)\ \chi\left(M^{-1}\varepsilon_k^{2\nu_0}\mathcal{R}\right) u^{\varepsilon_k}(x)\ \overline{u^{\varepsilon_k}(x)} dx,$$
$$I_1^{k,M} := \int_G \phi(x)\ (1-\chi)\left(M^{-1}\varepsilon_k^{2\nu_0}\mathcal{R}\right) u^{\varepsilon_k}(x)\ \overline{u^{\varepsilon_k}(x)} dx.$$

As (u^ε) is ε-oscillating, we have

$$\lim_{M\to+\infty}\lim_{k\to+\infty} I_0^{k,M} = 0.$$

For the other integral, it is known that $\phi(x)\ (1-\chi)\left(M^{-1}\pi(\mathcal{R})\right) \in \mathcal{A}_0$, see Section **??**. So, by the definition of $\Gamma d\gamma$, Theorem 5.1 implies

$$\lim_{M\to+\infty}\lim_{k\to+\infty} I_1^{k,M} = \lim_{M\to+\infty}\int_{G\times\widehat{G}} \phi(x)\mathrm{Tr}_{\mathcal{H}_\pi}\left((1-\chi)\left(M^{-1}\pi(\mathcal{R})\right)\Gamma(x,\pi)\right) d\gamma(x,\pi)$$
$$= \int_{G\times\widehat{G}} \phi(x)\mathrm{Tr}_{\mathcal{H}_\pi}\left(\Gamma(x,\pi)\right) d\gamma(x,\pi).$$

Combining the limits shows the statement since $\mathrm{Tr}_{\mathcal{H}_\pi}\left(\Gamma(x,\pi)\right) = 1$. □

5.3. **Examples.** Let us first describe different examples of bounded families of $L^2(\mathbb{R}^d)$ with different types of semi-classical measures.

5.3.1. *Concentration on a given point of G.* Let $a \in \mathcal{S}(G)$ and $x_0 \in G$, we set

$$u^\varepsilon(x) = \varepsilon^{-\frac{Q}{2}} a(\delta_{\varepsilon^{-1}}(x_0^{-1}x)), \quad x \in G.$$

Proposition 5.5 (Concentration)**.** *The family u^ε is a bounded family in $L^2(G)$ with $\|u^\varepsilon\|_{L^2(G)} = \|a\|_{L^2(G)}$. Moreover, u^ε is ε-oscillating and any semi-classical measure of the family u^ε is equivalent to the pair (Γ, γ) with*

$$\Gamma(\pi) = \mathcal{F}a(\pi)\mathcal{F}a(\pi)^*, \quad \gamma(x,\pi) = \ \delta_{x_0}(x) \otimes \mu(\pi).$$

Proof. The calculus of $\|u^\varepsilon\|_{L^2(G)}$ is straightforward. Moreover, u^ε satisfies the Sobolev criterium of Proposition 5.3 and thus is ε-oscillating. Let us now calculate its semi-classical measures in the case $x_0 = 0$. We have

$$(\mathrm{Op}(\sigma)u^\varepsilon, u^\varepsilon)_{L^2(G)} = \varepsilon^{-2Q} \int_{G\times G} \kappa_x(\delta_{\varepsilon^{-1}}(y^{-1}x)) a(\delta_{\varepsilon^{-1}}y)\overline{a(\delta_{\varepsilon^{-1}}x)} dx\, dy.$$

The change of variable $\delta_{\varepsilon^{-1}}x \to x$, $\delta_{\varepsilon^{-1}}y \to y$ gives

$$(\mathrm{Op}(\sigma)u^\varepsilon, u^\varepsilon)_{L^2(G)} = \int_{G\times G} \kappa_{\delta_\varepsilon x}(y^{-1}x) a(y)\overline{a(x)} dxdy.$$

