Model completion of scaled lattices and co-Heyting algebras of p-adic semi-algebraic sets*

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Abstract

Let p be prime number, K be a p-adically closed field, $X \subseteq K^m$ a semi-algebraic set defined over K and $L(X)$ the lattice of semi-algebraic subsets of X which are closed in X. We prove that the complete theory of $L(X)$ eliminates quantifiers in a certain language $\mathcal{L}_{\text{ASC}}$, the $\mathcal{L}_{\text{ASC}}$-structure on $L(X)$ being an extension by definition of the lattice structure. Moreover it is decidable, contrary to what happens over a real closed field for $m > 1$. We classify these $\mathcal{L}_{\text{ASC}}$-structures up to elementary equivalence, and get in particular that the complete theory of $L(K^m)$ only depends on m, not on K nor even on p. As an application we obtain a classification of semi-algebraic sets over countable p-adically closed fields up to so-called "pre-algebraic" homeomorphisms.

1 Introduction

This paper explores the model-theory of various classes of lattices coming from algebraic geometry, real geometry or p-adic geometry, with special emphasis on the p-adic case. We obtain model-completion and decidability results for some of them. Before entering in technical details let us present the main motivations for this, coming from geometry and model-theory of course, but also from proof theory and non-classical logics.

Given an expansion of a topological field K and a definable[1] set $X \subseteq K^n$, we consider the lattice $L(X)$ of all definable subsets of X which are closed in X, and the ring $\mathcal{C}(X)$ of all continuous definable functions from X to K. These rings are central objects nowadays in functional analysis, topology and geometry. To name an example, they are rings of sections for the sheaf of continuous (say, real valued) functions on a topological space and as such play the algebraic part in the study of topological (Hausdorff) spaces. In most cases $L(X)$ is interpretable in $\mathcal{C}(X)$, and the prime filter spectrum of $L(X)$ is homeomorphic to the prime ideal spectrum of $\mathcal{C}(X)$. Thus $L(X)$ is a first-order structure interpretable in $\mathcal{C}(X)$, which captures all the topological (hence second-order) information on the spectrum of the ring $\mathcal{C}(X)$. For the real field $\mathbf{R}$ for example, it is known since [10] that $L(\mathbf{R}^n)$ is undecidable for every $n \geq 2$, hence can be held liable for the undecidability of $\mathcal{C}(\mathbf{R}^n)$. On the contrary $L(\mathbf{R})$ is decidable, and so is the lattice of all closed subsets of the real line [13]. Recently this has been strengthen and widely generalised in [15]. However these undecidability results for $L(X)$ strongly depend on the existence of irreducible or connected components, hence do not apply to the p-adic case. But even in that case it is proved in [7] that $\mathcal{C}(\mathbf{Q}_p^n)$ is undecidable, this time for every $n \geq 1$. On the contrary, our main result implies that for every $n \geq 1$:

- $L(\mathbf{Q}_p^n)$ is decidable, and;
- The theory of $L(\mathbf{Q}_p^n)$ eliminates the quantifier in a natural expansion by definition of the lattice language.

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[1] We assume the reader to be familiar with basic notions from model theory, in particular definable sets and functions. In simplest cases "definable" boils down to "semi-algebraic" over the field $\mathbf{R}$ of real numbers, or the field $\mathbf{Q}_p$ of p-adic numbers.

In another direction, the model-theory of these geometric lattices $L(X)$ is tightly connected to the existence of uniform interpolants for propositional calculus in certain intermediate[2] and modal logics. Indeed, thanks to the one-to-one correspondence between intermediate logics and varieties of Heyting algebras, the existence of uniform interpolants for a logic $\mathcal{L}$ can be rephrased, *mutatis mutandis*, as the existence of a model-completion for the theory of the corresponding variety $\mathcal{V}(\mathcal{L})$ (see [9]). As lattices of closed sets, all our lattices $L(X)$ are co-Heyting algebras, that is Heyting algebras with the order reversed. Moreover, their structure is mostly determined by the *geometry* of X. This geometric intuition coming from X was essential to our model-completion results, with natural axiomatisations, for certain theories of (expansions of) co-Heyting algebras (theorems B and C below). See also [1], [6] for related results.

Now we are going to present our results in more detail. They are based on a careful study of certain expansions of lattices, all inspired by the geometric examples of lattices of closed sets over an o-minimal, P-minimal or C-minimal expansion of a field K. More general structures will be considered in the appendix section 11, where precise definitions are given of what we call here "tame" topological structures. The point is that there a good dimension theory for definable sets over such structures.

Example 1.1 Let $\mathcal{K} = (K, \dots)$ be a tame topological structure. For every definable sets $A, B \subseteq X \subseteq K^m$ let $A - B = \overline{A \setminus B} \cap X$ where the overline stands for the topological closure. For every $a \in A$, the **local dimension** of A at a is the maximum of the dimensions of the definable neighborhoods of a in A, and A is called **pure dimensional** if it has the same local dimension at every point. For every non-negative integer i let

$$\mathrm{C}^i(A) = \overline{\{a \in A \mid \dim(A, a) = i\}} \cap A.$$

This is a definable subset of A, closed in A, called the ***i*-pure component** of A. We call $\mathrm{L}_{\mathrm{def}}(X)$ the lattice of all the definable subsets of X which are closed in X, enriched with the above functions "$-$" and C^i for every i. This is a typical example (Proposition 11.6) of what we are going to call a d-scaled lattice.

Let $\mathcal{L}_{\mathrm{lat}} = \{\mathbf{0}, \mathbf{1}, \vee, \wedge\}$ be the language of lattices, and $\mathcal{L}_{\mathrm{SC}} = \mathcal{L}_{\mathrm{lat}} \cup \{-, (C^i)_{i \geq 0}\}$ be its expansion by the above function symbols. Finally let $\mathrm{SC}_{\mathrm{def}}(\mathcal{K}, d)$ be the class of the $\mathcal{L}_{\mathrm{SC}}$-structures $\mathrm{L}_{\mathrm{def}}(X)$ of Example 1.1, for all the sets X of dimension at most d definable over $\mathcal{K}$. A similar construction can be done over a pure field K, with the Zariski topology on K^m. We let $\mathrm{SC}_{\mathrm{Zar}}(K, d)$ denote the corresponding class of $\mathcal{L}_{\mathrm{SC}}$-structures. Surprisingly enough, we prove that in most cases the universal theory of $\mathrm{SC}_{\mathrm{def}}(\mathcal{K}, d)$ (resp. $\mathrm{SC}_{\mathrm{Zar}}(K, d)$) does not depend on $\mathcal{K}$ (resp. K)!

Theorem A *Given any non-negative integer d, the universal theories of* $\mathrm{SC}_{\mathrm{def}}(\mathcal{K}, d)$ *(resp.* $\mathrm{SC}_{\mathrm{Zar}}(K, d)$*) in the language* $\mathcal{L}_{\mathrm{SC}}$ *are* the same *for every tame expansion $\mathcal{K}$ of a topological field K (resp. for every infinite field K).*

In order to prove this we give in Section 2 an explicit list of universal axioms for a theory T_d in $\mathcal{L}_{\mathrm{SC}}$, the models of which we call d-subscaled lattices. All the examples given above are d-scaled lattices, a natural subclass of d-subscaled lattices (the class of d-scaled lattices is elementary but not universal). After some technical preliminaries in Section 3 we prove in Section 4 that every finitely generated d-subscaled lattice is finite. Combining this with a linear representation for finite d-subscaled lattices (Proposition 5.3) and with the model-theoretic compactness theorem, we then prove in Section 5 that, whatever is K or $\mathcal{K}$ in Example 1.1, the theory of d-subscaled lattices is exactly the universal theory of $\mathrm{SC}_{\mathrm{def}}(\mathcal{K}, d)$ and of $\mathrm{SC}_{\mathrm{Zar}}(K, d)$ (Theorem 5.3).

A detailed study of the minimal finitely generated extensions of finite d-subscaled lattices, achieved in Section 6, leads us in Section 7 to the next result (Theorem 7.3 and Corollary 7.5).

Theorem B *For every non-negative integer d, the theory of d-subscaled lattices admits a model-completion $\bar{T}_d$ which is finitely axiomatizable and $\aleph_0$-categorical. Moreover, $\bar{T}_d$ has finitely many prime models, hence it is decidable as well as all its completions.*

[2] An intermediate logic is logic which stands between classical and intuitionist logic.

The axiomatization of $\bar{T}_d$ given in Section 7 consists of a pair of axioms expressing a “Catenarity” and a “Splitting” property which both have a natural topological and geometric meaning. In particular the Splitting Property expresses a very strong form of disconnectedness, which implies that the models of $\bar{T}_d$ are atomless.

Remark 1.2 Since 0-subscaled lattices are exactly non-trivial boolean algebras, the above model-completion result for subscaled lattices is a generalisation to arbitrary finite dimension d of the classical theorem on the model-completion of boolean algebras.

We develop in Sections 8 and 9 a variant of this quantifier elimination result in a language $\mathcal{L}_{\rm ASC} = \mathcal{L}_{\rm SC} \cup \{{\rm At}_k\}_{k\geq 1}$, where each ${\rm At}_k$ is a unary predicate symbol, to be interpreted as the set of elements which are the join of exactly k atoms. The model-completion $\bar{T}_d^{At}$ that we obtain is axiomatized by the Catenarity Property and a small restriction of the Splitting Property which preserves the atoms. This theory $\bar{T}_d^{At}$ has $\aleph_0$ prime models which can easily be classified in terms of the prime models of $\bar{T}_d$, from which it follows that it is decidable as well as all its completions (Theorem 9.4).

In the initial version of this paper [3] we conjectured that ${\rm L}_{\rm def}(\mathbf{Q}_p^d)$ might be a natural model of $\bar{T}_d^{At}$. This intuition proved to be crucial in the proof of the triangulation of semi-algebraic sets over a p-adically closed field [4]. Conversely it follows from this triangulation that ${\rm L}_{\rm def}(X)$ is indeed a model of $\bar{T}_d^{At}$, for every semi-algebraic[3] set $X \subseteq K^m$ of dimension $\leq d$, from which we derive the following result in the last section (Theorem 10.2).

Theorem C *Let K be a p-adically closed field, $X \subseteq K^m$ a semi-algebraic set. Then the complete theory of $L(X)$ is decidable, and eliminates quantifier in $\mathcal{L}_{\rm ASC}$.*

The prime $\mathcal{L}_{\rm ASC}$-substructure of ${\rm L}_{\rm def}(X)$ (which is generated by the empty set) is finite. By Theorem C it determines the complete theory of ${\rm L}_{\rm def}(X)$. We expect this invariant to play also a decisive role in the classification of semi-algebraic sets over p-adically closed fields up to semi-algebraic homeomorphisms. Such a classification is far from being achieved, but a weaker classification, up to “pre-algebraic” homeomorphisms over countable p-adically closed fields, is done here by means of this invariant (Theorem 10.5).

2 Notation and definitions

$\mathbf{N}$ denotes the set of non-negative integers, and $\mathbf{N}^* = \mathbf{N} \setminus \{0\}$. If $\mathcal{N}$ is an unbounded non-empty subset of $\mathbf{N}$ (resp. the empty subset) we set $\max \mathcal{N} = +\infty$ (resp. $\max \mathcal{N} = -\infty$). The symbols $\subseteq$ and $\subset$ denote respectively inclusion and strict inclusion. The logical connectives ‘or’, ‘and’ and their iterated forms will be denoted by $\bigvee$, $\bigwedge$, $\bigvee\!\!\!\bigvee$ and $\bigwedge\!\!\!\bigwedge$ respectively.

2.1 Lattices and dimension

In this paper a **lattice** is a partially ordered set in which every finite subset has a greatest lower element and a least greater bound. This applies in particular to the empty subset, hence our lattices must have a least and a greatest bound. We let $\mathcal{L}_{\rm lat} = \{\mathbf{0}, \mathbf{1}, \vee, \wedge\}$ is the language of lattices, each symbol having its obvious meaning. As usual $b \leq a$ is an abbreviation for $a \vee b = a$ and similarly for $b < a$, $b \geq a$ and $b > a$. Iterated $\vee$ and $\wedge$ operations are denoted by $\bigvee\!\!\!\bigvee_{i\in I} a_i$ and $\bigwedge\!\!\!\bigwedge_{i\in I} a_i$ respectively. If the index set I is empty then $\bigvee\!\!\!\bigvee_{i\in I} a_i = \mathbf{0}$ and $\bigwedge\!\!\!\bigwedge_{i\in I} a_i = \mathbf{1}$. Given a subset S of a lattice L, the **upper semi-lattice** generated in L by S is the set of finite joins of elements of S.

The **spectrum** of a lattice L is the set ${\rm Spec}(L)$ of all prime filters of L, endowed with the so-called Zariski topology, defined by taking as a basis of closed sets all the sets

$$P(a) = \{\mathbf{p} \in {\rm Spec}(L) \,|\, a \in \mathbf{p}\}$$

[3] A generalization to definable sets over more general P-minimal fields, if possible, has still to be done.

for a ranging over L. Stone's duality asserts that if L is distributive (which is always the case in this paper) the map $a \mapsto P(a)$ is an isomorphism between L and the lattice of closed subsets S of $\mathrm{Spec}(L)$ such that the complement of S in $\mathrm{Spec}(L)$ is compact.

We call a lattice **noetherian** if it is isomorphic to the lattice of closed sets of a noetherian topological space. By Stone's duality a lattice L is noetherian if and only if its spectrum is a noetherian topological space. In such a lattice every filter is principal and every element a can be written uniquely as the join of its (finitely many) **$\vee$-irreducible components**, which are the maximal elements in the set of $\vee$-irreducible[4] elements of L smaller than a. We denote by $\mathcal{I}(L)$ the set of all $\vee$-irreducible elements of L.

We define the **dimension of an element a in a lattice** L, denoted $\dim_L a$, as the least upper bound (in $\mathbf{N} \cup \{-\infty, +\infty\}$) of the set of non-negative integers n such that

$$\exists \mathbf{p}_0 \subset \cdots \subset \mathbf{p}_n \in P(a).$$

This is nothing but the ordinary topological or Krull dimension (defined by chains of irreducible closed subsets) of the spectral space $P(a)$. By construction $\dim_L \mathbf{0} = -\infty$ and $\dim_L a \vee b = \max(\dim_L a, \dim_L b)$. The subscript L is necessary since $\dim_L a$ is not preserved by $\mathcal{L}_{\mathrm{lat}}$-embeddings, but we will omit it whenever the ambient lattice is clear from the context. We let the **dimension of** L be the dimension of $\mathbf{1}_L$ in L.

The following definable relation will give us a first-order definition of the dimension of the elements of L inside L, when L is a co-Heyting algebra (see Fact 2.4 below):

$$b \ll a \iff \forall c\,(c < a \Rightarrow b \vee c < a)$$

This is a strict order on $L \setminus \{\mathbf{0}\}$ (but not on L because $\mathbf{0} \ll a$ for every a, including $a = \mathbf{0}$).

