The nonlinear Schrödinger equation
A little bit of pop culture

Lysianne Hari
LMB, University of Bourgogne Franche Comté

AlgDynQua Days - September, 16th 2020

laboratoire de mathématiques de besançon
UNIVERSITÉ DE FRANCHE-COMTÉ • CNRS • UMR 6623

Solving ODEs

General first order ODE with given data: consider $f : \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ continuous and

$$(*) \ \ \dot{u}(t) = f\left(t, u(t)\right), \quad u(0) = u_0, \ t \in]-T, T[\subset \mathbf{R}.$$

$\rightarrow$ is called **"Cauchy problem"**. Solving locally the Cauchy problem is finding $T > 0$ and $u \in C^1(]-T, T[)$ such that $(*)$ is true.
If $T = +\infty$, we have a **global** solution.

Cauchy-Lipschitz theorem gives (local) existence of unique solution for each choice of u_0 under conditions on f.

As a consequence, it ensures (local) existence of the flow Φ on $]-T, T[$, where Φ is the application $u_0 \mapsto u(t)$, and for all $t \in]-T, T[$, $\Phi(u_0, t) = u(t)$ the solution associated to initial data u_0.

Solving ODEs

General first order ODE with given data: consider $f : \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ continuous and

$$(*) \quad \dot{u}(t) = f\left(t, u(t)\right), \quad u(0) = u_0, \ t \in]-T, T[\subset \mathbf{R}.$$

$\to$ is called **"Cauchy problem"**. Solving locally the Cauchy problem is finding $T > 0$ and $u \in C^1(]-T, T[)$ such that $(*)$ is true.
If $T = +\infty$, we have a **global** solution.

Cauchy-Lipschitz theorem gives (local) existence of unique solution for each choice of u_0 under conditions on f.

As a consequence, it ensures (local) existence of the flow Φ on $]-T, T[$, where Φ is the application $u_0 \mapsto u(t)$, and for all $t \in]-T, T[$, $\Phi(u_0, t) = u(t)$ the solution associated to initial data u_0.

Solving ODEs

General first order ODE with given data: consider $f : \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ continuous and

$$(*)\ \ \dot{u}(t) = f\left(t, u(t)\right), \quad u(0) = u_0,\ t \in]-T, T[\subset \mathbf{R}.$$

$\rightarrow$ is called **"Cauchy problem"**. Solving locally the Cauchy problem is finding $T > 0$ and $u \in C^1(]-T, T[)$ such that $(*)$ is true.
If $T = +\infty$, we have a **global** solution.

Cauchy-Lipschitz theorem gives (local) existence of unique solution for each choice of u_0 under conditions on f.

As a consequence, it ensures (local) existence of the flow Φ on $]-T, T[$, where Φ is the application $u_0 \mapsto u(t)$, and for all $t \in]-T, T[$, $\Phi(u_0, t) = u(t)$ the solution associated to initial data u_0.

Fixed point or Contraction Mapping Theorem

Consider X a Banach space with norm $\| \cdot \|$ and $T : I \to \mathbf{R}$ a contraction, that is

$$\|T(x) - T(y)\| \leq c\|x - y\|, \ 0 \leq c < 1.$$

Then T has a unique fixed point in I and all sequences $x_0 \in I, \ x_{n+1} = T(x_n)$ converge to this unique fixed point.

Example: consider

$$\dot{u} = a(t)u, \quad u(0,\cdot) = u_0, \quad a \text{ is continuous.}$$

Then $U(t)u_0 = \exp\left(\int_0^t a(s)\, ds\right) u_0$ is solution.

We can deduce solutions to:

$$\dot{u} = a(t)u + b(t), \quad u(0,\cdot) = u_0, \quad a, b \text{ are continuous.}$$

Then a solution is given by

$$\text{(Duhamel formula)} \quad \Phi\left(u_0, t\right) = U(t)u_0 + \int_0^t U(t-s)b(s)\, ds$$
$$= \text{general solution} + \text{particular solution.}$$

The Nonlinear Schrödinger equation

$$(\mathrm{S}): \quad i\partial_t u + \Delta u = \lambda |u|^\alpha u \quad ; \quad u(0,.) = u_0 \in \mathcal{H},$$

where :

- $(t,x) \in \mathbf{R} \times \mathbf{R}^d,\ d \geq 1$,
- $u = u(t,x) \in \mathbf{C}$,
- $\lambda \in \{-1, 0, 1\}$,
- $\mathcal{H}$ a Hilbert space.

I. Immediate information : conserved quantities.

Mass conservation: $\|u(t)\|_{L^2} = \|u_0\|_{L^2}$.