By Lebesgue dominated convergence, we obtain

$$\lim_{\varepsilon\to 0} (\mathrm{Op}(\sigma)u^\varepsilon, u^\varepsilon)_{L^2(G)} = \int_{G\times G} \kappa_0(y^{-1}x) a(y)\overline{a(x)} dxdy,$$

whence the result in view of

$$\begin{aligned}\int_{G\times G} \kappa_0(y^{-1}x) a(y)\overline{a(x)} dxdy &= \int_{G\times G\times\widehat{G}} \mathrm{Tr}_{\mathcal{H}_\pi}\left(\pi(y^{-1}x)\sigma(0,\pi)\right) a(y)\overline{a(x)} dxdyd\mu(\pi)\\ &= \int_{\widehat{G}} \mathrm{Tr}_{\mathcal{H}_\pi}\left(\sigma(0,\pi)\mathcal{F}a(\pi)\mathcal{F}a(\pi)^*\right) d\mu(\pi).\end{aligned}$$

□

5.3.2. *Spectral or dual concentration.* We assume that the group G admits a (unitary irreducible) representation π_0 which is square integrable modulo its centre.

Let us introduce some notations. We may assume that the basis $\{X_1, \ldots, X_n\}$ of the Lie algebra $\mathfrak{g}$ has been chosen so that a subset $\{X_{j_1}, \ldots, X_{j_{j_{n_\mathfrak{z}}}}\}$, form a basis for the centre $\mathfrak{z}$ of $\mathfrak{g}$. Therefore we can write any element x as

$$x = \exp_G(x_1X_1 + \ldots + x_nX_n) = x'x_\mathfrak{z} = x_\mathfrak{z}x',$$

where $x_\mathfrak{z} = \exp_G(x_{j_1}X_{j_1} + \ldots + x_{n_\mathfrak{z}}X_{n_\mathfrak{z}})$ and $x' = \exp_G\left(\sum_{j\notin\{j_1,\ldots,j_{n_\mathfrak{z}}\}} x_jX_j\right)$. Naturally, we identify the centre of the Lie algebra $\mathfrak{z}$ and the centre of the group $G_\mathfrak{z} := \exp_G \mathfrak{z}$ with $\mathbb{R}^{n_\mathfrak{z}}$. Note that we still consider anisotropic dilations in those directions. The quotient group $G' := G/G_\mathfrak{z}$ is also graded and we denote by Q' its homogeneous dimensions, also given by

$$Q' := \sum_{j\notin\{j_1,\ldots,j_{n_\mathfrak{z}}\}} \upsilon_j.$$

Let $\varphi \in \mathcal{H}_{\pi_0}$, in the domain of $\pi_0(\mathcal{R})$ for a Rockland operator $\mathcal{R}$, and set

$$e_0(x) = (\pi_0(x)\varphi, \varphi)_{\mathcal{H}_{\pi_0}}.$$

On the one hand, on the centre $G_{\mathfrak{z}}$ of the group, π_0 coincides with a character $e^{i\lambda_0 \cdot}$, i.e. $\pi(x_{\mathfrak{z}}) = e^{i\lambda_0 x_{\mathfrak{z}}}$ where we identify $x_{\mathfrak{z}}$ with an element of $\mathbb{R}^{n_{\mathfrak{z}}}$ and where $\lambda_0 x_{\mathfrak{z}}$ denotes the standard scalar product of the two elements λ_0 and $x_{\mathfrak{z}}$ of $\mathbb{R}^{n_{\mathfrak{z}}}$. Thus for any $x = x' x_{\mathfrak{z}}$ in G we have

$$e_0(x) = (\pi_0(x' x_{\mathfrak{z}})\varphi, \varphi)_{\mathcal{H}_{\pi_0}} = e^{i\lambda_0 x_{\mathfrak{z}}}(\pi_0(x')\varphi, \varphi)_{\mathcal{H}_{\pi_0}} = e^{i\lambda_0 x_{\mathfrak{z}}} e_0(x'). \tag{5.2}$$

On the other hand, $e_0\big|_{G'} \in \mathcal{S}(G')$. See [4, p. 169 and Theorem 4.5.11].

We denote by d_{π_0} the formal degree of π_0 for which we have for any $\varphi_1, \varphi_2, \varphi_3, \varphi_4 \in \mathcal{H}_{\pi_0}$:

$$d_{\pi_0} \int_{G/Z} (\pi_0(x')\varphi_1, \varphi_2)_{\mathcal{H}_{\pi_0}} \overline{(\pi_0(x')\varphi_3, \varphi_4)_{\mathcal{H}_{\pi_0}}} dx' = (\varphi_1, \varphi_2)_{\mathcal{H}_{\pi_0}} \overline{(\varphi_3, \varphi_4)_{\mathcal{H}_{\pi_0}}}, \tag{5.3}$$

see again [4, p. 169 and Theorem 4.5.11].