2.2 Co-Heyting algebras

We let $\mathcal{L}_{\mathrm{TC}} = \mathcal{L}_{\mathrm{lat}} \cup \{-\}$ with '$-$' a binary function symbol. A $\mathcal{L}_{\mathrm{TC}}$-structure L is a **co-Heyting algebra** if its $\mathcal{L}_{\mathrm{lat}}$-reduct is a lattice and if every element b has in L a **topological complement** relatively to every element, denoted $a - b$. By definition $a - b$ is the least element c such that $a \leq b \vee c$. Equivalently $P(a - b)$ is the topological closure of the relative complement $P(a) \setminus P(b)$, hence the notation $a - b$. Reversing the order of a co-Heyting algebra L gives a Heyting algebra L^*, with $b \to a$ in L^* corresponding to $a - b$ in L, and every co-Heyting algebra is of this form. From the theory of Heyting algebras (see for example [12]) we know that every co-Heyting algebra is distributive and that the class of all co-Heyting algebras is a variety (in the sense of universal algebra). Observe that in co-Heyting algebras the $\ll$ relation is quantifier-free definable since

$$b \ll a \iff b \leq a - b.$$

So it will be preserved by $\mathcal{L}_{\mathrm{TC}}$-embeddings. On the other hand, dimension will not be preserved in general by $\mathcal{L}_{\mathrm{TC}}$-embeddings.

We will use the following rules, the proof of which are elementary exercises (using either Stone's duality or corresponding properties of Heyting algebras).

TC$_1$: $a = (a \wedge b) \vee (a - b)$.
In particular if a is $\vee$-irreducible then $b < a \implies b \ll a$.

TC$_2$: $(a_1 \vee a_2) - b = (a_1 - b) \vee (a_2 - b)$.

TC$_3$: $(a - b) - b = a - b$.
In particular $(a - b) \wedge b \ll a - b \leq a$.

TC$_4$: More generally $a - (b_1 \vee b_2) = (a - b_1) - b_2$.
So if $a - b_1 = a$ then $a - (b_1 \vee b_2) = a - b_2$.

[4] An element x of a lattice L is $\vee$-irreducible if it is *non-zero* and if $a \vee b = x$ implies $a = x$ or $b = x$.

Fact 2.3 (Theorem 3.8 in [5]) *For every element $a \neq \mathbf{0}$ in a co-Heyting algebra L, $\dim_L a$ is the least upper bound of the set of positive integers n such that there exists $a_0, \dots, a_n \in L$ such that*

$$\mathbf{0} \neq a_0 \ll a_1 \ll \cdots \ll a_n \leq a.$$

In all the geometric examples given in the introduction, a set A is said to be pure dimensional if and only if $\dim U = \dim A$ for every non-empty definable subset U of A which is open in A. This motivates the next definition: given an integer k we say that an element a of a distributive lattice L is **k-pure in** L if and only if

$$\forall b \in L\ (a - b \neq \mathbf{0} \Rightarrow \dim_L a - b = k).$$

Then either $a = \mathbf{0}$ or $\dim_L a = k$. In the latter case we say that a **has pure dimension** k **in** L. If L is any of the lattices $\mathrm{L_{def}}(X)$ or $\mathrm{L_{Zar}}(X)$ in Example 1.1, for every $A \in L$ we will show in Section 11 that $\dim_L A$ is exactly the usual (geometric) dimension of A. It follows that A is pure dimensional in L if and only if it so in the geometric sense.

There is a well-established duality between (co-)Heyting algebras and so-called Esakia spaces with p-morphisms, from which we will pick up Fact 2.4 below. We first need a notation. Given an element x in a poset I and a subset X of I let

$$x^{\downarrow} = \{y \in I \mid y \leq x\} \qquad\qquad X{\downarrow} = \bigcup_{x \in I} x^{\downarrow}.$$

The dual notation $x^{\uparrow}$ and $X{\uparrow}$ is defined accordingly. The family $\mathcal{D}^{\downarrow}(I)$ of decreasing subsets of I (that is the sets $X \subseteq I$ such that $X{\downarrow} = X$) are the closed sets of a topology on I, hence a co-Heyting algebra with respect to the following operations.

$$X \vee Y = X \cup Y \qquad\qquad X \wedge Y = X \cap Y \qquad\qquad X - Y = (X \setminus Y){\downarrow}$$

The $\vee$-irreducible elements of $\mathcal{D}^{\downarrow}(I)$ are precisely the sets $x^{\downarrow}$ for $x \in I$.

Fact 2.4 *Let L be a finite co-Heyting algebra and $\mathcal{I}$ an ordered set. Assume that there is a surjective increasing map $\pi : \mathcal{I} \to \mathcal{I}(L)$ such that $\pi(x^{\uparrow}) \subseteq \pi(x){\uparrow}$ for every $x \in \mathcal{I}(L)$. Then there exists an $\mathcal{L}_{\mathrm{TC}}$-embedding φ of L into $\mathcal{D}^{\downarrow}(\mathcal{I})$ such that*[5] *$\pi(\varphi(a)) = a^{\downarrow} \cap \mathcal{I}(L)$ for every $a \in L$.*

2.5 (Sub)scaled lattices.

Recall that $\mathcal{L}_{\mathrm{SC}} = \mathcal{L}_{\mathrm{lat}} \cup \{-, \mathrm{C}^i\}_{i \in \mathbf{N}} = \mathcal{L}_{\mathrm{TC}} \cup \{\mathrm{C}^i\}_{i \in \mathbf{N}}$ where $\{\mathrm{C}^i\}_{i \in \mathbf{N}}$ is a family of new unary function symbols. With the examples of the introduction in mind, we define the **sc-dimension** of a non-zero element a of an $\mathcal{L}_{\mathrm{SC}}$-structure L as

$$\text{sc-dim}\, a = \min\Big\{k \in \mathbf{N} \mid a = \bigvee_{0 \leq i \leq k} \mathrm{C}^i(a)\Big\}.$$

Of course this is defined only if $\text{sc-dim}\, a = \bigvee_{0 \leq i \leq k} \mathrm{C}^i(a)$, for some k. If it is not defined we let $\text{sc-dim}\, a = +\infty$, and by convention $\text{sc-dim}\, \mathbf{0} = -\infty$. The **sc-dimension of** L, denoted $\text{sc-dim}(L)$, is the sc-dimension of $\mathbf{1}_L$. In general the dimension of an element in a co-Heyting algebra is not preserved by $\mathcal{L}_{\mathrm{TC}}$-embeddings. On the contrary the sc-dimension of an element is obviously preserved by $\mathcal{L}_{\mathrm{SC}}$-embeddings, and this is the “raison d'être” of this structure.

A **d-subscaled lattice** is an $\mathcal{L}_{\mathrm{SC}}$-structure whose $\mathcal{L}_{\mathrm{TC}}$-reduct is a co-Heyting algebra and which satisfies the following list of axioms:

$\mathbf{SS_1^d}$: $\bigvee_{0 \leq i \leq d} \mathrm{C}^i(a) = a$ and $\forall i > d,\ C^i(a) = \mathbf{0}$.

$\mathbf{SS_2^d}$: $\forall I \subseteq \{0, \dots, d\}$, $\forall k$:

$$\mathrm{C}^k\Big(\bigvee_{i \in I} \mathrm{C}^i(a)\Big) = \begin{cases} \mathbf{0} & \text{if } k \notin I \\ \mathrm{C}^k(a) & \text{if } k \in I \end{cases}$$

[5] Note that the composition $\pi \circ \varphi$ is not defined. In this proposition $\varphi(a)$ is a decreasing subset of $\mathcal{I}$ and $\pi(\varphi(a)) = \{\pi(\xi) \mid \xi \in \varphi(a)\}$.

SS$_3$: $\forall k \geq \max(\text{sc-dim}(a), \text{sc-dim}(b)), \quad \mathrm{C}^k(a \vee b) = \mathrm{C}^k(a) \vee \mathrm{C}^k(b)$

SS$_4$: $\forall i \neq j, \quad \text{sc-dim}\left(\mathrm{C}^i(a) \wedge \mathrm{C}^j(a)\right) < \min(i, j)$

SS$_5$: $\forall k \geq \text{sc-dim}(b), \quad \mathrm{C}^k(a) - b = \mathrm{C}^k(a) - \mathrm{C}^k(b)$.

SS$_6$: If $b \ll a$ then $\text{sc-dim}\, b < \text{sc-dim}\, a$.

It is a ***d*-scaled lattice** if it satisfies in addition the following property:

SC$_0$: $\text{sc-dim}\, a = \dim a$

All the geometric $\mathcal{L}_{\mathrm{SC}}$-structures in $\mathrm{SC}_{\mathrm{def}}(\mathcal{K}, d)$ or $\mathrm{SC}_{\mathrm{Zar}}(K, d)$ (defined after Example 1.1) are d-scaled lattices (see Proposition 11.6). However SC_0 does not follow from the other axioms as the following example shows.

Example 2.6 Let L be an arbitrary noetherian lattice, and $D\colon \mathcal{I}(L) \to \{0, \ldots, d\}$ be a strictly increasing map. For every $a, b \in L$, if $\mathcal{C}(a)$ denotes the set of all $\vee$-irreducible components of a, let

$$a - b = \bigvee\{c \in \mathcal{C}(a) \,|\, c \not\leq b\},$$
$$(\forall k) \qquad \mathrm{C}^k_D(a) = \bigvee\{c \in \mathcal{C}(a) \,|\, D(c) = k\}.$$

This is a typical example of a d-subscaled lattice in which the sc-dimension does not coincide with the dimension, except of course if $D(a) = \dim_L a$ for every $a \in \mathcal{I}(L)$. Conversely, every noetherian (in particular every finite) d-subscaled lattice is of that kind.

We call **(sub)scaled lattices** the $\mathcal{L}_{\mathrm{SC}}$-structures whose $\mathcal{L}_{\mathrm{SC}}$-reduct is a d-(sub)scaled for some $d \in \mathbf{N}$. Of course this is not an elementary class. On the contrary, for any fixed $d \in \mathbf{N}$, $\mathrm{SS}_1^{\mathrm{d}}$ to SS_6 are expressible by a universal formula and SC_0 by a first order formula in $\mathcal{L}_{\mathrm{SC}}$, hence d-scaled (resp. d-subscaled) lattices form elementary class. As the terminology suggests, we will see that d-subscaled lattices are precisely the $\mathcal{L}_{\mathrm{SC}}$-substructures of d-scaled lattices.

Remark 2.7 $\mathrm{SS}_1^{\mathrm{d}}$ to SS_5 are actually expressible by equations in $\mathcal{L}_{\mathrm{SC}}$, hence define a variety[6] (in the sense of universal algebra). This is clear for $\mathrm{SS}_1^{\mathrm{d}}$ and $\mathrm{SS}_2^{\mathrm{d}}$. The other ones can then be written as follows.

SS_3 $(\forall k \geq l), \quad \mathrm{C}^k(\bigvee_{i\leq l} \mathrm{C}^l(a) \vee \mathrm{C}^l(b)) = \mathrm{C}^k(\bigvee_{i\leq l} \mathrm{C}^l(a)) \vee \mathrm{C}^k(\bigvee_{i\leq l} \mathrm{C}^l(b))$

SS_4 $\forall i > j, \quad \mathrm{C}^i(a) \wedge \mathrm{C}^j(a) = \bigvee_{k<j} \mathrm{C}^k(\mathrm{C}^i(a) \wedge \mathrm{C}^j(a))$

SS_5 $\forall k \geq 0, \quad \mathrm{C}^k(a) - \bigvee_{l\leq k} \mathrm{C}^l(b) = \mathrm{C}^k(a) - \mathrm{C}^k(b)$

By analogy with our guiding geometric examples, we say that an element a in a d-subscaled lattice is ***k*-sc-pure** if

$$\forall b \in L \ (a - b \neq \mathbf{0} \Rightarrow \text{sc-dim}(a - b) = k).$$

We will see that a is k-sc-pure if and only if $a = \mathrm{C}^k(a)$ (this is SS_{13} in Section 3). Then either $a = \mathbf{0}$ or $\text{sc-dim}\, a = k$. In the latter case we say that a **has pure sc-dimension** k. For any a, the element $\mathrm{C}^k(a)$ is called the ***k*-sc-pure component** of a, or simply its ***k*-pure component** if L is a scaled lattice. By construction these notions coincide with their geometric counterparts in $\mathrm{SC}_{\mathrm{def}}(\mathcal{K}, d)$ and $\mathrm{SC}_{\mathrm{Zar}}(K, d)$.

The following notation will be convenient in induction arguments. If $\mathcal{L}$ is any of our languages $\mathcal{L}_{\mathrm{lat}}$, $\mathcal{L}_{\mathrm{TC}}$ or $\mathcal{L}_{\mathrm{SC}}$ we let $\mathcal{L}^* = \mathcal{L} \setminus \{\mathbf{1}\}$. Given an $\mathcal{L}$-structure L whose reduct to $\mathcal{L}_{\mathrm{lat}}$ is a lattice, for any $a \in L$ we let

$$L(a) = \{b \in L \,|\, b \leq a\}.$$

$L(a)$ is a typical example of $\mathcal{L}^*$-substructure of L.

[6] Is this the variety generated by d-scaled lattices? This question might be of importance for further developments in non-classical logics.

3 Basic properties and embeddings

The next properties follow easily from the axioms of d-subscaled lattices.

SS$_7$: $\text{sc-dim}\, a = \max\{k \mid \mathrm{C}^k(a) \neq \mathbf{0}\}$.
In particular $\forall k$, $\text{sc-dim}\, \mathrm{C}^k(a) = k \iff \mathrm{C}^k(a) \neq \mathbf{0}$.

SS$_8$: $\forall k \geq \text{sc-dim}(a)$, $\text{sc-dim}(b \wedge a) < k \implies \mathrm{C}^k(a) - b = \mathrm{C}^k(a)$.

SS$_9$: $\text{sc-dim}\, a \vee b = \max(\text{sc-dim}\, a, \text{sc-dim}\, b)$.
In particular $b \leq a \Rightarrow \text{sc-dim}\, b \leq \text{sc-dim}\, a$.

SS$_{10}$: $\forall k \geq \text{sc-dim}(a)$, $\mathrm{C}^k(a)$ is the largest k-sc-pure element smaller than a.

SS$_{11}$: $\dim a \leq \text{sc-dim}\, a$.

SS$_{12}$: $\forall I \subseteq \{0, \dots, d\}$, $\quad a - \bigvee_{i \in I} \mathrm{C}^i(a) = \bigvee_{i \notin I} \mathrm{C}^i(a)$.
In particular $\text{sc-dim}\left(a - \bigvee_{i \geq k} \mathrm{C}^i(a)\right) < k$.

SS$_{13}$: $\forall k, \quad \mathrm{C}^k(a) = a \iff \forall b \left(a - b \neq \mathbf{0} \Rightarrow \text{sc-dim}\, a - b = k\right)$.
That is a is k-sc-pure if and only if $a = \mathrm{C}^k(a)$.
In particular if a is $\vee$-irreducible then a is sc-pure by $\mathrm{SS}_1^{\mathrm{d}}$.

Proof: (Sketch) SS_7 follows from $\mathrm{SS}_1^{\mathrm{d}}$ and $\mathrm{SS}_2^{\mathrm{d}}$; SS_8 from SS_5 and SS_7; SS_9 from $\mathrm{SS}_2^{\mathrm{d}}$, SS_3 and SS_7; SS_{10} from $\mathrm{SS}_2^{\mathrm{d}}$ and SS_3; SS_{11} from SS_6 by Fact 2.3. Only the two last properties require a little effort.