Energy conservation: $\|\nabla u(t)\|_{L^2} + \dfrac{\lambda}{\alpha+2}\|u(t)\|_{L^{\alpha+2}} = E(u(t)) = E(u_0)$.

Sobolev space $H^1(\mathbf{R}^d)$ is appearing naturally; $H^1 \subset L^q$ for all $q \in [2, \frac{2d}{d-2}]$.

The Nonlinear Schrödinger equation

$$(\mathrm{S}): \quad i\partial_t u + \Delta u = \lambda |u|^\alpha u \quad ; \quad u(0,.) = u_0 \in \mathcal{H},$$

where :

- $(t,x) \in \mathbf{R} \times \mathbf{R}^d, \; d \geq 1,$
- $u = u(t,x) \in \mathbf{C},$
- $\lambda \in \{-1, 0, 1\},$
- $\mathcal{H}$ a Hilbert space.

I. Immediate information : conserved quantities.

Mass conservation: $\|u(t)\|_{L^2} = \|u_0\|_{L^2}.$

Energy conservation: $\|\nabla u(t)\|_{L^2} + \dfrac{\lambda}{\alpha+2}\|u(t)\|_{L^{\alpha+2}} = E(u(t)) = E(u_0).$

Sobolev space $H^1(\mathbf{R}^d)$ is appearing naturally; $H^1 \subset L^q$ for all $q \in [2, \frac{2d}{d-2}]$.

The Nonlinear Schrödinger equation

$$(\mathrm{S}): \quad i\partial_t u + \Delta u = \lambda |u|^\alpha u \quad ; \quad u(0,.) = u_0 \in \mathcal{H},$$

where :

- $(t, x) \in \mathbf{R} \times \mathbf{R}^d,\ d \geq 1,$
- $u = u(t, x) \in \mathbf{C},$
- $\lambda \in \{-1, 0, 1\},$
- $\mathcal{H}$ a Hilbert space.

I. Immediate information : conserved quantities.

Mass conservation: $\|u(t)\|_{L^2} = \|u_0\|_{L^2}.$

Energy conservation: $\|\nabla u(t)\|_{L^2} + \dfrac{\lambda}{\alpha+2}\|u(t)\|_{L^{\alpha+2}} = E(u(t)) = E(u_0).$

Sobolev space $H^1(\mathbf{R}^d)$ is appearing naturally; $H^1 \subset L^q$ for all $q \in [2, \frac{2d}{d-2}]$.

II. $\lambda = 0$: The fundamental solution

$i\partial_t u = -\Delta u$, "taking Fourier" : $i\partial_t \mathcal{F}(u) = 4\pi^2|\xi|^2\mathcal{F}(u)$.

$$\partial_t \mathcal{F}(u) = -4\pi^2 i|\xi|^2\mathcal{F}(u) \Rightarrow \mathcal{F}(u)(t,\xi) = e^{-4\pi^2 it|\xi|^2}\mathcal{F}(u_0)(\xi)$$

$$u(t,x) =: U(t)u_0 = \mathcal{F}^{-1}\left(e^{-4\pi^2 it|\cdot|^2}\mathcal{F}(u_0)(\cdot)\right)(x)$$

$$U(t)u_0 =: \mathcal{F}^{-1}\left(e^{-4\pi^2 it|\cdot|^2}\mathcal{F}(u_0)\right) = (4\pi it)^{-1/2}e^{i|x|^2/(4t)} * u_0$$

III. Some properties of the fundamental solution $U(t)$.

Young : $p,q,r \in [1,\infty]$, $\frac{1}{p}+\frac{1}{q} = 1+\frac{1}{r}$ $\|f*g\|_{L^r} \leq \|f\|_{L^p}\|g\|_{L^q}$. With $p = r = \infty$, $q = 1$, we have

Dispersion : $\|U(t)f\|_{L^\infty} \leq (4\pi|t|)^{-d/2}\|f\|_{L^1}$

Unitary: $\|U(t)f\|_{L^2} = \|f\|_{L^2}$

II. $\lambda = 0$: The fundamental solution

$i\partial_t u = -\Delta u$, "taking Fourier" : $i\partial_t \mathcal{F}(u) = 4\pi^2|\xi|^2\mathcal{F}(u)$.

$$\partial_t \mathcal{F}(u) = -4\pi^2 i|\xi|^2\mathcal{F}(u) \Rightarrow \mathcal{F}(u)(t,\xi) = e^{-4\pi^2 it|\xi|^2}\mathcal{F}(u_0)(\xi)$$

$$u(t,x) =: U(t)u_0 = \mathcal{F}^{-1}\left(e^{-4\pi^2 it|\cdot|^2}\mathcal{F}(u_0)(\cdot)\right)(x)$$

$$U(t)u_0 =: \mathcal{F}^{-1}\left(e^{-4\pi^2 it|\cdot|^2}\mathcal{F}(u_0)\right) = (4\pi it)^{-1/2}e^{i|x|^2/(4t)} * u_0$$