Finally, we fix $a \in \mathcal{S}(\mathbb{R}^{n_{\mathfrak{z}}})$. The family $(v^\varepsilon)_{\varepsilon>0}$ is defined by

$$v^\varepsilon(x) = \varepsilon^{-\frac{Q'}{2}} a(x_{\mathfrak{z}}) e_0(\delta_{\varepsilon^{-1}} x).$$

Proposition 5.6. *The family $(v^\varepsilon)_{\varepsilon>0}$ is uniformly bounded in $L^2(G)$ and ε-oscillating. Moreover, it has only one semi-classical measure $\Gamma d\gamma$ such that*

$$\gamma(x, \pi) = \|\varphi\|^4_{\mathcal{H}_{\pi_0}} \left(\frac{|a(x_{\mathfrak{z}})|^2}{d_{\pi_0}} dx_{\mathfrak{z}} \otimes \delta_{x'=0} \right) \otimes \delta_{\pi=\pi_0},$$

and Γ being the orthogonal projection on $\mathbb{C}\varphi$.

This example can be generalised (see Section 6.4 in [10]) in order to prove that one can find families that admits as semi-classical measure a Dirac mass on a given representation. If the representation is finite dimensional, then it is of dimension 1 (see [4]) and we may proceed as in the Euclidean case. If π_0 is any irreducible representation of infinite dimension, the properties of square integrability can be replaced by concepts introduced by Pedersen [22] and general results obtained therein about representations of nilpotent Lie groups

Proof. First let us show that each function v^ε is square integrable:

$$\begin{aligned} \|v^\varepsilon\|^2_{L^2(G)} &= \varepsilon^{-Q} \int_{G'} \int_{\mathbb{R}^{n_{\mathfrak{z}}}} |e_0(\delta_{\varepsilon^{-1}} x' \delta_{\varepsilon^{-1}} x_{\mathfrak{z}}) a(x_z)|^2 dx_{\mathfrak{z}} dx' \\ &= \varepsilon^{-Q'} \int_{G'} |e_0(\delta_{\varepsilon^{-1}} x')|^2 dx' \int_{\mathbb{R}^{n_{\mathfrak{z}}}} |a(x_z)|^2 dx_{\mathfrak{z}} \\ &= \int_{G'} |e_0(x')|^2 dx' \int_{\mathbb{R}^{n_{\mathfrak{z}}}} |a(x_z)|^2 dx_{\mathfrak{z}}, \end{aligned}$$

having used the change of variable $x' \mapsto \delta_{\varepsilon^{-1}} x'$. As the functions e_0 and a are Schwartz on G' and $\mathbb{R}^{n_{\mathfrak{z}}}$ respectively, the quantity above is finite, and $v^\varepsilon \in L^2(G)$. Moreover, by (5.3),

$$\int_{G'} |e_0(x')|^2 dx' = \frac{1}{d_{\pi_0}} \|\varphi\|^4_{\mathcal{H}_{\pi_0}}.$$

We deduce

$$\|v^\varepsilon\|_{L^2(G)} = \frac{1}{\sqrt{d_{\pi_0}}} \|\varphi\|^2_{\mathcal{H}_{\pi_0}} \|a\|_{L^2(G_{\mathfrak{z}})}.$$

Let us now consider the Rockland operator $\mathcal{R}$ for which $\varphi \in \mathrm{Dom}(\pi_0(\mathcal{R}))$

$$\varepsilon^{2\nu_0} \mathcal{R} v^\varepsilon = \varepsilon^{-Q'} a(x_{\mathfrak{z}}) (\pi_0(\delta_{\varepsilon^{-1}} x) \pi_0(\mathcal{R}) \varphi, \varphi)_{\mathcal{H}_{\pi_0}} + O(\varepsilon)$$

in $L^2(G)$ and is thus bounded uniformly in ε therein. Therefore $(v^\varepsilon)_{\varepsilon>0}$ satsifies the Sobolev criterium of Proposition 5.3 and thus is ε-oscillating.