SS_{12}**:** For every $l \in I$, $\mathrm{C}^l(a) \leq \bigvee_{i\in I} \mathrm{C}^i(a)$ hence $\mathrm{C}^l(a) - \bigvee_{i \in I} \mathrm{C}^i(a) = \mathbf{0}$. On the other hand for every $l \notin I$ and every $i \in I$, $\mathrm{C}^l(a) - \mathrm{C}^i(a) = \mathrm{C}^l(a)$ by SS_4 and SS_5. So $\mathrm{C}^l(a) - \bigvee_{i\in I} \mathrm{C}^i(a) = \mathrm{C}^l(a)$ by TC_4. Finally by $\mathrm{SS}_1^{\mathrm{d}}$ and TC_2,

$$a - \bigvee_{i \in I} \mathrm{C}^i(a) = \bigvee_{l \leq d} \Big(\mathrm{C}^l(a) - \bigvee_{i \in I} \mathrm{C}^i(a)\Big) = \bigvee_{l \notin I} \mathrm{C}^l(a).$$

SS_{13}**:** Assume that $a = \mathrm{C}^k(a)$ and $a - b \neq \mathbf{0}$ for some b. Then $\text{sc-dim}\, a = k$ so $\text{sc-dim}(a - b) \leq k$ and $\text{sc-dim}(a \wedge b) \leq k$ by SS_9. Since $a = (a - b) \vee (a \wedge b)$ it follows by SS_3 that

$$\mathrm{C}^k(a) = \mathrm{C}^k\left((a - b) \vee (a \wedge b)\right) = \mathrm{C}^k(a - b) \vee \mathrm{C}^k(a \wedge b).$$

By assumption $\mathrm{C}^k(a) = a$, and $\mathrm{C}^k(a \wedge b) \leq a \wedge b$ by $\mathrm{SS}_1^{\mathrm{d}}$. So $a \leq \mathrm{C}^k(a - b) \vee (a \wedge b) \leq \mathrm{C}^k(a - b) \vee b$ which implies that $a - b \leq \mathrm{C}^k(a - b)$. In particular $\mathrm{C}^k(a - b) \neq \mathbf{0}$. Since $\text{sc-dim}(a - b) \leq k$ it follows that $\text{sc-dim}(a - b) = k$ by SS_7.

Conversely assume that $a \neq \mathrm{C}^k(a)$ (hence $a \neq \mathbf{0}$). For $b = \mathrm{C}^k(a)$ we then have $a - b \neq \mathbf{0}$ on one hand and $\text{sc-dim}(a - b) \neq k$ by SS_7 on the other hand, because $\mathrm{C}^k(a - b) = \mathbf{0}$ by SS_{12} and $\mathrm{SS}_2^{\mathrm{d}}$.
■

Proposition 3.1 *The $\mathcal{L}_{\mathrm{SC}}$-structure of a d-scaled lattice L is uniformly definable in the $\mathcal{L}_{\mathrm{lat}}$-structure of L. In particular it is uniquely determined by this $\mathcal{L}_{\mathrm{lat}}$-structure.*

Proof: Clearly the $\mathcal{L}_{\mathrm{TC}}$-structure is an extension by definition of the lattice structure of L. For every positive integer k the class of k-pure elements is uniformly definable, using the definability of $\ll$ and Fact 2.3. Then so is the function C^k for every k, by decreasing induction on k. Indeed by SS_{10} and SS_{12}, $\mathrm{C}^k(a)$ is the largest k-pure element c such that $c \leq a - \bigvee_{i>k} \mathrm{C}^i(a)$.
■

We need a reasonably easy criterion for an $\mathcal{L}_{\mathrm{lat}}$-embedding of subscaled lattices to be an $\mathcal{L}_{\mathrm{SC}}$-embedding. In the special case of a noetherian[7] embedded lattice, it is given by Proposition 3.3 below, whose proof will use the following characterisation of sc-pure components.

[7] Although we won't use it, let us mention that in the general case of an $\mathcal{L}_{\mathrm{lat}}$-embedding $\varphi : L \to L'$ between arbitrary subscaled lattices, one may easily derive from Proposition 3.3, by means of the Local Finiteness Theorem 4.1 and the model-theoretic compactness theorem, that φ is an $\mathcal{L}_{\mathrm{SC}}$-embedding if and only if it preserves sc-dimension and sc-purity, that is for every $a \in L$ and every $k \in \mathbf{N}$, $\mathrm{C}^k(a) = a \Rightarrow \mathrm{C}^k(\varphi(a)) = \varphi(a)$.

Proposition 3.2 *Let L be a subscaled lattice and $a, a_0, \dots, a_d \in L$ be such that $a = \bigvee_{i \le d} a_i$, each a_i is i-sc-pure and* $\text{sc-dim}(a_i \wedge a_j) < \min(i,j)$ *for every $i \ne j$. Then $\mathrm{C}^i(a) = a_i$ for every i.*

Proof: Note first that $\mathrm{C}^k(a_k) = a_k$ for every $k \le d$ by SS_{13}. Hence for every $k \le i \le d$ we have by SS_3 and $\mathrm{SS}_2^{\mathrm{d}}$

$$\mathrm{C}^i\Big(\bigvee_{k \le i} a_k\Big) = \bigvee_{k \le i} \mathrm{C}^i(a_k) = \bigvee_{k \le i} \mathrm{C}^i\big(\mathrm{C}^k(a_k)\big) = \mathrm{C}^i(a_i) = a_i. \tag{1}$$

In particular $\mathrm{C}^d(a) = a_d$. Now assume that for some $i < d$ we have proved that $\mathrm{C}^j(a) = a_j$ for $i < j \le d$. By SS_{12} and $\mathrm{SS}_2^{\mathrm{d}}$ we then have

$$\mathrm{C}^i\big(a - \bigvee_{j > i} a_j\big) = \mathrm{C}^i\big(a - \bigvee_{j > i} \mathrm{C}^j(a)\big) = \mathrm{C}^i\big(\bigvee_{j \le i} \mathrm{C}^j(a)\big) = \mathrm{C}^i(a). \tag{2}$$

On the other hand $a - \bigvee_{j>i} a_j = \bigvee_{k \le d}(a_k - \bigvee_{j>i} a_j)$ by TC_2. For $k > i$ obviously $a_k - \bigvee_{j>i} a_j = \mathbf{0}$. For $k < i$, $a_k = \mathrm{C}^k(a)$ and $a_j = \mathrm{C}^j(a)$ imply that $\text{sc-dim}(a_k \wedge a_j) < k$ for $j > i$ by SS_4. Hence $a_k - a_j = a_k$ by SS_8 and finally $a_k - \bigvee_{j>i} a_j = a_k$ by TC_4, so

$$a - \bigvee_{j > i} a_j = \bigvee_{k \le i} a_k. \tag{3}$$

By (1), (2), (3) we conclude that $\mathrm{C}^i(a) = a_i$. The result follows for every i by decreasing induction. ∎

Proposition 3.3 *Let L_0 be a noetherian subscaled lattice, L a subscaled lattice, and $\varphi : L_0 \to L$ an $\mathcal{L}_{\mathrm{lat}}$-embedding such that for every $a \in \mathcal{I}(L_0)$, $\varphi(a)$ is sc-pure and has the same sc-dimension as a. Then φ is an $\mathcal{L}_{\mathrm{SC}}$-embedding.*

Remark 3.4 Clearly the same statement remains true with $\mathcal{L}_{\mathrm{lat}}$ and $\mathcal{L}_{\mathrm{SC}}$ replaced respectively by $\mathcal{L}_{\mathrm{lat}}^*$ and $\mathcal{L}_{\mathrm{SC}}^*$. We will freely use these variants.

Proof: We have that L_0 and L are d-subscaled lattices for some $d \in \mathbf{N}$. Given $a \in L_0$ and k a non-negative integer, we first check that $\varphi(\mathrm{C}^k(a))$ is k-sc-pure. Note that every $\vee$-irreducible component c of $\mathrm{C}^k(a)$ in L_0 has sc-pure dimension k. Indeed $\mathrm{C}^k(a)$ is k-sc-pure by SS_{13}, and $c = \mathrm{C}^k(a) - b \ne \mathbf{0}$ where b is the join of all the other $\vee$-irreducible components of $\mathrm{C}^k(a)$, hence $\text{sc-dim}(c) = \text{sc-dim}(\mathrm{C}^k(a) - b) = k$. Moreover c is sc-pure because it is $\vee$-irreducible, hence c is k-pure. By our assumption on φ it follows that $\varphi(c)$ is k-sc-pure. Every finite union of k-sc-pure elements being k-sc-pure by SS_3, it follows that

$$\varphi\big(\mathrm{C}^k(a)\big) \text{ is } k\text{-sc-pure}. \tag{4}$$

Now for every $l \ne k$ we have $\text{sc-dim}(\mathrm{C}^k(a) \wedge \mathrm{C}^l(a)) < \min(k,l)$ by SS_4. It follows that each $\vee$-irreducible component c of $\mathrm{C}^k(a) \wedge \mathrm{C}^l(a)$ has sc-dimension strictly less than $\min(k,l)$, hence so does $\varphi(c)$ by assumption. By SS_9 we conclude that

$$\text{sc-dim}\big(\mathrm{C}^k(a) \wedge \mathrm{C}^l(a)\big) < \min(k,l) \quad (\forall l \ne k). \tag{5}$$

We have that $\varphi(a) = \bigvee_{k \le d} \varphi(\mathrm{C}^k(a))$ by $\mathrm{SS}_1^{\mathrm{d}}$ and because φ is an $\mathcal{L}_{\mathrm{lat}}$-embedding. By (4), (5) and Proposition 3.2 it follows that $\mathrm{C}^k(\varphi(a)) = \varphi(\mathrm{C}^k(a))$ for every $k \le d$. Since φ is injective, this implies by SS_7 that for every $a \in L_0$

$$\text{sc-dim}\, a = \text{sc-dim}\, \varphi(a). \tag{6}$$

It only remains to check that $\varphi(a - b) = \varphi(a) - \varphi(b)$ for every $a, b \in L_0$. By TC_2, replacing if necessary a by its $\vee$-irreducible components, we may assume w.l.o.g. that a itself is $\vee$-irreducible in L_0. This implies that $a = \mathrm{C}^k(a)$ for some k, hence $\varphi(a)$ is k-sc-pure by assumption on φ. It then remains two possibilities for $a - b$:

- If $b \ge a$ then $\varphi(b) \ge \varphi(a)$, hence $a - b = \mathbf{0}$ and $\varphi(a) - \varphi(b) = \mathbf{0}$, so $\varphi(a - b) = \varphi(\mathbf{0}) = \mathbf{0} = \varphi(a) - \varphi(b)$.

- Otherwise $b \wedge a < a$ hence $a - b = a$ by TC_1. So we have to prove that $\varphi(a) - \varphi(b) = \varphi(a)$. By SS_6 $\operatorname{sc-dim} b \wedge a < \operatorname{sc-dim} a$, hence $\operatorname{sc-dim}(\varphi(b) \wedge \varphi(a)) < \operatorname{sc-dim}(\varphi(a)) = k$ by (6). Since $\varphi(a)$ is k-sc-pure it follows that $\varphi(a) - \varphi(b) = \varphi(a)$ by SS_8.

■

Corollary 3.5 *Let L_0 be a noetherian lattice embedded in a subscaled lattice L. Assume that every $b' < b \in \mathcal{I}(L_0)$ are sc-pure in L and $\operatorname{sc-dim} b' < \operatorname{sc-dim} b$ in L. Then the restrictions to L_0 of the $\mathcal{L}_{\mathrm{SC}}$-operations "$-$" and "$\mathrm{C}_i$" of L turn L_0 into an $\mathcal{L}_{\mathrm{SC}}$-substructure which is a subscaled lattice.*

Proof: The assumptions imply that the map $D : a \mapsto \operatorname{sc-dim} a$ is a strictly increasing map from $\mathcal{I}(L_0)$ to $\mathbf{N}$. Endow L_0 with the structure of subscaled lattice determined by D as in Example 2.6. By construction the inclusion map from L_0 to L is an $\mathcal{L}_{\mathrm{lat}}$-embedding which preserves the sc-purity and sc-dimension of every $b \in \mathcal{I}(L_0)$, hence is an $\mathcal{L}_{\mathrm{SC}}$-embedding by Proposition 3.3.
■

4 Local finiteness

We prove in this section that every finitely generated subscaled lattice is finite. This result is far from obvious, due to the lack of any known normal form for terms in $\mathcal{L}_{\mathrm{SC}}$. It contrasts with the general situation in co-Heyting algebras, which can be both infinite and generated by a single element. Our main ingredient, which explains this difference, is the uniform bound given *a priori* for the sc-dimension of any element in a given d-subscaled lattice.

Theorem 4.1 *Any d-subscaled lattice L generated by n elements is finite. More precisely, the cardinality of $\mathcal{I}(L)$ is bounded by the function $\mu(n, d)$ defined by*

$$\mu(n, d) = 2^n + \mu(2^{n+1}, d - 1).$$

for $d \geq 0$, and $\mu(n, d) = 0$ for $d < 0$.

Proof: The only subscaled lattice of sc-dimension $d < 0$ is the one-element lattice $\{\mathbf{0}\}$, so the result is trivial in this case. Assume that $d \geq 0$ and that the result is proved for every $d' < d$ and every non-negative integer n.

Let L be a subscaled lattice of sc-dimension d generated by elements $x_1, \dots, x_n$. Let Ω_n be the family of all subsets of $\{1, \dots, n\}$ (so $\Omega_0 = \{\emptyset\}$). For every $I \in \Omega_n$ let $I^c = \Omega_n \setminus I$ and

$$y_I = \Big(\bigwedge_{i \in I} x_i\Big) - \Big(\bigvee_{i \in I^c} x_i\Big), \qquad z_I = \mathrm{C}^d(y_I).$$

The family of all $\mathcal{Y}_I = \bigcap_{i \in I} P(x_i) \cap \bigcap_{i \in I^c} P(x_i)^c$ is a partition of $\operatorname{Spec}(L)$. Indeed the $\mathcal{Y}_i$'s are the atoms of the boolean algebra generated in the power set $\mathcal{P}(\operatorname{Spec}(L))$ by the $P(x_i)$'s. Moreover each $P(y_I)$ is the topological closure $\overline{\mathcal{Y}}_I$ of $\mathcal{Y}_I$ in $\operatorname{Spec}(L)$ hence for every $x \in L$

$$P(x) = \bigcup_{I \in \Omega_n} P(x) \cap \mathcal{Y}_I \subseteq \bigcup_{I \in \Omega_n} P(x) \cap \overline{\mathcal{Y}}_I = P\Big(\bigvee_{I \in \Omega_n} x \wedge y_I\Big).$$

So $x \leq \bigvee_{I \in \Omega_n} (x \wedge y_I)$ by Stone's duality. The reverse inequality being obvious we have proved that

$$\forall x \in L, \quad x = \bigvee_{I \in \Omega_n} (x \wedge y_I). \tag{7}$$

In particular SS_3 also gives

$$\mathrm{C}^d(\mathbf{1}) = \mathrm{C}^d\Big(\bigvee_{I \in \Omega_n} y_I\Big) = \bigvee_{I \in \Omega_n} z_I. \tag{8}$$

For every $I \neq J \in \Omega_n$, if for example $I \not\subseteq J$ choose any $i \in I \setminus J$ and observe that $y_I \leq x_i$ and $y_J \leq \mathbf{1} - x_i$ so $y_I \wedge y_J \ll \mathbf{1} - x_i$ by TC_3. By SS_6 and the d-sc-purity of the z_I's it follows that

$$\operatorname{sc-dim} z_I \wedge z_J < d \quad \text{hence} \quad z_I - z_J = z_I. \tag{9}$$

It follows from SS_9, SS_{12} and (9) above, that the element

$$a = \big(\mathbf{1} - \mathrm{C}^d(\mathbf{1})\big) \vee \Big(\bigvee\!\!\!\bigvee_{I \in \Omega_n} (y_I - z_I) \Big) \vee \Big(\bigvee\!\!\!\bigvee_{I \neq J \in \Omega_n} (z_I \wedge z_J) \Big)$$

has sc-dimension strictly smaller than d. So the induction hypothesis applies to the $\mathcal{L}_{\mathrm{SC}}$-substructure L_0^- of $L(a)$ generated by the $(y_I - z_I)$'s and the $(z_I \wedge a)$'s: L_0^- is finite, with at most $\mu(2|\Omega_n|, d-1)$ $\vee$-irreducible elements. Note that L_0^- is an $\mathcal{L}_{\mathrm{SC}}^*$-substructure of L (recall that $\mathcal{L}_{\mathrm{SC}}^* = \mathcal{L}_{\mathrm{SC}} \setminus \{\mathbf{1}\}$). Finally let L_1 be the upper semi-lattice generated in L by $L_0^- \cup \{z_I\}_{I \in \Omega_n}$. By construction L_1 is finite and $\mathcal{I}(L_1) \subseteq \mathcal{I}(L_0^-) \cup \{z_I\}_{I \in \Omega_n}$, so $|\mathcal{I}(L_1)| \leq 2^n + \mu(2^{n+1}, d-1) = \mu(n,d)$. It is then sufficient to show that $L_1 = L$.