III. Some properties of the fundamental solution $U(t)$.

Young : $p,q,r \in [1,\infty]$, $\frac{1}{p}+\frac{1}{q} = 1+\frac{1}{r}$ $\|f*g\|_{L^r} \leq \|f\|_{L^p}\|g\|_{L^q}$. With $p = r = \infty, q = 1$, we have

Dispersion : $\|U(t)f\|_{L^\infty} \leq (4\pi|t|)^{-d/2}\|f\|_{L^1}$

Unitary: $\|U(t)f\|_{L^2} = \|f\|_{L^2}$

II. $\lambda = 0$: The fundamental solution

$i\partial_t u = -\Delta u$, "taking Fourier" : $i\partial_t \mathcal{F}(u) = 4\pi^2|\xi|^2\mathcal{F}(u)$.

$$\partial_t \mathcal{F}(u) = -4\pi^2 i|\xi|^2\mathcal{F}(u) \Rightarrow \mathcal{F}(u)(t,\xi) = e^{-4\pi^2 it|\xi|^2}\mathcal{F}(u_0)(\xi)$$

$$u(t,x) =: U(t)u_0 = \mathcal{F}^{-1}\left(e^{-4\pi^2 it|\cdot|^2}\mathcal{F}(u_0)(\cdot)\right)(x)$$

$$U(t)u_0 =: \mathcal{F}^{-1}\left(e^{-4\pi^2 it|\cdot|^2}\mathcal{F}(u_0)\right) = (4\pi it)^{-1/2}e^{i|x|^2/(4t)} * u_0$$

III. Some properties of the fundamental solution $U(t)$.

Young : $p, q, r \in [1,\infty]$, $\dfrac{1}{p} + \dfrac{1}{q} = 1 + \dfrac{1}{r}$ $\|f * g\|_{L^r} \le \|f\|_{L^p}\|g\|_{L^q}$. With $p = r = \infty, q = 1$, we have

Dispersion : $\|U(t)f\|_{L^\infty} \le (4\pi|t|)^{-d/2}\|f\|_{L^1}$

Unitary: $\|U(t)f\|_{L^2} = \|f\|_{L^2}$

II. $\lambda = 0$: The fundamental solution

$i\partial_t u = -\Delta u$, "taking Fourier" : $i\partial_t \mathcal{F}(u) = 4\pi^2 |\xi|^2 \mathcal{F}(u)$.

$$\partial_t \mathcal{F}(u) = -4\pi^2 i |\xi|^2 \mathcal{F}(u) \Rightarrow \mathcal{F}(u)(t,\xi) = e^{-4\pi^2 it|\xi|^2} \mathcal{F}(u_0)(\xi)$$

$$u(t,x) =: U(t)u_0 = \mathcal{F}^{-1}\left(e^{-4\pi^2 it|\cdot|^2}\mathcal{F}(u_0)(\cdot)\right)(x)$$

$$U(t)u_0 =: \mathcal{F}^{-1}\left(e^{-4\pi^2 it|\cdot|^2}\mathcal{F}(u_0)\right) = (4\pi it)^{-1/2} e^{i|x|^2/(4t)} * u_0$$

III. Some properties of the fundamental solution $U(t)$.

Young : $p, q, r \in [1,\infty]$, $\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}$ $\|f * g\|_{L^r} \leq \|f\|_{L^p} \|g\|_{L^q}$. With $p = r = \infty, q = 1$, we have

Dispersion : $\|U(t)f\|_{L^\infty} \leq (4\pi|t|)^{-d/2} \|f\|_{L^1}$

Unitary: $\|U(t)f\|_{L^2} = \|f\|_{L^2}$

Consequence 1. For all compact region $K \subset \mathbf{R}^d$, $\lim_{|t|\to\infty} \|1_K u(t)\|_{L^2} = 0$.

Remark on this property on compact manifold (on board. Sorry people outside the CIRM).

Consequence 2: Strichartz estimates. "Interpolation" between unitary and dispersion gives: "for admissible couples (p,q), $(\overline{p},\overline{q})$"

1. $\|U(t)f\|_{L_t^p L_x^q} \leq C(q)\|f\|_{L_x^2}$
2. $\|U(t) *_t f\|_{L_t^p L_x^q} \leq C(q,\overline{q})\|f\|_{L_t^{\overline{p'}} L_x^{\overline{q'}}}$

VERY VERY VERY IMPORTANT to deal with many nonlinear problems.