Let us now compute the semi-classical measure of $(v^\varepsilon)_{\varepsilon>0}$. Let $\sigma\in\mathcal{A}_0$ with convolution kernel κ_x. We have

$$
\begin{aligned}
(\mathrm{Op}_\varepsilon(\sigma)v^\varepsilon,v^\varepsilon)_{L^2(G)} &= \varepsilon^{-Q-Q'}\int_{G\times G}\kappa_x(\delta_{\varepsilon^{-1}}(y^{-1}x))\bar v^\varepsilon(x)v^\varepsilon(y)dxdy\\
&= \varepsilon^{-Q-Q'}\int_{G\times G}\kappa_x(\delta_{\varepsilon^{-1}}(y^{-1}x))\overline{e_0(\delta_{\varepsilon^{-1}}x)}e_0(\delta_{\varepsilon^{-1}}y)\bar a(x_{\mathfrak z})a(y_{\mathfrak z})dxdy\\
&= \varepsilon^{-Q'}\int_{G\times G}\kappa_x(w)\overline{e_0(\delta_{\varepsilon^{-1}}x)}e_0((\delta_{\varepsilon^{-1}}x)w^{-1})\bar a(x_{\mathfrak z})a((x\delta_\varepsilon w^{-1})_{\mathfrak z})dxdw\\
&= \varepsilon^{-Q'}\int_{G\times G}\kappa_x(w)\overline{e_0(\delta_{\varepsilon^{-1}}x)}e_0((\delta_{\varepsilon^{-1}}x)w^{-1})|a(x_{\mathfrak z})|^2dxdw+O(\varepsilon).
\end{aligned}
$$

We have taken advantage that the functions $(x,z)\mapsto\kappa_x(w)$ is Schwartz, which justifies the approximation. We observe that for $x,z\in G$, we have

$$
\begin{aligned}
\int_G\kappa_x(w)e_0(zw^{-1})dw &= \left(\pi_0(z)\left(\int_G\kappa_x(w)\pi_0(w)^*dw\right)\varphi,\varphi\right)_{\mathcal{H}_{\pi_0}}\\
&= (\pi_0(z)\sigma(x,\pi_0)\varphi,\varphi)_{\mathcal{H}_{\pi_0}},
\end{aligned}
$$

where we have used the Fourier inversion formula for the regular symbol σ. We are left with

$$
(\mathrm{Op}_\varepsilon(\sigma)v^\varepsilon,v^\varepsilon)_{L^2(G)}=\int_G|a(x_{\mathfrak z})|^2\overline{e_0(\delta_{\varepsilon^{-1}}x)}\,(\pi_0(\delta_{\varepsilon^{-1}}x)\sigma(x,\pi_0)\varphi,\varphi)_{\mathcal{H}_{\pi_0}}\,dx+O(\varepsilon).
$$

The property (5.2) yields that for $x=x'x_{\mathfrak z}$,

$$
\begin{aligned}
&\overline{e_0(\delta_{\varepsilon^{-1}}x)}\,(\pi_0(\delta_{\varepsilon^{-1}}x)\sigma(x,\pi_0)\varphi_1,\varphi_2)_{\mathcal{H}_{\pi_0}}\\
&\qquad=\overline{(\pi_0(\delta_{\varepsilon^{-1}}x')\varphi,\varphi)_{\mathcal{H}_{\pi_0}}}\,\left(\pi_0(\delta_{\varepsilon^{-1}}x')\sigma(x'x_{\mathfrak z},\pi_0)\varphi,\varphi\right)_{\mathcal{H}_{\pi_0}}.
\end{aligned}
$$