We first prove that L_1 is a lattice. By (8), $\mathrm{C}^d(\mathbf{1}) \vee a = \bigvee\!\!\!\bigvee_{I \in \Omega_n} z_I \vee a \in L_1$ hence $\mathbf{1} = \mathrm{C}^d(\mathbf{1}) \vee a \in L_1$. For every $I \in \Omega_n$ and every $b \in L_0^-$, $z_I \wedge b = (z_I \wedge a) \wedge b \in L_0^-$. For every $I \neq J \in \Omega_n$, $z_I \wedge z_J = (z_I \wedge a) \wedge (z_J \wedge a) \in L_0^-$. So by the distributivity law, L_1 is a sublattice of L.

In order to conclude that L_1 is an $\mathcal{L}_{\mathrm{SC}}$-substructure of L, by Corollary 3.5 it only remains to check that for every $b' < b \in \mathcal{I}(L_1)$, b is sc-pure in L and $\text{sc-dim}\, b' < \text{sc-dim}\, b$ in L. Since $\mathcal{I}(L_1) \subseteq \mathcal{I}(L_0^-) \cup \{z_I\}_{I \in \Omega_n}$ we can distinguish two cases.

Case 1: $b \in \mathcal{I}(L_0^-)$. Then b is sc-pure in L_0^- by SS_{13} hence also in L since L_0^- is an $\mathcal{L}_{\mathrm{SC}}^*$-substructure of L. Similarly $b' \ll b$ in L_0^- by TC_1 that is $b - b' = b$ in L_0^- hence also in L. Thus $b' \ll b$ in L which implies that $\text{sc-dim}\, b' < \text{sc-dim}\, b$ in L by SS_6.

Case 2: $b = z_I$ for some $I \in \Omega_n$. Then $b = \mathrm{C}^d(y_I)$ is sc-pure in L and $\text{sc-dim}\, b = d$. If $b' = z_J$ for some other $J \in \Omega_n$ then on one hand $\text{sc-dim}(b') = d$ and on the other hand $I \neq J$ hence $b' = b' \wedge b = z_I \wedge z_J$ has sc-dimension $< d$ by (9), a contradiction. So necessarily $b' \in \mathcal{I}(L_0^-)$, in particular $b' \leq a$ hence $\text{sc-dim}(b') \leq \text{sc-dim}(a) < d$

So L_1 is indeed an $\mathcal{L}_{\mathrm{SC}}$-substructure of L. Finally every $y_I = (y_I - z_I) \vee z_I \in L_1$ and (7) gives, for every $i \leq n$,

$$x_i = \bigvee\!\!\!\bigvee_{I \in \Omega_n} x_i \wedge y_I \leq \bigvee\!\!\!\bigvee_{\substack{I \in \Omega_n \\ i \in I}} y_I \leq x_i.$$

So equality holds, hence each $x_i \in L_1$, which finally proves that $L_1 = L$.

■

Corollary 4.2 *For every n, d there are finitely many non-isomorphic subscaled lattices of sc-dimension d generated by n elements.*

Proof: Any such subscaled lattice L is finite, with $|\mathcal{I}(L)| \leq \mu(n,d)$ by Theorem 4.1. Clearly there are finitely many non-isomorphic lattices such that $|\mathcal{I}(L)| \leq \mu(n,d)$ and each of them admits finitely many non-isomorphic $\mathcal{L}_{\mathrm{SC}}$-structures of d-subscaled lattices.

■

5 Linear representation

In this section we prove that the theory of d-subscaled lattices is the universal theory of various natural classes of geometric d-scaled lattices, including $\mathrm{SC}_{\mathrm{def}}(\mathcal{K}, d)$ in Example 1.1 as well as $\mathrm{SC}_{\mathrm{Zar}}(K, d)$. The argument is based on an elementary representation theorem for d-subscaled lattices, combined with the local finiteness result of Section 4.

Given an arbitrary field K, a non-empty linear variety $X \subseteq K^m$ is determined by the data of an arbitrary point $P \in X$ and the vector subspace $\overrightarrow{X}$ of K^m, *via* the relation $X = P + \overrightarrow{X}$ (the orbit of P under the action of $\overrightarrow{X}$ by translation). We call X a **special linear variety** (resp. a **special linear set**) if X is a linear variety such that $\overrightarrow{X}$ is generated by a subset of the canonical basis of K^m (resp. if X is a finite union of special linear varieties). The family $\mathrm{L}_{\mathrm{lin}}(X)$ of all special linear subsets of X is the family of closed sets of a noetherian topology on X, hence a noetherian lattice. For every $A \in \mathrm{L}_{\mathrm{lin}}(X)$ we let $D(A)$ be the dimension of A in the sense of linear algebra. This endows $\mathrm{L}_{\mathrm{lin}}(X)$ with a natural structure of scaled lattice as in Example 2.6.

Remark 5.1 For every $A \in \mathrm{L_{lin}}(X)$, if K is infinite then sc-dim $A = \dim_{\mathrm{L_{lin}}(X)} A$ = the dimension of A as defined in linear algebra. It coincides with the Krull dimension as well. If moreover A is $\vee$-irreducible in $\mathrm{L_{lin}}(X)$ then it is pure dimensional, hence it is sc-pure both in $\mathrm{L_{lin}}(X)$ and $\mathrm{L_{Zar}}(X)$. By Proposition 3.3 it follows that $\mathrm{L_{lin}}(X)$ is an $\mathcal{L}_{\mathrm{SC}}$-substructure of $\mathrm{L_{Zar}}(X)$. Similarly if $\mathcal{K}$ is a tame expansion of a topological field then $\mathrm{L_{lin}}(X)$ is an $\mathcal{L}_{\mathrm{SC}}$-substructure of $\mathrm{L_{def}}(X)$.

In what follows K^m is identified with $K^m \times \{0\}^r \subseteq K^{m+r}$. The very easy result below prepares the proof of Proposition 5.3.

Proposition 5.2 *For every two special linear sets $C \subseteq B \subseteq K^m$ and every non-negative integer $n \geq \dim C$ there exists a special linear set $A \subseteq K^{m+n}$ of pure dimension n such that $A \cap B = C$.*

Proof: The result being rather trivial if C is empty, we can assume w.l.o.g. that $C \neq \emptyset$. Let $(e_1, \dots, e_{m+n})$ be the canonical basis of K^{m+n}. If I is a subset of $\{1, \dots, m+n\}$ we let $\overrightarrow{E}(I)$ denote the vector space generated in K^n by $(e_i)_{i\in I}$. Decompose C as a union of special linear varieties $C_1, \dots, C_p$, and write each $C_i = P_i + \overrightarrow{E}(J_i)$ with $|J_i| = \dim C_i \leq n$. Let $I_i = J_i \cup \{m + 1, \dots, m+n-|J_i|\}$ and $A_i = P_i + \overrightarrow{E}(I_i)$ for every $i \leq p$. Finally let $A = A_1 \cup \dots \cup A_p$. By construction each A_i has pure dimension $|I_i| = n$, hence A has pure dimension n. Clearly each $A_i \cap K^m = C_i$, hence $A \cap K^m = C$ and *a fortiori* $A \cap B = C$.
■

Proposition 5.3 (Linear representation) *Let K be an infinite field, $d \geq 0$ an integer and L a finite d-subscaled lattice. Then there exists a special linear set X over K of dimension $\leq d$ and an $\mathcal{L}_{\mathrm{SC}}$-embedding $\varphi : L \to \mathrm{L_{lin}}(X)$.*

Proof: By induction on the number r of $\vee$-irreducible elements of an arbitrary d-subscaled lattice L, we prove that there exists an $\mathcal{L}^*_{\mathrm{SC}}$-embedding φ of L into $\mathrm{L_{lin}}(K^m)$ for some m depending on L. Taking $X = \varphi(\mathbf{1}_L)$ then gives the conclusion. Indeed X is a special linear set over K, $\dim X =$ sc-dim$(\mathbf{1}_L) \leq d$ because φ preserves the sc-dimension, and φ is obviously an $\mathcal{L}_{\mathrm{SC}}$-embedding of L into $\mathrm{L_{lin}}(X)$.

If $r = 0$ then L is the one-element lattice $\{\mathbf{0}\}$, hence an $\mathcal{L}^*_{\mathrm{SC}}$-substructure of $\mathrm{L_{lin}}(K)$. So, given a fixed $r \geq 1$, we can assume by induction that the result is proved for $r-1$. Let L be a d-subscaled lattice with $\vee$-irreducible elements $a_1, \dots, a_r$. Let $a = a_r$ and $b = \bigvee\!\!\!\vee_{1\leq i<r} a_i$.

Renumbering if necessary we may assume that a_r is maximal among the a_i's. By maximality, the $\vee$-irreducible elements of $L(b)$ are $a_1, \dots, a_{r-1}$. Let $c = a \wedge b$ and φ an $\mathcal{L}^*_{\mathrm{SC}}$-embedding of $L(b)$ into some $\mathrm{L_{lin}}(K^m)$ given by the induction hypothesis. Since a is $\vee$-irreducible in L it is sc-pure. Moreover $c \leq a$ by TC_1, hence a has pure sc-dimension n for some $n \geq$ sc-dim(c) by SS_6. Let B, C be the respective images of b, c by φ. Proposition 5.2 gives a special linear set $A \subseteq K^{m+n}$ of pure dimension n such that $A \cap B = C$. Identifying K^m with $K^m \times \{0\}^n \subseteq K^{m+n}$ turns φ into an $\mathcal{L}^*_{\mathrm{SC}}$-embedding of $L(b)$ into $\mathrm{L_{lin}}(K^{m+n})$.

Every $x \in L$ can be written uniquely as $x_a \vee x_b$ with $x_a \in \{\mathbf{0}, a\}$ and $x_b \in L(b)$ by grouping appropriately the $\vee$-irreducible components of x. So we can let

$$\bar{\varphi}(x) = \begin{cases} \varphi(x_b) & \text{if } x_a = \mathbf{0}, \\ A \cup \varphi(x_b) & \text{if } x_a = a. \end{cases}$$

This is a well-defined $\mathcal{L}^*_{\mathrm{lat}}$-embedding of L into $\mathrm{L_{lin}}(K^{m+n})$. Moreover $\bar{\varphi}$ is an $\mathcal{L}^*_{\mathrm{SC}}$-embedding by Proposition 3.3. This finishes the induction.
■

Given an infinite field K and positive integer d let $\mathrm{SC_{lin}}(K, d)$ be the class of d-scaled lattices $\mathrm{L_{lin}}(X)$ with X ranging over the special linear sets over K of dimension at most d.

Theorem 5.4 *The universal theories of $\mathrm{SC_{def}}(\mathcal{K}, d)$ (resp. $\mathrm{SC_{Zar}}(K, d)$, $\mathrm{SC_{lin}}(K, d)$) in the language $\mathcal{L}_{\mathrm{SC}}$ are the same for every fixed integer $d \geq 0$ and every tame expansion $\mathcal{K}$ of a topological field K (resp. for every infinite field K). This is the theory of d-subscaled lattices.*

Proof: As explained in Section 11, for every such expansion $\mathcal{K}$ of K the good properties of the dimension theory for definable sets $X \subseteq K^m$ ensure that $\mathrm{L_{def}}(X)$ is a d-scaled lattice. Obviously the same holds true for $\mathrm{L_{lin}}(X)$ and $\mathrm{L_{Zar}}(X)$. So the universal theory of any of these classes contains the theory of d-subscaled lattices. For the converse, thanks to Remark 5.1 it suffices to prove that every d-subscaled lattice L embeds into a model of the theory of $\mathrm{SC_{lin}}(K, d)$. If L is finite this is Proposition 5.3. The general case then follows from the model-theoretic compactness theorem, because L is locally finite by Theorem 4.1.
■

6 Minimal extensions

Minimal proper extensions[8] of any finite subscaled lattices are entirely determined by so-called "SC-signatures" (see below). Since this is a special case of minimal extensions of finite co-Heyting algebras, we first recall the main results of [6] on this subject, and try to reduce to them as much as possible.

We need some specific notation and definitions. Given a finite lattice L_0, a $\mathcal{L}_{\mathrm{lat}}$-extension L, elements $a \in L_0$ and $x \in L$ we write:

- $a^- = \bigvee\{b \in L_0 \,|\, b < a\}$.
- $g(x, L_0) = \bigwedge\{a \in L_0 \,|\, x \leq a\}$.

Clearly $a \in \mathcal{I}(L_0)$ if and only if a^- is the unique predecessor of a in L_0 (otherwise $a^- = a$).

Assume that L_0 and L are co-Heyting algebras (or topologically complemented lattices, or TC-lattice for short). A **TC-signature** in L_0 is a triple (g, H, r) where $g \in \mathcal{I}(L_0)$, H is a set of one or two elements $h_1, h_2 \in L_0$ and $r \in \{1, 2\}$ are such that:

- either $r = 1$ and $h_1 = h_2 < g$;
- or $r = 2$ and $h_1 \vee h_2$ is the unique predecessor of g.

A couple (x_1, x_2) of non-zero elements of L is **TC-primitive over** L_0 if there is $g \in \mathcal{I}(L_0)$ such that

P1 $g^- \wedge x_1$ and $g^- \wedge x_2$ belong to L_0.

P2 One of the following happens:

1. $x_1 = x_2$ and $g^- \wedge x_1 \ll x_1 \ll g$.
2. $x_1 \neq x_2$, $x_1 \wedge x_2 \in L_0$ and $g - x_1 = x_2$, $g - x_2 = x_1$.