Consequence 1. For all compact region $K \subset \mathbf{R}^d$, $\lim_{|t|\to\infty} \|1_K u(t)\|_{L^2} = 0$.

Remark on this property on compact manifold (on board. Sorry people outside the CIRM).

Consequence 2: Strichartz estimates. "Interpolation" between unitary and dispersion gives: "for admissible couples (p, q), $(\overline{p}, \overline{q})$"

1. $\|U(t)f\|_{L^p_t L^q_x} \leq C(q)\|f\|_{L^2_x}$
2. $\|U(t) *_t f\|_{L^p_t L^q_x} \leq C(q, \overline{q})\|f\|_{L^{\overline{p'}}_t L^{\overline{q'}}_x}$

VERY VERY VERY IMPORTANT to deal with many nonlinear problems.

IV. $\lambda \neq 0$ **: about** $\pm|u|^\alpha u$

Important features?

IV. $\lambda \neq 0$: about $\pm|u|^{\alpha}u$

Important features?

Sign of λ:

- $+1$: defocusing case

- -1 : focusing case

IV. $\lambda \neq 0$: about $\pm|u|^\alpha u$

Important features?

Sign of λ:

- $+1$: defocusing case
- -1 : focusing case

Criticality

- Existence problem: α not too big, α below $\overline{\alpha_c} = \dfrac{4}{d-2}$ $(d \geq 3)$.
- Long time behaviour (scattering): α not too small, α above $\underline{\alpha_c} = \dfrac{4}{d}$.

Dealing with $\overline{\alpha_c}$ and $\underline{\alpha_c}$ can be tedious !

IV. $\lambda \neq 0$: about $\pm|u|^\alpha u$

Important features?

Sign of λ:

- $+1$: defocusing case
- -1 : focusing case

Criticality

- Existence problem: α not too big, α below $\overline{\alpha_c} = \dfrac{4}{d-2}$ $(d \geq 3)$.
- Long time behaviour (scattering): α not too small, α above $\underline{\alpha_c} = \dfrac{4}{d}$.

Dealing with $\overline{\alpha_c}$ and $\underline{\alpha_c}$ can be tedious !

V. $\lambda \neq 0$: Existence of solution(s)

Local/global solutions

Consider a data in $\mathcal{H}$. We are looking for

1. local existence in a Banach X,
2. uniqueness of the solution in X,
3. The flow $\Phi : u_0 \mapsto u(t)$ is continuous,
4. global existence in a Banach X.

$\longrightarrow$ locally/globally **well-posed** problem in X.

Fixed point method gives $(1, 2, 3)$. Conservation laws gives (4).

An example

$d = 3$, $\alpha = 2$, $\lambda = \pm 1$. We write

$$\Phi u(t) = U(t)u_0 - i \int_0^t U(t-s)(|u|^2 u)(s)\, ds$$

and (NLS) is equivalent to $\Phi(u(t)) = u(t)$.

Aim: apply the fixed point theorem in a Banach space X, by proving

- $\Phi(X) \subset X$,
- $\|\Phi(u(t)) - \Phi(v(t))\|_X < c\|u - v\|_X, \quad c < 1.$

We'll see how Strichartz are used in $(1, 2, 3)$. Believe me :$(8/3, 4)$ is admissible, and so is (trivially) $(\infty, 2)$.

An example

$d = 3$, $\alpha = 2$, $\lambda = \pm 1$. We write

$$\Phi u(t) = U(t)u_0 - i\int_0^t U(t-s)(|u|^2 u)(s)\, ds$$

and (NLS) is equivalent to $\Phi(u(t)) = u(t)$.

Aim: apply the fixed point theorem in a Banach space X, by proving

- $\Phi(X) \subset X$,
- $\|\Phi(u(t)) - \Phi(v(t))\|_X < c\|u - v\|_X, \quad c < 1.$

We'll see how Strichartz are used in $(1, 2, 3)$**.** Believe me :$(8/3, 4)$ is admissible, and so is (trivially) $(\infty, 2)$.

An example

$d = 3$, $\alpha = 2$, $\lambda = \pm 1$. We write

$$\Phi u(t) = U(t)u_0 - i \int_0^t U(t-s)(|u|^2 u)(s)\, ds$$

and (NLS) is equivalent to $\Phi(u(t)) = u(t)$.

Aim: apply the fixed point theorem in a Banach space X, by proving

- $\Phi(X) \subset X$,
- $\|\Phi(u(t)) - \Phi(v(t))\|_X < c\|u - v\|_X, \quad c < 1.$

We'll see how Strichartz are used in $(1, 2, 3)$**.** Believe me :$(8/3, 4)$ is admissible, and so is (trivially) $(\infty, 2)$.