Therefore, after a change of variables, we obtain

$$
\begin{aligned}
(\mathrm{Op}_\varepsilon(\sigma)v^\varepsilon,v^\varepsilon)_{L^2(G)} &= \int_{G'\times G_{\mathfrak z}}|a(x_{\mathfrak z})|^2\overline{(\pi_0(x')\varphi_1,\varphi_2)_{\mathcal{H}_{\pi_0}}}\left(\pi_0(x')\sigma(\delta_\varepsilon(x')x_{\mathfrak z},\pi_0)\varphi,\varphi\right)_{\mathcal{H}_{\pi_0}}dx'dx_{\mathfrak z}+O(\varepsilon)\\
&\underset{\varepsilon\to0}{\longrightarrow}\int_{G'\times G_{\mathfrak z}}|a(x_{\mathfrak z})|^2\overline{(\pi_0(x')\varphi,\varphi)_{\mathcal{H}_{\pi_0}}}\left(\pi_0(x')\sigma(x_{\mathfrak z},\pi_0)\varphi,\varphi\right)_{\mathcal{H}_{\pi_0}}dx'dx_{\mathfrak z}\\
&= \frac{1}{d_{\pi_0}}\|\varphi\|^2_{\mathcal{H}_{\pi_0}}\int_{G_{\mathfrak z}}|a(x_{\mathfrak z})|^2\,(\sigma(x_{\mathfrak z},\pi_0)\varphi,\varphi)_{\mathcal{H}_{\pi_0}}\,dx_{\mathfrak z}
\end{aligned}
$$

where we have used (5.3), This terminates the proof. □

5.3.3. *Wave packets.* Let $a\in\mathcal{S}(G)$, $(x_0,\pi_0)\in G\times\widehat G$, $\varphi_1,\varphi_2\in\mathcal{H}_{\pi_0}$. We assume that there exists a Rockland operator $\mathcal{R}$ of homogeneous degree $2\nu_0$ such that the harmonics φ_1 and φ_2 are in the intersection of the set $\mathcal{H}^\infty_{\pi_0}$ defined as

$$
\mathcal{H}^\infty_{\pi_0}=\bigcap_{\ell\in\mathbb{N}}\mathrm{dom}\left(\pi_0(\mathcal{R})^\ell\right).
$$

One defines $g^\varepsilon_{x,\pi}$ as in (4.7):

$$
g^\varepsilon_{x_0,\pi_0}(y):=\left(\mathrm{WP}^\varepsilon_{x_0,\pi_0}[a](y)\varphi_1,\varphi_2\right)_{\mathcal{H}_{\pi_0}},\ y\in G,
$$

Note that we have made no assumption on the function a, such as being normalized is non-negative or satisfies some symmetries relations. We are interested here in the properties of the function g_{x_0,π_0} itself.

Wave packets are bounded families in $L^2(\mathbb{R}^d)$ that enjoy localisation properties. In the next statement, we assume that φ_1 and φ_2 enjoy enough regularity so that everything makes sense, which is meant by $\varphi_1, \varphi_2 \in \mathcal{H}_{\pi_0}^\infty$.

Lemma 5.7. *Assume $\varphi_1, \varphi_2 \in \mathcal{H}_{\pi_0}^\infty$. The family $(g^\varepsilon_{x,\pi})_{\varepsilon>0}$ is a bounded ε-oscillating family in $L^2(G)$ and satisfies the following properties*

(1) *For $\sigma \in \mathcal{A}_0$, we have in $L^2(G)$,*
$$\mathrm{Op}_\varepsilon(\sigma) g^\varepsilon_{x_0,\pi_0} = \left(\mathrm{WP}^\varepsilon_{x_0,\pi_0}[a]\sigma(x_0,\pi_0)\varphi_1,\varphi_2\right)_{\mathcal{H}_{\pi_0}} + O(\sqrt{\varepsilon}).$$
More precisely, if $N\in\mathbb{N}$ and $\mathbb{P}_{a,0,N} = \sum_{[\alpha]\leq N} c_\alpha z^\alpha \mathbb{X}^\alpha a(0)$ denotes the Taylor polynomial associated with a in 0 at order N, then in $L^2(G)$,
$$\mathrm{Op}_\varepsilon(\sigma) g^\varepsilon_{x_0,\pi_0} = \sum_{[\alpha]\leq N} \varepsilon^{\frac{[\alpha]}{2}} c_\alpha \left(\mathrm{WP}^\varepsilon_{x_0,\pi_0}[\mathbb{X}^\alpha a]\Delta_\alpha\sigma(x_0,\pi_0)\varphi_1,\varphi_2\right)_{\mathcal{H}_{\pi_0}} + O\left(\varepsilon^{\frac{\lceil N\rfloor+1}{2}}\right).$$