This implies that each $x_i \notin L_0$, that $g = g(x_1, L_0) = g(x_2, L_0)$ and that the triple $\sigma_{\mathrm{TC}}(x_1, x_2) = (g, H, r)$ defined as follows is a SC-signature in L_0, called the **SC-signature of** (x_1, x_2) **in** L_0.

$$g = g(x_1, L_0) \qquad H = \{g^- \wedge x_1, g^- \wedge x_2\} \qquad r = \mathrm{Card}\{x_1, x_2\}$$

Finally we say that L is a **TC-primitive extension** of L_0 if it is $\mathcal{L}_{\mathrm{TC}}$-generated over L_0 by a TC-primitive couple. For the convenience of the reader we collect here all the properties of TC-signatures and TC-primitive extensions that we are going to use.

We will refer to the k-th item of the next proposition as Proposition 6.1.k.

Proposition 6.1 ([6]) *Let L_0 be a finite co-Heyting algebra and L an $\mathcal{L}_{\mathrm{TC}}$-extension*[9].

1. *([6, Theorem 3.3]) If L is $\mathcal{L}_{\mathrm{TC}}$-generated over L_0 by a TC-primitive tuple (x_1, x_2), then L is exactly the upper semi-lattice generated over L_0 by x_1 and x_2. It is a finite co-Heyting algebra and one of the following holds:*

[8] When we talk about an **extension** L of a lattice, a co-Heyting algebra or a subscaled lattice L_0, it is always understood that L is also a lattice, a co-Heyting algebra or a subscaled lattice respectively.

[9] See Footnote 8

(a) $x_1 = x_2$ *and* $\mathcal{I}(L) = \mathcal{I}(L_0) \cup \{x_1\}$.

(b) $x_1 \neq x_2$ *and* $\mathcal{I}(L) = (\mathcal{I}(L_0) \setminus \{g\}) \cup \{x_1, x_2\}$.

2. *([6, Remark 3.6]) The TC-signatures in* L_0 *and the TC-primitive extensions of* L_0 *are in one-to-one correspondence: every TC-signature in* L_0 *is the TC-signature of a TC-primitive extension, and two TC-primitive extensions of* L_0 *are* $\mathcal{L}_{\mathrm{TC}}$*-isomorphic over* L_0 *if and only if they have the same TC-signature in* L_0*.*

3. *([6, Corollary 3.4]) If* L *is finite, the following are equivalent.*

   (a) L *is a minimal proper extension of* L_0*.*

   (b) L *is a TC-primitive extension of* L_0*.*

   (c) $\mathrm{Card}(\mathcal{I}(L)) = \mathrm{Card}(\mathcal{I}(L_0)) + 1$.

   As a consequence every finite $\mathcal{L}_{\mathrm{TC}}$*-extension* L' *of* L_0 *is the union of a tower of TC-primitive extensions* $L_0 \subset L_1 \subset \cdots \subset L_n = L'$ *with* $n = \mathrm{Card}(\mathcal{I}(L')) - \mathrm{Card}(\mathcal{I}(L_0))$.

If L is a TC-primitive extension of a finite co-Heyting algebra L_0, by Proposition 6.1.1 it is $\mathcal{L}_{\mathrm{TC}}$-generated over L_0 by a *unique* (up to permutation) TC-primitive tuple (x_1, x_2). We then call $\sigma_{\mathrm{TC}}(x_1, x_2)$ the **TC-signature of** L **in** L_0 and denote it $\sigma_{\mathrm{TC}}(L)$.

Now let L_0 be a finite subscaled lattice and L a $\mathcal{L}_{\mathrm{SC}}$-extension. A **SC-signature** in L_0 is a triple $\sigma = (g, H, q)$ where $g \in \mathcal{I}(L_0)$, H is a set of one or two elements $h_1, h_2 \in L_0$ and $q \in \mathbf{N}$ are such that:

- either $\text{sc-dim}\, h_1 < q < \text{sc-dim}\, g$ and $h_1 = h_2 < g$;
- or $q = \text{sc-dim}\, g$ and $h_1 \vee h_2 = g^-$.

Let $r_\sigma = 1$ if $q < \text{sc-dim}\, g$, $r_\sigma = 2$ if $q = \text{sc-dim}\, g$, and $\sigma^{\mathrm{TC}} = (g, H, r_\sigma)$. By construction this is a TC-signature in L_0. Given a $\mathcal{L}_{\mathrm{SC}}$-extension L of L_0, a tuple (x_1, x_2) of elements of L is **SC-primitive over** L_0 if it is TC-primitive over L_0^{TC} and if in addition

P3 x_1, x_2 are sc-pure of the same sc-dimension.

Such a SC-primitive couple (x_1, x_2) determines its so-called **SC-signature in** L_0, denoted by $\sigma_{\mathrm{SC}}(x_1, x_2) = (g, H, q)$ and defined as follows.

$$g = g(x_1, L_0) \qquad H = \{g^- \wedge x_1, g^- \wedge x_2\} \qquad q = \text{sc-dim}\, x_1$$

Note that, by condition **P2** of the definition of TC-signatures, $x_1 = x_2$ if and only if $x_1 \ll g$, and otherwise $\text{sc-dim}\, x_1 = \text{sc-dim}\, x_2 = \text{sc-dim}\, g$. This ensures that $\sigma_{\mathrm{TC}}(x_1, x_2) = (\sigma_{\mathrm{SC}}(x_1, x_2))^{\mathrm{TC}}$.

Let L_0^{TC} and L^{TC} denote the respective $\mathcal{L}_{\mathrm{TC}}$-reducts of L_0 and L. For every subset X_0 of L we let:

- $L_0\langle X_0\rangle$ = the $\mathcal{L}_{\mathrm{SC}}$-structure generated by $L_0 \cup X_0$ in L;
- $L_0^{\mathrm{TC}}\langle X_0\rangle$ = the $\mathcal{L}_{\mathrm{TC}}$-structure generated by $L_0^{\mathrm{TC}} \cup X_0$ in L^{TC}.

We say that L is a **SC-primitive extension** of L_0, if there exists a tuple (x_1, x_2) SC-primitive over L_0 such that $L = L_0\langle x_1, x_2\rangle$ (then clearly $L = L_0\langle x_1\rangle = L_0\langle x_2\rangle$). By Lemma 6.2 below and Proposition 6.1.1 such a tuple is necessarily unique.

Lemma 6.2 *Let* L_0 *be finite subscaled lattice, and* L *a* $\mathcal{L}_{\mathrm{SC}}$*-extension*[10] *generated over* L_0 *by an SC-primitive tuple* (x_1, x_2)*. Then* $L^{\mathrm{TC}} = L_0^{\mathrm{TC}}\langle x_1, x_2\rangle$*,* (x_1, x_2) *is TC-primitive over* L_0^{TC} *and* $\sigma_{\mathrm{TC}}(x_1, x_2) = (\sigma_{\mathrm{SC}}(x_1, x_2))^{\mathrm{TC}}$*.*

[10] See Footnote 8

Proof: That (x_1, x_2) is TC-primitive over L_0^{TC} and $\sigma_{\mathrm{TC}}(x_1, x_2) = (\sigma_{\mathrm{SC}}(x_1, x_2))^{\mathrm{TC}}$ is only a reminder: it follows directly from the definitions. Let $L_1 = L_0^{\mathrm{TC}}\langle x_1, x_2\rangle$, in order to conclude that $L_1 = L$ it only remains to prove that L_1 is an $\mathcal{L}_{\mathrm{SC}}$-substructure of L. By Corollary 3.5 it suffices to check that for every $b' < b \in \mathcal{I}(L_1)$, b is sc-pure in L and $\text{sc-dim}\, b' < \text{sc-dim}\, b$ in L.

If $b \in \mathcal{I}(L_0)$, then b is sc-pure in L_0 by SS_{13}, hence also in L because L_0 is an $\mathcal{L}_{\mathrm{SC}}$-substructure of L. Otherwise $b = x_i$ for some $i \in \{1, 2\}$. Then b is sc-pure in L by definition of SC-primitive tuples over L_0.

In both cases $b' \ll b$ in L_1 by TC_1, that is $b - b' = b$ in L_1, hence also in L because L_1 is an $\mathcal{L}_{\mathrm{TC}}$-substructure of L. So $b' \ll b$ in L hence $\text{sc-dim}(b') < \text{sc-dim}(b)$ in L by SS_6.
■

Lemma 6.3 *Let L_0 be finite subscaled lattice, L_1 a $\mathcal{L}_{\mathrm{TC}}$-extension generated over L_0^{TC} by a TC-primitive tuple (x_1, x_2), and $\tau = (g, \{h_1, h_2\}, q)$ a SC-signature in L_0 such that $\tau^{\mathrm{TC}} = \sigma_{\mathrm{TC}}(x_1, x_2)$. Then there exists a unique structure of subscaled lattice expanding L_1 which makes it a $\mathcal{L}_{\mathrm{SC}}$-extension of L_0 such that (x_1, x_2) is SC-primitive over L_0 and $\sigma_{\mathrm{SC}}(x_1, x_2) = \tau$.*

Proof: By Proposition 6.1.1, $\mathcal{I}(L_1) \subseteq \mathcal{I}(L_0) \cup \{x_1, x_2\}$. For every $x \in \mathcal{I}(L_0)$ let $D(x) = \text{sc-dim}\, x$, and let $D(x_1) = D(x_2) = q$. This defines by restriction a function from $\mathcal{I}(L_1)$ to $\mathbf{N}$. Assume that D is strictly increasing. Then it determines as in Example 2.6 an $\mathcal{L}_{\mathrm{SC}}$-structure on L_1 expanding its $\mathcal{L}_{\mathrm{TC}}$-structure. Let us denote it L, so that $L^{\mathrm{TC}} = L_1$. Every $\vee$-irreducible element of L_0 remains sc-pure in L with the same sc-dimension, hence by Proposition 3.3 the inclusion of L_0 into L is an $\mathcal{L}_{\mathrm{SC}}$-embedding. This is clearly the only possible $\mathcal{L}_{\mathrm{SC}}$-structure on L_1 which makes it an $\mathcal{L}_{\mathrm{SC}}$-extension of L such that $\text{sc-dim}\, x_1 = \text{sc-dim}\, x_2 = q$. So it only remains to prove that D is strictly increasing.

Let $b < a$ in $\mathcal{I}(L)$, if $a, b \in \mathcal{I}(L_0)$ then $D(b) < D(a)$ by SS_6. So we can assume that a or b does not belong to $\mathcal{I}(L_0)$. By Proposition 6.1.1 one of them must belong to $\{x_1, x_2\}$ and the other one to $\mathcal{I}(L_0)$. Note that our assumption $\sigma_{\mathrm{TC}}(x_1, x_2) = \sigma^{\mathrm{TC}}$ implies that (for $i = 1, 2$) $g = g(x_i, L_0)$, and $h_i = x_i \wedge g^-$ (up to re-numbering) and: either $x_1 = x_2$, $h_1 = h_2 < g$ and $\text{sc-dim}\, h_1 < q < \text{sc-dim}\, g$; or $x_1 \neq x_2$, $h_1 \vee h_2 = g^-$ and $q = \text{sc-dim}\, g$.

Case 1: $b = x_1$ or $b = x_2$, hence $D(b) = q$. Then $a \in \mathcal{I}(L_0)$, in particular $a \in L_0$, hence $g \leq a$ and so $\text{sc-dim}\, g \leq \text{sc-dim}\, a$. If $x_1 = x_2$ then $q < \text{sc-dim}\, g \leq \text{sc-dim}\, a$ hence $D(b) < D(a)$. If $x_1 \neq x_2$ then $q = \text{sc-dim}\, g$, and g is not $\vee$-irreducible in L hence $g \neq a$. So $g \ll a$ (because $g < a$ and a is $\vee$-irreducible) hence $\text{sc-dim}\, g < \text{sc-dim}\, a$ by SS_6, that is $D(b) < D(a)$.

Case 2: $a = x_1$ or $a = x_2$, hence $D(a) = q$. Then again $b \in L_0$, and $b < a \leq g$ hence $b \leq g^-$. If $x_1 = x_2$, since $b \leq a \wedge g^- = h_1$ we get $\text{sc-dim}\, b \leq \text{sc-dim}\, h_1 < q$, hence $D(b) < D(a)$. If $x_1 \neq x_2$ then $\text{sc-dim}\, g = q$. Since $b < g$ we have $b \ll g$ (because g is $\vee$-irreducible) hence $\text{sc-dim}\, b < \text{sc-dim}\, g$ by SS_6. So $\text{sc-dim}\, b < q$, that is $D(b) < D(a)$.
■

We can now pack all this together. We will refer to the k-th item of the above proposition as Proposition 6.4.k.

Proposition 6.4 *Let L_0 be a finite subscaled lattice and L a $\mathcal{L}_{\mathrm{SC}}$-extension*[11].

1. *If L is $\mathcal{L}_{\mathrm{SC}}$-generated over L_0 by a SC-primitive tuple (x_1, x_2), then L is exactly the upper semi-lattice generated over L_0 by x_1 and x_2. It is a finite subscaled lattice and one of the following holds:*
   - *(a) $x_1 = x_2$ and $\mathcal{I}(L) = \mathcal{I}(L_0) \cup \{x_1\}$.*
   - *(b) $x_1 \neq x_2$ and $\mathcal{I}(L) = (\mathcal{I}(L_0) \setminus \{g\}) \cup \{x_1, x_2\}$.*
2. *SC-signatures in L_0 and SC-primitive extensions of L_0 are in one-to-one correspondence: every SC-signature in L_0 is the SC-signature of a SC-primitive extension, and two SC-primitive extensions of L_0 are $\mathcal{L}_{\mathrm{SC}}$-isomorphic over L_0 if and only if they have the same SC-signature in L_0.*
3. *If L is finite, the following are equivalent.*

[11] See Footnote 8.

(a) L is a minimal proper $\mathcal{L}_{\rm SC}$-extension of L_0.

(b) L is a SC-primitive extension of L_0.

(c) $\mathrm{Card}(\mathcal{I}(L)) = \mathrm{Card}(\mathcal{I}(L_0)) + 1$.

As a consequence every finite $\mathcal{L}_{\rm SC}$-extension L' of L_0 is the union of a tower of SC-primitive extensions $L_0 \subset L_1 \subset \cdots \subset L_n = L'$ with $n = \mathrm{Card}(\mathcal{I}(L')) - \mathrm{Card}(\mathcal{I}(L_0))$.

If L is a SC-primitive extension of a finite subscaled lattice L_0, by Proposition 6.4.1 it is generated over L_0 by a *unique* (up to permutation) SC-primitive couple (x_1, x_2). We call $\sigma_{\rm SC}(x_1, x_2)$ the **SC-signature of L in** L_0 and denote it $\sigma_{\rm SC}(L)$.

Proof: (1) If L is $\mathcal{L}_{\rm SC}$-generated over L_0 by an SC-primitive tuple (x_1, x_2), then by Lemma 6.2 $L^{\rm TC}$ is also $\mathcal{L}_{\rm TC}$-generated over $L_0^{\rm TC}$ by (x_1, x_2), which is TC-primitive. The first item the follows from Proposition 6.1.1.