Let $T > 0$. Maybe T small (we'll see later). Strichartz estimates on Φ on $[0, T]$ yield

$$
\begin{aligned}
\|\Phi u(t)\|_{L^\infty L^2 \cap L^{8/3} L^4} &\leq C \left[\|u_0\|_{L^2} + \| \, |u|^2 u\|_{L^{5/3} L^{4/3}}\right] \\
&\leq C \left[\|u_0\|_{L^2} + \|u\|^2_{L^\theta L^4} \|u\|_{L^{8/3} L^4}\right] \\
&\leq C \left[\|u_0\|_{L^2} + \|u\|^2_{L^\theta H^1} \|u\|_{L^{8/3} L^4}\right] \\
&\leq C \left[\|u_0\|_{L^2} + T^{2/\theta} \|u\|^2_{L^\infty H^1} \|u\|_{L^{8/3} L^4}\right],
\end{aligned}
$$

with Hölder, $H^1 \subset L^4$.

The green term makes us handle ∇u too.

$$
i\partial_t \nabla u + \Delta \nabla u = \kappa \nabla(|u|^2 u) \quad ; \quad \nabla u(0, .) = \nabla u_0 \in H^1,
$$

and again

$$
\|\nabla \Phi u(t)\|_{L^\infty L^2 \cap L^{8/3} L^4} \leq C \left[\|\nabla u_0\|_{L^2} + T^{2/\theta} \|u\|^2_{L^\infty H^1} \|\nabla u\|_{L^{8/3} L^4}\right]
$$

i. We sum everything

$$\|\Phi u(t)\|_{X_T} + \|\nabla\Phi u(t)\|_{X_T} \leq C\left[\|u_0\|_{H^1} + T^{2/\theta}\|u\|_{X_T}^3\right].$$

ii. We restrict the analysis to the ball $B = \{u \in X_T \big| \|u\|_{X_T} \leq 2C\|u_0\|_{H^1}\}$.
Why ? Because then

$$2C\|u_0\|_{H^1}\left(1/2 + T^{2/\theta}\|u_0\|_{H^1}^2\right) \leq 2C\|u_0\|_{H^1} \Leftrightarrow T^{2/\theta}\|u_0\|_{H^1}^2 \leq 1/2,$$

is true for somme T_1 small enough.

iii. Same computations to prove
$\|\Phi(u(t)) - \Phi(v(t))\|_B < c\|u-v\|_B, \quad c < 1$, and it gives a small T_2.

iv. We take $T = \min(T_1, T_2)$ and the windmill turns ! A continuity argument extends the solution from B to X_T.

i. We sum everything

$$\|\Phi u(t)\|_{X_T} + \|\nabla \Phi u(t)\|_{X_T} \leq C \left[\|u_0\|_{H^1} + T^{2/\theta} \|u\|_{X_T}^3 \right].$$

ii. We restrict the analysis to the ball $B = \left\{ u \in X_T \middle| \|u\|_{X_T} \leq 2C\|u_0\|_{H^1} \right\}$.
Why ? Because then

$$2C\|u_0\|_{H^1} \left(1/2 + T^{2/\theta} \|u_0\|_{H^1}^2 \right) \leq 2C\|u_0\|_{H^1} \Leftrightarrow T^{2/\theta} \|u_0\|_{H^1}^2 \leq 1/2,$$

is true for somme T_1 small enough.

iii. Same computations to prove
$\|\Phi\left(u(t)\right) - \Phi\left(v(t)\right)\|_B < c\|u - v\|_B, \quad c < 1$, and it gives a small T_2.

iv. We take $T = \min(T_1, T_2)$ and the windmill turns ! A continuity argument extends the solution from B to X_T.

i. We sum everything

$$\|\Phi u(t)\|_{X_T} + \|\nabla \Phi u(t)\|_{X_T} \leq C \left[\|u_0\|_{H^1} + T^{2/\theta} \|u\|_{X_T}^3 \right].$$

ii. We restrict the analysis to the ball $B = \left\{ u \in X_T \middle| \|u\|_{X_T} \leq 2C\|u_0\|_{H^1} \right\}$.
Why ? Because then

$$2C\|u_0\|_{H^1} \left(1/2 + T^{2/\theta} \|u_0\|_{H^1}^2 \right) \leq 2C\|u_0\|_{H^1} \Leftrightarrow T^{2/\theta} \|u_0\|_{H^1}^2 \leq 1/2,$$

is true for somme T_1 small enough.

iii. Same computations to prove
$\|\Phi(u(t)) - \Phi(v(t))\|_B < c\|u - v\|_B, \quad c < 1$, and it gives a small T_2.