(2) *Let $\mathcal{R}$ a positive Rockland operator of order $2\nu_0$, then*
$$\forall\alpha\in\mathbb{N}^n,\ \ \forall k\in\mathbb{N},\ \ \exists C_{\alpha,k}>0,\ \ \forall\varepsilon>0,\ \ \|x^\alpha(\varepsilon^{2\nu_0}\mathcal{R})^k g^\varepsilon_{x,\pi}\|_{L^2(G)} \leq C_k.$$

(3) *Any semi-classical measure $\Gamma d\gamma$ of the family $(g^\varepsilon_{x,\pi})_{\varepsilon>0}$ satisfies*
$$\gamma = c\,\delta(x-x_0)\otimes\delta(\pi-\pi_0),\ \ c>0.$$

Proof. In view of (4.2), we have
$$\|g^\varepsilon_{x_0,\pi_0}\|_{L^2} \leq \|a\|_{L^2}\|\varphi_1\|_{\mathcal{H}_{\pi_0}}\|\varphi_2\|_{\mathcal{H}_{\pi_0}}.$$
Moreover, the ε-oscillation comes from the Sobolev criterium of Proposition 5.3, as for the dual concentration of Proposition 5.6. Points (2) and (3) are consequences of the calculus of Point (1).

It remains to prove Point (1). We write
$$\begin{aligned}
&\mathrm{Op}_\varepsilon(\sigma) g^\varepsilon_{x_0,\pi_0}(x)\\
&= \varepsilon^{-Q}\varepsilon^{-\frac{Q}{4}}\int_{G\times\widehat{G}} \mathrm{Tr}_{\mathcal{H}_\pi}\left(\pi(\delta_{\varepsilon^{-1}}(y^{-1}x)\sigma(x,\pi)\right) a\left(\delta_{\varepsilon^{-\frac12}}(x_0^{-1}y)\right)\left(\pi_0(\delta_{\varepsilon^{-1}}y)\varphi_1,\varphi_2\right)_{\mathcal{H}_{\pi_0}} dy d\mu(\pi)\\
&= \varepsilon^{-\frac{Q}{4}}\int_{G\times\widehat{G}} \mathrm{Tr}_{\mathcal{H}_\pi}\left(\pi(z)\sigma(x,\pi)\right) a\left(\delta_{\varepsilon^{-\frac12}}(x_0^{-1}x)\delta_{\varepsilon^{\frac12}} z^{-1}\right)\left(\pi_0(\delta_{\varepsilon^{-1}}x)\pi_0(z^{-1})\varphi_1,\varphi_2\right)_{\mathcal{H}_{\pi_0}} dz d\mu(\pi)
\end{aligned}$$
Set $x_\varepsilon = \delta_{\varepsilon^{-\frac12}}(x_0^{-1}x)$ and consider the Taylor polynomial $\mathbb{P}_{a,x_\varepsilon,N}$ of a at x_ε at order N. Then, the function
$$\begin{aligned}
r_\varepsilon(x) &:= \mathrm{Op}_\varepsilon(\sigma) g^\varepsilon_{x_0,\pi_0}\\
&\quad - \varepsilon^{-\frac{Q}{4}}\int_{G\times\widehat{G}} \mathrm{Tr}_{\mathcal{H}_\pi}\left(\pi(z)\sigma(x,\pi)\right)\mathbb{P}_{a,x_\varepsilon,N}(-\delta_{\varepsilon^{\frac12}} z)\left(\pi_0(\varepsilon^{-1}x)\pi_0(z^{-1})\varphi_1,\varphi_2\right)_{\mathcal{H}_{\pi_0}} dz d\mu(\pi)
\end{aligned}$$
satisfies
$$\begin{aligned}
|r_\varepsilon(x)| &\leq \varepsilon^{\frac{\lceil N\rfloor+1}{2}}\|\varphi_1\|_{\mathcal{H}_{\pi_0}}\|\varphi_2\|_{\mathcal{H}_{\pi_0}}\int_G |z|^{\lceil N\rfloor+1}\left|\int_{\widehat{G}}\mathrm{Tr}_{\mathcal{H}_\pi}(\pi(z)\sigma(x,\pi)d\mu(\pi)\right| dz\\
&\leq \varepsilon^{\frac{\lceil N\rfloor+1}{2}}\|\varphi_1\|_{\mathcal{H}_{\pi_0}}\|\varphi_2\|_{\mathcal{H}_{\pi_0}}\int_G |z|^{\lceil N\rfloor+1}|\kappa_x(z)|\, dz
\end{aligned}$$
where $\mathcal{F}\kappa_x(\pi) = \sigma(x,\pi)$. We deduce that $\varepsilon^{-\frac{\lceil N\rfloor+1}{2}}\|r_\varepsilon\|_{L^2(G)}$ is uniformly bounded and we observe that, using $\pi_0(z^{-1}) = \pi_0(z)^*$ and
$$\int_{G\times\widehat{G}} z^\alpha \mathrm{Tr}_{\mathcal{H}_\pi}\left(\pi(z)\sigma(x,\pi)\right)\pi_0(z)^* d\mu(\pi)dz = \Delta_\alpha\sigma(x,\pi_0),\ \ \alpha\in\mathbb{N}^n,$$