(2) Let σ be an SC-signature in L_0. Then $\sigma^{\rm TC}$ is a TC-signature in L_0. Proposition 6.1.2 gives a TC-primitive $\mathcal{L}_{\rm TC}$-extension L_1 of $L_0^{\rm TC}$ with TC-signature $\sigma^{\rm TC}$ in L_0. Lemma 6.3 then gives a unique structure of subscaled lattice expanding L_1 which makes it an SC-primitive extension of L_0 with signature σ in L_0. Let us denote it L, so that $L^{\rm TC} = L_1$. Now if L' is another SC-primitive extension with signature σ in L_0, by Proposition 6.1.2 $L'^{\rm TC}$ is $\mathcal{L}_{\rm TC}$-isomorphic to $L^{\rm TC}$ over L_0. The image of L' *via* this endomorphism defines an $\mathcal{L}_{\rm SC}$-structure expanding $L^{\rm TC}$, which makes it an SC-primitive extension of L_0 with the same signature as L. By the uniqueness of such a structure, given by Lemma 6.3, it follows that this $\mathcal{L}_{\rm TC}$-isomorphism from $L'^{\rm TC}$ to $L^{\rm TC}$ is actually an $\mathcal{L}_{\rm SC}$-isomorphism, which proves the result.

(3) We prove (3a)⇒(3b)⇒(3c)⇒(3a). Note that (3b)⇒(3c) follows from item 1).

(3c)⇒(3a). Let L' be a proper $\mathcal{L}_{\rm SC}$-extension of L_0 contained in L. Then $L'^{\rm TC}$ is a proper $\mathcal{L}_{\rm TC}$-extension of $L_0^{\rm TC}$ contained in $L^{\rm TC}$. By Proposition 6.1.3, (3c) implies that $L^{\rm TC}$ is a minimal proper $\mathcal{L}_{\rm TC}$-extension of $L_0^{\rm TC}$. So $L'^{\rm TC} = L^{\rm TC}$, thus necessarily $L' = L$, which proves that L is minimal.

(3a)⇒(3b). Let x_1 be a minimal element in $\mathcal{I}(L) \setminus \mathcal{I}(L_0)$. Let $g = g(x_1, L_0)$,; if $x_1 \ll g$ let $x_2 = x_1$, otherwise let $x_2 = g - x_1$. The proof of Corollary 3.4 in [6] shows that (x_1, x_2) is TC-primitive over $L_0^{\rm TC}$. In particular $x_1, x_2 \in \mathcal{I}(L)$ so they are sc-pure by SS_{13}. The same holds true for g, hence if $x_1 \neq x_2$ then $x_1 = g - x_2$ and $x_2 = g - x_1$ have the same dimension (the dimension of g, by definition of the sc-purity of g). So (x_1, x_2) is actually SC-primitive. Since $L_0^{\rm TC}\langle x_1, x_2\rangle = L^{\rm TC}$, *a fortiori* $L_0\langle x_1, x_2\rangle = L$, hence L is SC-primitive over L_0.

■

7 Model-completion of scaled lattices

We say that a subscaled lattice L is a **super scaled lattice**, if L satisfies the following additional properties, both of which are clearly axiomatizable by ∀∃-formulas in $\mathcal{L}_{\rm SC}$. Moreover, if sc-dim $L \leq d$, we say that L is a **super d-scaled lattice**.

Catenarity For every non-negative integers $r \leq q \leq p$ and every elements $c \leq a \neq \mathbf{0}$, if c is r-sc-pure and a is p-sc-pure then there exists a non-zero q-sc-pure element b such that $c \leq b \leq a$.

If Spec$L$ is noetherian this property is equivalent to the usual notion of catenarity, namely that any two maximal chains in Spec$L$ having the same first and last elements have the same length. In particular every d-scaled lattice L of type $\mathrm{L}_{\rm Zar}(X)$ or $\mathrm{L}_{\rm lin}(X)$ satisfies this property. If $\mathcal{K}$ is an o-minimal field and $X \subseteq K^m$ is any definable set, then $\mathrm{L}_{\rm def}(X)$ also satisfies the Catenarity Property: given $A, C \in \mathrm{L}_{\rm def}(X)$, respectively p-pure and r-pure, the Triangulation Theorem reduces to the case where A is a simplex and C one of its faces, and it then suffices to take for B a face of A of dimension p containing C In contrast, none of these scaled lattices satisfy the next property, as it implies that L is atomless.

Splitting For every elements b_1, b_2, a, if $b_1 \vee b_2 \ll a \neq \mathbf{0}$ then there exists non-zero elements $a_1 \geq b_1$ and $a_2 \geq b_2$ such that:

$$\begin{cases} a_1 = a - a_2 \\ a_2 = a - a_1 \\ a_1 \wedge a_2 = b_1 \wedge b_2 \end{cases}$$

We will then say a_1, a_2 **split** a **along** b_1, b_2.

Remark 7.1 If $r < p \leq q$ in the Catenarity axiom, the conclusion can be strengthen to $c \ll b \leq a$. Indeed b has pure sc-dimension q and $c \wedge b = c$ has sc-dimension $< q$ hence $b - c = b$ by SS_8. In particular *every subscaled lattice satisfying the Catenarity axiom is a scaled lattice.* Indeed, given any element a of sc-dimension $d \geq 1$, repeated applications of the Catenarity axiom to $\mathrm{C}^d(a)$, $c = \mathbf{0}$ and each integer p from 0 to d, gives a chain of sc-pure elements $a_0, \dots, a_d$ such that

$$\mathbf{0} \neq a_0 \ll a_1 \ll \cdots \ll a_d \leq a.$$

By Fact 2.3 it follows that $\dim a \geq d$, and by SS_{11} that $\dim a = d$.

Lemma 7.2 *Let a, b_1, b_2 be elements of a finite subscaled lattice L_0. If $b_1 \vee b_2 \ll a \neq \mathbf{0}$ then L_0 embeds in a finite subscaled lattice L containing non-zero elements a_1, a_2 which split a along b_1, b_2. Moreover, if $\mathrm{C}^0(a) = \mathbf{0}$ we can require*[12] *that all the atoms of L belong to L_0.*

Proof: We are going to prove by induction on $d = \text{sc-dim}\, a$ a slightly more precise result, namely that in addition $x \leq a$ for every $x \in \mathcal{I}(L) \setminus \mathcal{I}(L_0)$. Let $g_1, \dots, g_n$ be the $\vee$-irreducible components of a in L. Note that $n \geq 1$ because $a \neq \mathbf{0}$. If $d = 0$ our assumption that $b_1 \vee b_2 \ll a$ implies by SS_6 that $b_1 = b_2 = \mathbf{0}$. If $n = 1$, that is $a = g_1$ is $\vee$-irreducible, then $\sigma = (g, \{\mathbf{0}\}, 0)$ is a signature in L. Proposition 6.4.2 gives an SC-primitive couple (a_1, a_2) generating an $\mathcal{L}_{\mathrm{SC}}$-extension L_1 over L with signature σ. This signature ensures that (a_1, a_2) splits a along $(\mathbf{0}, \mathbf{0})$. If $n \geq 2$, $a_1 = g_1$ and $a_2 = a - a_1$ will do the job. So the result is proved for $d = 0$.

Now assume that $d \geq 1$ and the result is valid until $d-1$. Note that $g_1^- \vee \cdots \vee g_n^-$ is the greatest element $c \in L$ such that $c \ll a$, in particular

$$b_1 \vee b_2 \leq g_1^- \vee \cdots \vee g_n^-. \tag{10}$$

Let $u = (\bigvee_{i \leq n} g_i^-) - (b_1 \vee b_2)$ and $u^* = u - \mathrm{C}^0(u)$. Since $u \ll a$ we have $\text{sc-dim}\, u < d$ by SS_6.

We are claiming that L_0 embeds in a finite subscaled lattice L without new atoms, in which all the g_i's are still $\vee$-irreducible with the same predecessor as in L_0, and in which there are elements u_1^*, u_2^* which satisfy all the conditions to split u^* along $b_1 \wedge u^*$, $b_2 \wedge u^*$, except that u_1^*, u_2^* might be zero elements.

By TC_3, $(b_1 \vee b_2) \wedge u^* \ll u^*$ so if $\text{sc-dim}\, u \leq 0$ we can simply take $u_1^* = u_2^* = \mathbf{0}$ and $L_0 = L$. On the other hand, if $\text{sc-dim}\, u > 0$ the induction hypothesis applies to u^*, $b_1 \wedge u^*$, $b_2 \wedge u^*$. It gives a finite subscaled lattice L containing L_0 and elements $u_1^*, u_2^* \in L_0$ which split u^* along $b_1 \wedge u^*$, $b_2 \wedge u^*$. Moreover we can require that L do not contain any new atom because $\mathrm{C}^0(u^*) = \mathbf{0}$, and that $x \leq u^*$ for every $x \in \mathcal{I}(L) \setminus \mathcal{I}(L_0)$. For every $x \in \mathcal{I}(L)$ such that $x < g_i$ for some $i \leq n$, if $x \in L_0$ then $x \leq g_i^-$ (where g_i^- still denotes the predecessor of g_i in L_0). If $x \notin L_0$ then $x \leq u^*$ by construction hence $x \leq g_i \wedge u^*$. The latter belongs to L_0 and is strictly smaller than g_i, hence smaller than g_i^-, so $x < g_i$. It follows that g_i^- is still the unique predecessor of g_i in L. In particular g_i remains $\vee$-irreducible in L. This proves our claim in both cases.

Now let $u_1 = \mathrm{C}^0(u) \vee u_1^*$ and $u_2 = u_2^*$. We have in particular

$$u_1 \vee u_2 = \bigvee_{i \leq n} g_i^- - (b_1 \vee b_2). \tag{11}$$

Since $u - b_2 = u$ by TC_3 necessarily $b_2 \wedge c \ll c$ for every $\vee$-irreducible component c of u, hence $b_2 \wedge \mathrm{C}^0(u) \ll \mathrm{C}^0(u)$. By SS_6 it follows that $b_2 \wedge \mathrm{C}^0(u) = \mathbf{0}$ hence $b_2 \wedge u_1 = b_2 \wedge u_1^*$. Similarly $u^* \wedge \mathrm{C}^0(u) = \mathbf{0}$ because $u^* - \mathrm{C}^0(u) = u^*$ by SS_{12} and TC_3. *A fortiori* $u_2^* \wedge \mathrm{C}^0(u) = \mathbf{0}$ hence

[12] This additional requirement when $\mathrm{C}^0(a) = \mathbf{0}$ will be used only later, in Section 9.

$u_2^* \wedge u_1 = u_2^* \wedge u_1^*$. Note also that $b_1 \wedge u_2^* = b_1 \wedge u^* \wedge u_2^* \leq u_1^* \wedge u_2^*$, and symmetrically $b_2 \wedge u_1^* \leq u_1^* \wedge u_2^*$. Altogether, since $u_2 = u_2^*$ and $u_1^* \wedge u_2^* \leq b_1 \wedge b_2$ by construction, this gives

$$(b_1 \wedge u_2) \vee (b_2 \wedge u_1) \vee (u_1 \wedge u_2) \leq (b_1 \wedge b_2)$$

hence

$$(b_1 \vee u_1) \wedge (b_2 \vee u_2) = (b_1 \wedge b_2) \vee (b_1 \wedge u_2) \vee (b_2 \wedge u_1) \vee (u_1 \wedge u_2) = (b_1 \wedge b_2). \tag{12}$$

After this preparation, for each i let

$$h_{i,1} = g_i^- \wedge (b_1 \vee u_1), \qquad h_{i,2} = g_i^- \wedge (b_2 \vee u_2), \qquad \sigma_i = \big(g_i, \{h_{i,1}, h_{i,2}\}, \text{sc-dim}\, g_i\big)$$

Using (11) we get

$$\begin{aligned} h_{i,1} \vee h_{i,2} &= g_i^- \wedge (b_1 \vee u_1 \vee b_2 \vee u_2) \\ &= g_i^- \wedge \left[b_1 \vee b_2 \vee \left(\bigvee_{j \leq n} g_j^- - (b_1 \vee b_2) \right) \right] \\ &= g_i^- \wedge \bigvee_{j \leq n} g_j^- = g_i^-. \end{aligned}$$

So each σ_i is an SC-signature in L_0. In particular Proposition 6.4.2 gives an SC-primitive extension $L_1 = L_0\langle a_{1,1}, a_{1,2}\rangle$ with SC-signature σ_1 in L_0. By Proposition 6.4.1, $\mathcal{I}(L_1) = (\mathcal{I}(L_0) \setminus \{g_1\}) \cup \{a_{1,1}, a_{1,2}\}$. In particular $g_2 \in \mathcal{I}(L_1)$, hence σ_2 is still an SC-signature in L_1. Repeating the construction n times (note that $a \neq \mathbf{0}$ ensures that $n \geq 1$) gives a chain of $\mathcal{L}_{\text{SC}}$-extensions $(L_i)_{i \leq n}$ and for each $i > 0$, an SC-primitive couple $(a_{i,1}, a_{i,2})$ generating L_i over L_{i-1} with signature σ_i in L_{i-1}. Each $g_i = a_{i,1} \vee a_{i,2}$ and by Proposition 6.4.1

$$\mathcal{I}(L_n) = \big(\mathcal{I}(L_0) \setminus \{g_1, \dots, g_n\}\big) \cup \{a_{1,1}, a_{1,2}, \dots, a_{n,1}, a_{n,2}\} \tag{13}$$

so $a_{1,1}, a_{1,2}, \dots, a_{n,1}, a_{n,2}$ are the $\vee$-irreducible components of a in L_n. Moreover every $c \in \mathcal{I}(L_n)$ such that $c < a_{i,k}$ for some i, k must belong to L_0, hence the predecessor of $a_{i,k}$ is the same in every L_j and belongs to L_0. We can then denote it $a_{i,k}^-$ without ambiguity, and by construction we have

$$a_{i,k}^- = a_{i,k} \wedge g(a_i, L_{i-1}) = a_{i,k} \wedge g_i = h_{i,k}. \tag{14}$$

Let $a_1 = \bigvee_{i \leq n} a_{i,1}$, $a_2 = \bigvee_{i \leq n} a_{i,2}$, $h_1 = \bigvee_{i \leq n} h_{i,1}$ and $h_2 = \bigvee_{i \leq n} h_{i,2}$. We are going to check that a_1, a_2 split a along b_1, b_2. Both of them are non-zero and since the $a_{i,k}$'s are the $\vee$-irreducible components of a we have $a - a_1 = a_2$, $a - a_2 = a_1$. Each $a_{i,1} \geq h_{i,1}$ by construction, hence $a_1 \geq h_1$ and symmetrically $a_2 \geq h_2$. Moreover for $k \in \{1, 2\}$

$$h_k = \bigvee_{i \leq n} h_{i,k} \geq \bigvee_{i \leq n} g_i^- \wedge b_k = b_k$$

where the last equality comes from (10), so $a_k \geq b_k$. It remains to check that $a_1 \wedge a_2 = b_1 \wedge b_2$.

For $i \neq j$, $a_{i,1}$ and $a_{j,2}$ are mutually incomparable hence by (14)

$$a_{i,1} \wedge a_{j,2} = a_{i,1}^- \wedge a_{i,2}^- = h_{i,1} \wedge h_{j,2}.$$

On the other hand $a_{i,1} \wedge a_{i,2} = h_{i,1} \wedge h_{i,2}$ by construction. The conclusion follows, with $L = L_n$, using (12).

$$\begin{aligned} a_1 \wedge a_2 &= \bigvee_{i,j} a_{i,1} \wedge a_{j,2} = \bigvee_{i,j} h_{i,1} \wedge h_{j,2} \\ &= \bigvee_{i,j} \big[g_i^- \wedge (b_1 \vee u_1)\big] \wedge \big[g_j^- \wedge (b_2 \vee u_2)\big] \\ &= \bigvee_{i,j} (g_i^- \wedge g_j^-) \wedge \big[(b_1 \vee u_1) \wedge (b_2 \vee u_2)\big] \\ &= \Big(\bigvee_i g_i^-\Big) \wedge \Big(\bigvee_j g_j^-\Big) \wedge \big[(b_1 \vee u_1) \wedge (b_2 \vee u_2)\big] \\ &= (b_1 \vee u_1) \wedge (b_2 \vee u_2) = b_1 \wedge b_2. \end{aligned}$$

■

Theorem 7.3 *The theory of super d-scaled lattices is the model-completion of the theory of d-subscaled lattices. In particular, it eliminates the quantifiers in* $\mathcal{L}_{\mathrm{SC}}$.