iv. We take $T = \min(T_1, T_2)$ and the windmill turns ! A continuity argument extends the solution from B to X_T.

i. We sum everything

$$\|\Phi u(t)\|_{X_T} + \|\nabla \Phi u(t)\|_{X_T} \leq C\left[\|u_0\|_{H^1} + T^{2/\theta}\|u\|_{X_T}^3\right].$$

ii. We restrict the analysis to the ball $B = \left\{u \in X_T \middle| \|u\|_{X_T} \leq 2C\|u_0\|_{H^1}\right\}$.
Why ? Because then

$$2C\|u_0\|_{H^1}\left(1/2 + T^{2/\theta}\|u_0\|_{H^1}^2\right) \leq 2C\|u_0\|_{H^1} \Leftrightarrow T^{2/\theta}\|u_0\|_{H^1}^2 \leq 1/2,$$

is true for somme T_1 small enough.

iii. Same computations to prove
$\|\Phi\left(u(t)\right) - \Phi\left(v(t)\right)\|_B < c\|u - v\|_B, \quad c < 1$, and it gives a small T_2.

iv. We take $T = \min(T_1, T_2)$ and the windmill turns ! A continuity argument extends the solution from B to X_T.

i. We sum everything

$$\|\Phi u(t)\|_{X_T} + \|\nabla \Phi u(t)\|_{X_T} \leq C\left[\|u_0\|_{H^1} + T^{2/\theta}\|u\|_{X_T}^3\right].$$

ii. We restrict the analysis to the ball $B = \left\{u \in X_T \middle| \|u\|_{X_T} \leq 2C\|u_0\|_{H^1}\right\}$.
Why ? Because then

$$2C\|u_0\|_{H^1}\left(1/2 + T^{2/\theta}\|u_0\|_{H^1}^2\right) \leq 2C\|u_0\|_{H^1} \Leftrightarrow T^{2/\theta}\|u_0\|_{H^1}^2 \leq 1/2,$$

is true for somme T_1 small enough.

iii. Same computations to prove
$\|\Phi(u(t)) - \Phi(v(t))\|_B < c\|u-v\|_B, \quad c < 1$, and it gives a small T_2.

iv. We take $T = \min(T_1, T_2)$ and the windmill turns ! A continuity argument extends the solution from B to X_T.

$\rightarrow$ We have **local existence of a unique solution** in X_T.

The solution is either global ($T = \infty$), or it blows up in finite time: there is a maximal time $T_{max} < +\infty$ s.t. $\|u(t)\|_{H^1}$ is not bounded on $[0, T[$.

NB : if the solution blows up at finite time T_{max} in H^1, knowing that $\|u(t)\|_{L^2}$ is conserved, $\|\nabla u(t)\|_{L^2}$ is the quantity that blows up.

A simple example: $\lambda = +1$**.** Energy conservation gives

$$\|\nabla u(t)\|_{L^2} \leq \|\nabla u(t)\|_{L^2} + \frac{1}{4}\|u\|_{L^4}^4 = 2E(t) = 2E_0 < +\infty.$$

So $T = +\infty$.

$\rightarrow$ We have **local existence of a unique solution** in X_T.

The solution is either global ($T = \infty$), or it blows up in finite time: there is a maximal time $T_{max} < +\infty$ s.t. $\|u(t)\|_{H^1}$ is not bounded on $[0, T[$.

NB : if the solution blows up at finite time T_{max} in H^1, knowing that $\|u(t)\|_{L^2}$ is conserved, $\|\nabla u(t)\|_{L^2}$ is the quantity that blows up.

A simple example: $\lambda = +1$. Energy conservation gives

$$\|\nabla u(t)\|_{L^2} \leq \|\nabla u(t)\|_{L^2} + \frac{1}{4}\|u\|_{L^4}^4 = 2E(t) = 2E_0 < +\infty.$$

So $T = +\infty$.

$\rightarrow$ We have **local existence of a unique solution** in X_T.

The solution is either global ($T = \infty$), or it blows up in finite time: there is a maximal time $T_{max} < +\infty$ s.t. $\|u(t)\|_{H^1}$ is not bounded on $[0, T[$.

NB : if the solution blows up at finite time T_{max} in H^1, knowing that $\|u(t)\|_{L^2}$ is conserved, $\|\nabla u(t)\|_{L^2}$ is the quantity that blows up.

A simple example: $\lambda = +1$. Energy conservation gives

$$\|\nabla u(t)\|_{L^2} \leq \|\nabla u(t)\|_{L^2} + \frac{1}{4}\|u\|_{L^4}^4 = 2E(t) = 2E_0 < +\infty.$$

So $T = +\infty$.