we obtain the result. □

5.4. **Semi-classical measures and PDEs.** We assume here that the family $(\psi^\varepsilon)_{\varepsilon>0}$ in which we are interested satisfies a PDE. We consider a differential operator that reads

$$H^\varepsilon = {\rm Op}_\varepsilon(h_0)$$

with the notations of Proposition 3.5. For $(x,\pi)\in G\times\widehat G$, we denote by $\mathbb P$ the projector on ${\rm Ker}\, h_0$ and we assume $h_0^*=h_0$.

Proposition 5.8. *Let $(\psi^\varepsilon)_{\varepsilon>0}$ be a bounded family in $L^2(G)$.*

(1) *Assume $H^\varepsilon\psi^\varepsilon=o(1)$ in $L^2(G)$, then any semi-classical measure $\Gamma d\gamma$ of $(\psi^\varepsilon)_{\varepsilon>0}$ satisfies*

$$\Gamma(x,\pi)=\mathbb P(x,\pi)\Gamma(x,\pi)\mathbb P(x,\pi),\ \ d\gamma(x,\pi)-a.e.$$

(2) *Assume $H^\varepsilon\psi^\varepsilon=o(\varepsilon^{\upsilon_1})$ in $L^2(G)$, then any semi-classical measure $\Gamma d\gamma$ of $(\psi^\varepsilon)_{\varepsilon>0}$ satisfies the additional relation*

$$\sum_{j=1}^{d_1} X_jh_0\Delta_j\Gamma(x,\pi)-X_j\Gamma(x,\pi)\Delta_jh_0=0,\ \ d\gamma(x,\pi)-a.e.$$

Remark 5.9. The nature of the group plays a role in the information given by this equation and property (ii) may reduce to $0=0$. The case of step-2 stratified groups is treated in [**?**] for evolution equations. The arguments therein can be easily adapted to the context of Proposition 5.8.

Proof. Point (1). Let $\sigma\in\mathcal A_0$, then

$$({\rm Op}_\varepsilon(\sigma)H^\varepsilon\psi^\varepsilon,\psi^\varepsilon)=o(1)\ \text{ and }\ ({\rm Op}_\varepsilon(\sigma)\psi^\varepsilon,H^\varepsilon\psi^\varepsilon)=o(1).$$

We deduce that if $\Gamma d\gamma$ is a semi-classical measure of H^ε, then $h_0\Gamma=0$ and $\Gamma h_0=0$, which implies ${\rm Ran}\Gamma\subset{\rm Ker}h_0$ and $\Gamma({\rm Id}-\mathbb P)=0$. The result follows.

Point (2). One chooses σ such that $\sigma=\mathbb P\sigma\mathbb P$ and observes that

$$\frac{1}{\varepsilon^{\upsilon_1}}\left(({\rm Op}_\varepsilon(\sigma)H^\varepsilon-(H^\varepsilon)^*{\rm Op}_\varepsilon(\sigma))\psi^\varepsilon,\psi^\varepsilon\right)=o(1).$$

$$\frac{1}{\varepsilon^{\upsilon_1}}\left[{\rm Op}_\varepsilon(\sigma)\ ,\ H^\varepsilon\psi^\varepsilon\right]=\sum_{j=1}^{n_1}{\rm Op}_\varepsilon(\Delta_j\sigma X_jh_0-\Delta_jh_0X_j\sigma)+o(1)$$

□

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