Proof: The last statement follows from the first one, as is usual for the model-completion of a universal theory. By standard model-theoretic arguments it then suffices to prove that every existentially closed d-subscaled lattice is super d-scaled, and that for every super d-scaled lattice $\hat{L}$, every finitely generated d-subscaled lattice L and every common $\mathcal{L}_{\mathrm{SC}}$-substructure L_0, there is an embedding of L into $\hat{L}$ over L_0.

Let L be an existentially closed d-subscaled lattice, and L_0 a finitely generated substructure. By Theorem 4.1, L_0 is finite. By Proposition 5.3, L_0 $\mathcal{L}_{\mathrm{SC}}$-embeds the d-scaled lattice $\mathrm{L}_{\mathrm{def}}(X)$ of some special linear set X, which is in particular a Catenary lattice. By the model-theoretic compactness Theorem it follows that L is catenary. Similarly Theorem 4.1, Lemma 7.2 and the model-theoretic compactness Theorem prove that L has the Splitting property, hence L is super d-scaled.

Conversely assume that $\hat{L}$ is a super d-scaled lattice, L a finitely generated d-subscaled lattice, and L_0 is a common $\mathcal{L}_{\mathrm{SC}}$-substructure of both. By Theorem 4.1 and Proposition 6.4.3a, we are reduced to the case where L is a primitive extension of L_0. Let $\sigma = (g, \{h_1, h_2\}, q)$ be its SC-signature. By Proposition 6.4.2 it suffices to find a $x_1, x_2 \in \hat{L}$ such that (x_1, x_2) is SC-primitive over L_0 and $\sigma_{\mathrm{SC}}(x_1, x_2) = \sigma$. We distinguish two cases, and let g^- denotes the predecessor of g in L_0.

Case 1: $\text{sc-dim}\, h_1 < q < \text{sc-dim}\, g$ and $h_1 = h_2 < g$. Let $p = \text{sc-dim}\, g$ and $r = \text{sc-dim}\, h_1$. Let $y_1, y_2 \in \hat{L}$ which split g along h_1, g^-. For $0 \leq i \leq d$, either $i < q$ or $\mathrm{C}^i(h_1) = \mathbf{0}$ (because $\text{sc-dim}\, h_1 = r < q$), hence $\text{sc-dim}\, \mathrm{C}^i(h_1) < q < \text{sc-dim}\, g$. Recall that g is sc-pure and $y_1 = g - y_2 \neq \mathbf{0}$, so and y_1 has pure sc-dimension p like g. The Catenarity property then applies to $\mathrm{C}^i(h_1) \leq y_1 = \mathrm{C}^p(y_1)$ and gives $x_i \in \hat{L}$ such that $\mathrm{C}^i(h_1) \leq x_i \leq y_1$ and x_i has pure sc-dimension q. Let $x = \bigvee_{0 \leq i \leq d} x_i$, by construction $h_1 = \bigvee_{i \leq d} \mathrm{C}^i(h_1) \leq x \leq y_1$ and x has pure sc-dimension q. In particular

$$h_1 \leq x \wedge g^- \leq y_1 \wedge y_2 = h_1 \wedge g^- = h_1$$

hence $x \wedge g^- = h_1 \in L_0$. Moreover $x \wedge g^- = h_1 \ll x$ because $\text{sc-dim}\, h_1 < q$ and x has pure sc-dimension q. Finally $x \ll g$ because $\dim x_i = q < p$ and g has pure sc-dimension p. Altogether this proves that (x, x) is an SC-primitive tuple over L_0 with SC-signature σ.

Case 2: $q = \text{sc-dim}\, g$ and $h_1 \vee h_2 = g^-$. Let $y_1, y_2 \in \hat{L}$ which split g along h_1, h_2. By construction $y_1 \vee y_2 = g$, and since g has pure sc-dimension q so does each y_i. In addition $y_1 \wedge y_2 = h_1 \wedge h_2 \in L_0$. Moreover

$$y_1 \wedge h_2 \leq y_1 \wedge y_2 = h_1 \wedge h_2$$

hence $y_1 \wedge (h_1 \vee h_2) = h_1 \vee (y_1 \wedge h_2) = h_1$. Since $h_1 \vee h_2 = g^-$ it follows that $y_1 \wedge g^- = h_1 \in L_0$, and symmetrically $y_2 \wedge g^- = h_2 \in L_0$. So (y_1, y_2) is an SC-primitive tuple over L_0 with SC-signature σ.

■

Remark 7.4 The proof of Theorem 7.3 shows that if L_0 is a finite $\mathcal{L}_{\mathrm{SC}}$-substructure of a super scaled lattice $\hat{L}$, then every signature σ in L_0 is the signature of an SC-primitive extension of L_0 in $\hat{L}$.

The completions of the theory of super d-scaled lattices are easy to classify. Let us say that a d-subscaled lattice is **prime** if it does not contain any proper d-subscaled lattice, or equivalently if it is generated by the empty set. Every prime d-subscaled lattice is finite. By Corollary 4.2 there exists finitely many prime d-subscaled lattices up to isomorphism.

Corollary 7.5 *The theory of super d-scaled lattices containing (a copy of) a given prime d-subscaled lattice is* $\aleph_0$*-categorical, hence complete. It is also recursively axiomatizable, hence decidable. Every completion of the theory of super d-scaled lattices is of that kind, and the theory of super d-scaled lattices is decidable.*

Proof: Let L, L' be any two countable super d-scaled lattices containing isomorphic prime d-subscaled lattices L_0 and L_0'. By Remark 7.4 any partial isomorphism between L and L', extending the given isomorphism between L_0 and L_0', can be extended by a back and forth process. This proves the first statement. The other ones are immediate consequences.

■

8 Atomic scaled lattices

Every super scaled lattice is atomless because of the Splitting Property, hence none of the geometric scaled lattice amongst $\mathrm{SC}_{\mathrm{def}}(\mathcal{K},d)$, $\mathrm{SC}_{\mathrm{Zar}}(K,d)$, $\mathrm{SC}_{\mathrm{lin}}(K,d)$ can be super scaled. In order to apply our study to some of them, we now introduce a variant of subscaled lattices intended to protect atoms against splitting.

Let $\mathcal{L}_{\mathrm{ASC}} = \mathcal{L}_{\mathrm{SC}} \cup \{\mathrm{At}_k\}_{k\in\mathbf{N}^*}$, with each At_k a new unary predicate symbol. For any $\mathcal{L}_{\mathrm{ASC}}$-structure L we denote by $\mathrm{At}_k(L)$ the set of elements a in L such that $L \models \mathrm{At}_k(a)$, and we let $\mathrm{At}_0(L) = L \setminus \bigcup_{k>0} \mathrm{At}_k(L)$. We call L an **ASC-lattice** if its $\mathcal{L}_{\mathrm{SC}}$-reduct is a scaled lattice and if it satisfies the following condition.

ASC$_0$: $(\forall k > 0)$, $a \in \mathrm{At}_k(L)$ if and only if a is the join of exactly k atoms in L.

Remark 8.1 This condition can be expressed by $\forall\exists$ formulas in $\mathcal{L}_{\mathrm{ASC}}$ by saying first that $\mathrm{At}_1(L)$ is the set of atoms of L, and then that $\mathrm{At}_k(L)$ is the set of elements of L which are the join of exactly k elements of $\mathrm{At}_1(L)$.

Every ASC-lattice obviously satisfies also the following schemes (for $k, l > 0$) of universal axioms:

ASC$_1$: $(\forall k, l > 0,\ k \neq l),\quad \forall a,\ \mathrm{At}_k(a) \to \neg\, \mathrm{At}_l(a)$

ASC$_2$: $(\forall k > 0),\quad \forall a, a_0, \dots, a_{2^k},\quad \mathrm{At}_k(a) \longrightarrow$

$$\left[\bigwedge\!\!\!\bigwedge_{0\le i\le 2^k} (a_i \le a) \longrightarrow \bigvee\!\!\!\bigvee_{0\le i<j\le 2^k} (a_i = a_j) \right] \bigwedge \text{sc-dim}\, a = 0$$

ASC$_3$: $(\forall k > 0),\quad \forall a, a_1, a_2,$

$$\Big[(a = a_1 \vee a_2) \bigwedge (a_1 \wedge a_2 = \mathbf{0}) \bigwedge (a_1 \neq \mathbf{0}) \bigwedge (a_2 \neq \mathbf{0}) \Big]$$
$$\longrightarrow \left[\mathrm{At}_k(a) \longleftrightarrow \bigvee\!\!\!\bigvee_{0<l<k} \big(\mathrm{At}_l(a_1) \bigwedge \mathrm{At}_{k-l}(a_2) \big) \right]$$

We call **sub-ASC-lattices** the $\mathcal{L}_{\mathrm{ASC}}$-structures L whose $\mathcal{L}_{\mathrm{SC}}$-reduct is a subscaled lattice and which satisfy ASC_1 to ASC_3 (but not necessarily ASC_0).

The scheme ASC_1 obviously means that $(\mathrm{At}_k(L))_{k\in\mathbf{N}}$ is a partition[13] of L. For any $a \in L$ we then define $\mathrm{asc}(a)$ as the unique $k \in \mathbf{N}$ such that $a \in \mathrm{At}_k(L)$.

The scheme ASC_2 says that if $\mathrm{asc}(a) = k > 0$ then $L(a)$ has at most 2^k element and $\text{sc-dim}(a) = 0$. Then $\dim(a) = 0$ by SS_{11} so $L(a)$ is a co-Heyting algebra with dimension 0, hence a Boolean algebra. So ASC_2 actually says that $\text{sc-dim}\, a = 0$ and $L(a)$ is a Boolean algebra with n atoms for some non zero $n \le k$. In particular every $a \in \mathrm{At}_1(L)$ is an atom of L.

The scheme ASC_3 says that if a is the join of two non-zero disjoint elements a_1, a_2 then $\mathrm{asc}(a)$ is non-zero if and only if $\mathrm{asc}(a_1)$ and $\mathrm{asc}(a_2)$ are non-zero, in which case $\mathrm{asc}(a) = \mathrm{asc}(a_1) + \mathrm{asc}(a_2)$. By a straightforward induction this extends to any decomposition of a as the join of finitely many pairwise disjoint elements. In view of ASC_2 it then says that $\mathrm{asc}(a) > 0$ if and only if a is the join of finitely many atoms $a_1, \dots, a_n$ of L such that each $\mathrm{asc}(a_i) > 0$, in which case $\mathrm{asc}(a) = \sum_{1\le i\le n} \mathrm{asc}(a_i)$.

Remark 8.2 It follows immediately that a $\mathcal{L}_{\mathrm{SC}}$-embedding of sub-ASC-lattices $\varphi\colon L \to L'$ is an $\mathcal{L}_{\mathrm{ASC}}$-embedding if and only if $\mathrm{asc}(a) = \mathrm{asc}(\varphi(a))$ for every *atom* $a \in L$.

Remark 8.3 Obviously every finitely generated substructure of a sub-ASC-lattices is finite by the Local Finiteness Theorem 4.1, because $\mathcal{L}_{\mathrm{ASC}}$ expands $\mathcal{L}_{\mathrm{SC}}$ only by relational symbols.

Every scaled lattice L admits a unique structure of ASC-lattice which is an expansion by definition of its lattice structure. We denote by L^{At} this expansion of L.

[13]By a "partition" a set S, we mean here a collection of disjoint sets X covering S. In particular, we do not require these sets X to be non-empty.

Proposition 8.4 (Linear representation) *Let K be an infinite field and L_0 be a finite sub-ASC-lattice. For every integer $N \geq 0$ there exists a special linear set X_N over K and a $\mathcal{L}^*_{\mathrm{ASC}}$-embedding $\varphi_N \colon L_0 \to \mathrm{L}^{\mathrm{At}}_{\mathrm{lin}}(K^m)$ such that for every atom a of L_0 we have:*

- *If* $\mathrm{asc}(a) > 0$ *then* $\mathrm{asc}(\varphi_N(a)) = \mathrm{asc}(a)$.
- *If* $\mathrm{asc}(a) = 0$ *then* $\varphi_N(a)$ *is greater than at least* N *atoms.*

Proof: By induction on lexicographically ordered tuples of integers (r, s) we prove that the result is true for every finite sub-ASC-lattice L_0 having r $\vee$-irreducible elements, s of which have the same sc-dimension as L_0.

If $r = 0$ then $s = 0$ and the unique embedding of $L_0 = \{\mathbf{0}\}$ into $\mathrm{L}^{\mathrm{At}}_{\mathrm{lin}}(P)$, for an arbitrary point P of K, has the required property. So let us assume that $r \geq 1$ and that the result is proved for every $(r', s') < (r, s)$. Let $d = \text{sc-dim}\, L_0$ and $a_1, \ldots, a_r$ be the elements of $\mathcal{I}(L_0)$ ordered by increasing sc-dimension, so that $\text{sc-dim}\, a_r = d \geq 0$.

Case 1: $d = 0$. Then L_0 is a boolean algebra and $a_1, \ldots, a_r$ are its atoms. Let $A_1, \ldots, A_r$ be pairwise disjoint subsets of K such that:

- If $\mathrm{asc}(a_i) > 0$ then A_i has $\mathrm{asc}(a_i)$ elements, so $\mathrm{asc}(A_i) = \mathrm{asc}(a_i)$.
- If $\mathrm{asc}(a_i) = 0$ then A_i has N elements, so $\mathrm{asc}(A_i) = N$.

Let X be the union of all these A_i's. Clearly the map φ which maps each a_i to A_i extends uniquely to an $\mathcal{L}_{\mathrm{SC}}$-embedding of L_0 into $\mathrm{L}^{\mathrm{At}}_{\mathrm{lin}}(X)$ which has the required properties.

Case 2: $d > 0$. The upper semi-lattice L_0^- generated by $a_1, \ldots, a_{r-1}$ is an $\mathcal{L}^*_{\mathrm{ASC}}$-substructure of L_0 to which the induction hypothesis applies. This gives for some integer m a special linear set $B \subseteq K^m$ over K and an $\mathcal{L}_{\mathrm{SC}}$-embedding $\psi \colon L_0^- \to \mathrm{L}^{\mathrm{At}}_{\mathrm{lin}}(B)$ having the required properties. Let $C = \varphi(\mathbf{1}_{L_0^-} \wedge a_r)$ and $n = \text{sc-dim}\, a_r$. Proposition 5.2 gives a special linear set $A \subseteq K^{m+n}$ such that $A \cap B = C$. One can extend ψ to an $\mathcal{L}_{\mathrm{SC}}$-embedding φ of L_0 into $\mathrm{L}^{\mathrm{At}}_{\mathrm{lin}}(A \cup B)$ exactly like in the proof of Proposition 5.3. Then φ inherits from ψ the required properties because all the elements $x \in L_0$ such that $\mathrm{asc}(x) \neq 0$ already belong to L_0^-. Indeed, a_r is the only $\vee$-irreducible element of L_0 which doesn't belong to L_0, so every $a \in L_0 \setminus L_0^-$ is greater than a_r. But $\text{sc-dim}\, a_r = d > 0$ implies that $\text{sc-dim}\, a > 0$, hence $\mathrm{asc}(a) = 0$ by ASC_2. Moreover $\varphi(a) \geq \varphi(a_r) = A$ contains infinitely many atoms (because $\dim A = d > 0$), and the conclusion follows.