$\rightarrow$ We have **local existence of a unique solution** in X_T.

The solution is either global ($T = \infty$), or it blows up in finite time: there is a maximal time $T_{max} < +\infty$ s.t. $\|u(t)\|_{H^1}$ is not bounded on $[0, T[$.

NB : if the solution blows up at finite time T_{max} in H^1, knowing that $\|u(t)\|_{L^2}$ is conserved, $\|\nabla u(t)\|_{L^2}$ is the quantity that blows up.

A simple example: $\lambda = +1$. Energy conservation gives

$$\|\nabla u(t)\|_{L^2} \leq \|\nabla u(t)\|_{L^2} + \frac{1}{4}\|u\|_{L^4}^4 = 2E(t) = 2E_0 < +\infty.$$

So $T = +\infty$.

VI. Qualitative study of long-time behaviour

$$(**) \lim_{|t|\to\pm\infty} \|u(t) - U(t)u_\pm\|_{H^1} = 0.$$

Definitions

Every u_0 in H^1 gives a unique global solution u to (NLS), with

$$u, \nabla u \in C(\mathbf{R}, L^2) \cap L^p(\mathbf{R}, L^q), \quad \text{for some } (p, q).$$

Moreover

Asymptotic completeness: For all $u_0 \in H^1$, one can produce a $u_\pm \in H^1$ s.t. $(**)$ is satisfied.

Existence of the wave operator: For all $u_\pm \in H^1$, one can associate a solution $u(t)$ to (NLS), satisfying $(**)$.

Why are global in time Strichartz estimates crucial here ?

$$(**) \lim_{|t|\to\pm\infty} \|u(t) - U(t)u_\pm\|_{H^1} = 0$$

is equivalent to

$$(**) \lim_{|t|\to\pm\infty} \|U(-t)u(t) - u_\pm\|_{H^1} = 0.$$

So $U(-t)u(t)$ has to converge in H^1.

Duhamel $\rightarrow$

$$u(t) = U(t)u_0 - i\lambda \int_0^t U(t-s)|u|^\alpha u(s)\, ds$$

$$U(-t)u(t) = u_0 - i\lambda \int_0^t U(-s)|u|^\alpha u(s)\, ds.$$

H^1–scattering if and only if $\lambda \int_0^\infty U(-s)|u|^\alpha u(s)\, ds$ converges in H^1.

Why are global in time Strichartz estimates crucial here ?

$$(**) \lim_{|t|\to\pm\infty} \|u(t) - U(t)u_\pm\|_{H^1} = 0$$

is equivalent to

$$(**) \lim_{|t|\to\pm\infty} \|U(-t)u(t) - u_\pm\|_{H^1} = 0.$$

So $U(-t)u(t)$ has to converge in H^1.

Duhamel $\rightarrow$

$$u(t) = U(t)u_0 - i\lambda \int_0^t U(t-s)|u|^\alpha u(s)\, ds$$

$$U(-t)u(t) = u_0 - i\lambda \int_0^t U(-s)|u|^\alpha u(s)\, ds.$$

H^1-scattering if and only if $\lambda \int_0^\infty U(-s)|u|^\alpha u(s)\, ds$ converges in H^1.

VII. Small initial data.

"There is a $E_0 > 0$ s.t. for all $u_0 \in H^1$, $\|u_0\|_{H^1} \leq E_0$, $(**)$ holds."

Idea: If the data is small enough (E_0), then GWP comes easily and it gives $\|u\|_{L^p L^q} < +\infty$.

Then prove that $\|U(-t)u(t) - U(-\tau)u(\tau)\|_{H^1}$ tends to zero as t, τ tend to infinity ("Cauchy sequence").

Thanks to Strichartz (and other tools),

$$\|U(-t)u(t) - U(-\tau)u(\tau)\|_{H^1} \leq C \, \|u\|_X^{\theta} \cdot \|u\|_{L^p_{[t,\tau]} L^q}^{1-\theta}.$$

VII. Small initial data.

"There is a $E_0 > 0$ s.t. for all $u_0 \in H^1$, $\|u_0\|_{H^1} \leq E_0$, $(**)$ holds."

Idea: If the data is small enough (E_0), then GWP comes easily and it gives $\|u\|_{L^p L^q} < +\infty$.

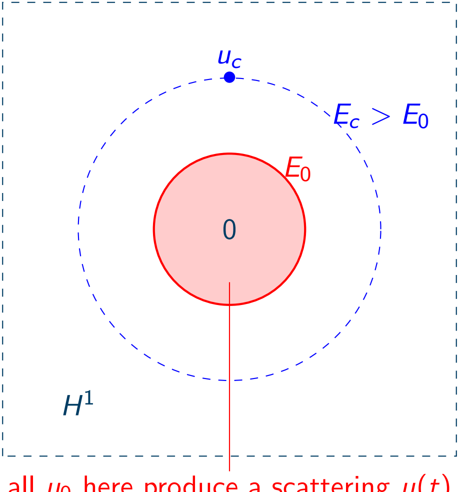
Then prove that $\|U(-t)u(t) - U(-\tau)u(\tau)\|_{H^1}$ tends to zero as t, τ tend to infinity ("Cauchy sequence").

Thanks to Strichartz (and other tools),

$$\|U(-t)u(t) - U(-\tau)u(\tau)\|_{H^1} \leq C \, \|u\|_X^{\theta} \, \cdot \, \|u\|_{L^p_{[t,\tau]} L^q}^{1-\theta}.$$

VIII. One famous machinery for arbitrarily large data.

Example in an easy case (for example defocusing subcritical, for which we expect scattering on whole H^1). Performed by contradiction.



The Kenig-Merle machinery

i. Small data scattering : "There is a $E_0 > 0$ s.t. for all $u_0 \in H^1$, $\|u_0\|_{H^1} \leq E_0$, $(**)$ holds."

ii. We suppose there is a level $E_c > E_0$ s.t. there is no scattering : "there is at least ONE $u_0 \in H^1$, $\|u_0\|_{H^1} = E_0$" and the associated solution $u_c(t)$ does not scatter.
$\rightarrow$ so s.t. $\|u_c\|_{L^p L^q}$ is not bounded.

iii. Construction of this u_c (technical part !) and study of its properties.

iv. The properties cannot be fulfilled unless $u_c = 0$ which is impossible because of *i*.

The Kenig-Merle machinery

i. Small data scattering : "There is a $E_0 > 0$ s.t. for all $u_0 \in H^1$, $\|u_0\|_{H^1} \leq E_0$, $(**)$ holds."

ii. We suppose there is a level $E_c > E_0$ s.t. there is no scattering : "there is at least ONE $u_0 \in H^1$, $\|u_0\|_{H^1} = E_0$" and the associated solution $u_c(t)$ does not scatter.
$\rightarrow$ so s.t. $\|u_c\|_{L^p L^q}$ is not bounded.

iii. Construction of this u_c (technical part !) and study of its properties.

iv. The properties cannot be fulfilled unless $u_c = 0$ which is impossible because of *i*.

The Kenig-Merle machinery

i. Small data scattering : "There is a $E_0 > 0$ s.t. for all $u_0 \in H^1$, $\|u_0\|_{H^1} \leq E_0$, $(**)$ holds."

ii. We suppose there is a level $E_c > E_0$ s.t. there is no scattering : "there is at least ONE $u_0 \in H^1$, $\|u_0\|_{H^1} = E_0$" and the associated solution $u_c(t)$ does not scatter.
$\rightarrow$ so s.t. $\|u_c\|_{L^p L^q}$ is not bounded.

iii. Construction of this u_c (technical part !) and study of its properties.

iv. The properties cannot be fulfilled unless $u_c = 0$ which is impossible because of *i*.

The Kenig-Merle machinery

i. Small data scattering : "There is a $E_0 > 0$ s.t. for all $u_0 \in H^1$, $\|u_0\|_{H^1} \leq E_0$, $(**)$ holds."

ii. We suppose there is a level $E_c > E_0$ s.t. there is no scattering : "there is at least ONE $u_0 \in H^1$, $\|u_0\|_{H^1} = E_0$" and the associated solution $u_c(t)$ does not scatter.
$\rightarrow$ so s.t. $\|u_c\|_{L^p L^q}$ is not bounded.

iii. Construction of this u_c (technical part !) and study of its properties.

iv. The properties cannot be fulfilled unless $u_c = 0$ which is impossible because of *i*.

The Kenig-Merle machinery

i. Small data scattering : "There is a $E_0 > 0$ s.t. for all $u_0 \in H^1$, $\|u_0\|_{H^1} \leq E_0$, $(**)$ holds."

ii. We suppose there is a level $E_c > E_0$ s.t. there is no scattering : "there is at least ONE $u_0 \in H^1$, $\|u_0\|_{H^1} = E_0$" and the associated solution $u_c(t)$ does not scatter.
$\rightarrow$ so s.t. $\|u_c\|_{L^p L^q}$ is not bounded.

iii. Construction of this u_c (technical part !) and study of its properties.

iv. The properties cannot be fulfilled unless $u_c = 0$ which is impossible because of *i*.

Gamebook/"talk dont vous êtes le héros":
* if you want to have details about *iii*., please ask (but will be on blackboard).
* if you want to stop, blink very fast.