■

Let $\mathrm{ASC}_{\mathrm{Zar}}(K, d)$, $\mathrm{ASC}_{\mathrm{lin}}(K, d)$, $\mathrm{ASC}_{\mathrm{def}}(K, d)$ denote the class of all ASC-lattices L^{At} for L ranging over $\mathrm{SC}_{\mathrm{Zar}}(K, d)$, $\mathrm{SC}_{\mathrm{lin}}(K, d)$, $\mathrm{SC}_{\mathrm{def}}(K, d)$ respectively.

Corollary 8.5 *For every integer $d \geq 0$, the universal theories of $\mathrm{ASC}_{\mathrm{def}}(\mathcal{K}, d)$ (resp. of $\mathrm{ASC}_{\mathrm{Zar}}(K, d)$ or $\mathrm{ASC}_{\mathrm{lin}}(K, d)$) is the same for every o-minimal or P-minimal expansion of a field K (resp. every infinite field K). This is the theory of sub-ASC-lattices.*

Proof: Since $\mathrm{ASC}_{\mathrm{lin}}(K, d)$ is contained in the other classes, all of which are contained in the class of ASC-lattices, it suffices to prove that conversely every sub-ASC-lattice $\mathcal{L}_{\mathrm{ASC}}$-embeds into an ultraproduct of elements of $\mathrm{ASC}_{\mathrm{lin}}(K, d)$. By the model-theoretic compactness theorem, it suffices to prove it for any finitely generated sub-ASC-lattice L_0.

By Theorem 4.1, L_0 is finite. For any integer $N \geq 0$ let $\varphi_N : L_0 \to \mathrm{L}^{\mathrm{At}}_{\mathrm{lin}}(X_N)$ be an $\mathcal{L}_{\mathrm{SC}}$-embedding given by Proposition 8.4. Let $\mathcal{U}$ be a non principal ultrafilter in the Boolean algebra of subsets of $\mathbf{N}$, and consider the ultraproduct $L = \prod_{N \in \mathbf{N}} \mathrm{L}^{\mathrm{At}}_{\mathrm{lin}}(X_N)/\mathcal{U}$. Then $\varphi = \prod_{N \in \mathbf{N}} \varphi_N/\mathcal{U}$ is an $\mathcal{L}_{\mathrm{SC}}$-embedding of L_0 into the L. In order to prove that it is an $\mathcal{L}_{\mathrm{ASC}}$-embedding, by Remark 8.2 it remains check that for every atom a of L_0, $\mathrm{asc}(\varphi(a)) = \mathrm{asc}(a)$. So let a be an atom of L_0 and $k = \mathrm{asc}(a)$.

If $k > 0$ then for every $N \geq k$, $\mathrm{L}^{\mathrm{At}}_{\mathrm{lin}}(X_N) \models \mathrm{At}_k(\varphi_N(a))$ by construction. So $L \models \mathrm{At}_k(\varphi(a))$, that is $\mathrm{asc}(\varphi(a)) = k$.

If $k = 0$, let l be any strictly positive integer. For every $N \geq l$, $\mathrm{L}^{\mathrm{At}}_{\mathrm{lin}}(X_N) \models \mathrm{At}_N(\varphi_N(a))$ by construction, hence $\mathrm{L}^{\mathrm{At}}_{\mathrm{lin}}(X_N) \not\models \mathrm{At}_l(\varphi_N(a))$. So $L \not\models \mathrm{At}_l(\varphi(a))$, and this being true for every $l > 0$ it follows that $\mathrm{asc}(\varphi(a)) = 0$.

■

9 Model-completion of atomic scaled lattices

Let us call **super ASC-lattices** those ASC-lattices which satisfy the following axioms, all of which are axiomatizable by $\forall\exists$-formulas in $\mathcal{L}_{\mathrm{ASC}}$. We are going to show that this theory is the model-completion of the theory of sub-ASC-lattices of dimension at most d (resp. exactly d).

Atomicity Every element x is the least upper bound of the set of atoms smaller than x.

Catenarity For every non-negative integers $r \leq q \leq p$ and every elements $c \leq a \neq \mathbf{0}$, if c is r-sc-pure and a is p-sc-pure then there exists a non-zero q-sc-pure element b such that $c \leq b \leq a$.

ASC-Splitting For every b_1, b_2, a, if $b_1 \vee b_2 \ll a \neq \mathbf{0}$ and $\mathrm{C}^0(a) = \mathbf{0}$ there exists non-zero elements $a_1 \geq b_1$ and $a_2 \geq b_2$ such that:

$$\begin{cases} a_1 = a - a_2 \\ a_2 = a - a_1 \\ a_1 \wedge a_2 = b_1 \wedge b_2 \end{cases}$$

Remark 9.1 An immediate consequence of the atomicity axiom is that for every elements x, y in a super ASC-lattice L such that $y < x$ and sc-dim$(x - y) \geq 1$, there are infinitely many atoms $a \in L$ such that $a \leq x$ and $a \wedge y = \mathbf{0}$. Indeed let A be the set of atoms $a \in L$ such that $a \leq x - y$, and B the subset of those a such that $a \wedge y = \mathbf{0}$. Assume for a contradiction that B is finite and let $b = \bigvee_{a\in B} a$. Note that $b \leq y$ and sc-dim $b = \dim b = 0$. Then by the Atomicity axiom

$$x - y = \bigvee_{a\in A} a \leq y \vee b, \text{ hence } x - y \leq (y \vee b) - y = b - y \leq b.$$

This implies that sc-dim$(x - y) \leq$ sc-dim $b = 0$, a contradiction.

The notions of ASC-primitive tuples and ASC-primitive extensions are defined for sub-ASC-lattices exactly like for subscaled lattices. Here is a typical example of what we are going to call an ASC-signature.

Example 9.2 Let L_0 be a finite sub-ASC-lattice, and L an $\mathcal{L}_{\mathrm{SC}}$-extension of L_0 generated by a (necessarily unique) SC-primitive tuple (x_1, x_2). Let $(g, \{h_1, h_2\}, q)$ be the SC-signature of L in L_0 and $k_i = \mathrm{asc}(x_i)$. The following properties are immediate.

1. If $q <$ sc-dim g then $k_1 = k_2$ (because $x_1 = x_2$ in that case).
2. If $q \neq 0$ then $k_1 = k_2 = 0$ (because each x_i has sc-pure dimension > 0 in that case) .
3. If $k_1 = 0$ or $k_2 = 0$ then $\mathrm{asc}(g) = 0$ (because $g \geq x_1 \vee x_2$).
4. If $k_1 \neq 0$, $k_2 \neq 0$ and sc-dim $g = 0$ then $\mathrm{asc}(g) = k_1 + k_2$ (because $g = x_1 \vee x_2$ in that case).

We define **ASC-signatures** in a finite sub-ASC-lattice L_0 as triples (g, H, q) with H a set of non-necessarily distinct couples (h_1, k_1), (h_2, k_2) in $L_0 \times \mathbf{N}$, such that $(g, \{h_1, h_2\}, q)$ is a SC-signature in the $\mathcal{L}_{\mathrm{SC}}$-reduct of L_0 and all the conditions enumerated in Example 9.2 hold true. In particular we call the ASC-signature in this example the **ASC-signature** of L and of (x_1, x_2) in L_0. Note that if $q <$ sc-dim g then $h_1 = h_2$ because $(g, \{h_1, h_2\}, q)$ is a SC-signature.

The same argument as in Proposition 6.4.2 shows (using Remark 8.2) that two SC-primitive extensions of a finite sub-ASC-lattice L_0 are $\mathcal{L}_{\mathrm{ASC}}$-isomorphic over L_0 if and only if they have the same ASC-signature in L_0.

Lemma 9.3 *Let L_0 be a finite $\mathcal{L}_{\mathrm{ASC}}$-substructure of a super ASC-lattice $\hat{L}$. Let $\sigma_{\mathrm{At}} = (g, q, \{(h_1, k_1), (h_2, k_2)\})$ be an ASC-signature in L_0. Assume that $q \neq 0$ or $k_1 k_2 \neq 0$. Otherwise assume that $\hat{L}$ is $\aleph_0$-saturated. Then there exists a primitive tuple $(x_1, x_2) \in \hat{L}$ over L_0 whose ASC-signature is σ_{At}.*

Proof: Let $\sigma = (g, \{h_1, h_2\}, q)$. This is a SC-signature in L_0 (more precisely in its $\mathcal{L}_{\rm SC}$-reduct).

Case 1: sc-dim $g \geq 1$ and $q \geq 1$. Then $\mathrm{C}^0(g) = 0$ and by definition of ASC-signatures $k_1 = k_2 = 0$. By Remark 7.4 there is an SC-primitive tuple (x_1, x_2) in $\hat{L}$ with signature σ in L_0. Moreover each asc $x_i = 0$ (because sc-dim $x_i = p \geq 1$) and each $k_i = 0$, so the ASC-signature of (x_1, x_2) is $\sigma_{\rm At}$.

Case 2: sc-dim $g \geq 1$ and $q = 0$. Then $\mathrm{C}^0(g) = 0$ again and since sc-dim$(h_1 \vee h_2) < q = 0$ by definition of SC-signatures we get that $h_1 = h_2 = \mathbf{0}$. Finally $k_1 = k_2$ by definition of ASC-signatures since $q = 0 <$ sc-dim g. By Remark 9.1 there are infinitely many atoms z in $\hat{L}$ such that $z \leq g$ and $z \wedge g^- = \mathbf{0}$. If $k_1 > 0$ let x be the join of k_1 such atoms of $\hat{L}$. Otherwise $\hat{L}$ is $\aleph_0$-saturated by assumption hence it contains an element $x \leq g$ of dimension 0 such that $x \wedge g^- = \mathbf{0}$ and $\hat{L}(x)$ has infinitely many atoms. By the Atomicity Property asc$(x) = 0$. So in both cases (x, x) is an SC-primitive tuple over L_0 with ASC-signature $\sigma_{\rm At}$.

Case 3: sc-dim $g = 0$. Then $q = 0$, g is an atom of L_0 and $h_1 = h_2 = \mathbf{0}$. In each of the two remaining sub-cases, we build a tuple (x_1, x_2) and leave as an exercise to check that (x_1, x_2) is SC-primitive over L_0 with ASC-signature $\sigma_{\rm At}$.

If k_1 and k_2 are non-zero then asc$(g) = k_1 + k_2$ hence $\hat{L}(g)$ contains $k_1 + k_2$ atoms. Let x_1 be the join of k_1 of them and x_2 be the join of the others.

Otherwise, by symmetry we can assume that $k_1 = 0$. Then asc$(g) = 0$ by definition of ASC-signatures so $\hat{L}(g)$ contains infinitely many atoms. By $\aleph_0$-saturation it follows that $\hat{L}$ contains an element x smaller than g such that both $\hat{L}(x)$ and $\hat{L}(g - x)$ contain infinitely many atoms, hence asc$(x) =$ asc$(g - x) = 0$. If $k_2 = 0$ let $(x_1, x_2) = (x, g - x)$. Otherwise let x_2 be the join of k_2 atoms in $\hat{L}(g)$ and let $x_1 = g - x_2$.

■

Theorem 9.4 *The theory of super ASC-lattices of sc-dimension at most d (resp. exactly d) is the model-completion of the theory of ASC-lattices of dimension at most d (resp. exactly d). In particular, it eliminates the quantifiers in $\mathcal{L}_{\rm ASC}$. It admits $\aleph_0$ completions, each of which is decidable, and it is decidable.*

Proof: We first only sketch the proof of the first statement, as it essentially the same as for Theorem 7.3.

On one hand, given a finite sub-ASC-lattice L_0, we can embed it in an extension satisfying the Atomicity and Catenarity Property by Proposition 8.4, and the ASC-Splitting Property by means of Lemma 7.2 applied to any $a, b_1, b_2 \in L_0$ such that $b_1 \vee b_2 \ll a \neq \mathbf{0}$ and $\mathrm{C}^0(a) = \mathbf{0}$ (note that this last assumption ensures that the extension built in Lemma 7.2 is an $\mathcal{L}_{\rm ASC}$-extension). That every existentially closed sub-ASC-lattice is a super ASC-lattice then follows, by the model-theoretic compactness theorem.

On the other hand, given an $\aleph_0$-saturated super ASC-lattice $\hat{L}$, a finite $\mathcal{L}_{\rm ASC}$-substructure L_0 and a finite extension L of L_0, we reduce to the case where L is SC-primitive and let σ be its ASC-signature in L_0. Lemma 9.3 gives an SC-primitive extension L_1 of L_0 in $\hat{L}$ with the same signature in L_0, hence an embedding of L_1 into $\hat{L}$ over L_0 (which maps L to L_1). This proves the first statement.

Quantifier elimination follows, as usual for the model-completion of a universal theory. Moreover there are finitely many $\emptyset$-generated subscaled lattices of dimension at most d (resp. exactly d). Each of them (except the trivial ones, in which $\mathbf{0} = \mathbf{1}$) can be enriched with $\aleph_0$ different structures of sub-ASC-lattices obtained as follows: given a finite d-subscaled lattice L and a partition[14] $(X_k)_{k \in \mathbf{N}}$ of the set of atoms a of L such that $\mathrm{C}^0(a) = a$, we let asc$(a) = k$ for every $a \in X_k$; we then expand L to an $\mathcal{L}_{\rm ASC}$-structure according to ASC_3. So the completions of the theory of super ASC-lattices, which are determined by their prime model, can be recursively enumerated.

■

We say that a sub-ASC-lattice L is **standard** if every element of sc-dimension 0 belongs to some $\mathrm{At}_k(L)$ for some $k > 0$. The existence of standard super ASC-lattices (see Section 10) and non-standard super ASC-lattices (by the model theoretic compactness theorem) implies that the theory of super ASC-lattices containing a given prime sub-ASC-lattice is not $\aleph_0$-categoric, contrary

[14]Necessarily $X_k = \emptyset$ for all but finitely many k's, see Footnote 13.

to what happens for super scaled lattice. However we can recover $\aleph_0$-categorical by restricting to standard models.

Proposition 9.5 *Let L_1, L_2 be two* standard *countable super ASC-lattices. Then every $\mathcal{L}_{\mathrm{ASC}}$-isomorphism from a finite sub-ASC-lattice $L_{1,0} \subset L_1$ to a sub-ASC-lattice $L_{2,0} \subset L_2$ extends to an $\mathcal{L}_{\mathrm{ASC}}$-isomorphism from L_1 to L_2. In particular L_1 and L_2 are isomorphic if and only if their prime $\mathcal{L}_{\mathrm{ASC}}$-substructures (those generated by the empty set) are isomorphic.*

Proof: Let φ be an $\mathcal{L}_{\mathrm{ASC}}$-isomorphism from $L_{1,0}$ to $L_{2,0}$. Pick any element $x \in L_1 \setminus L_{1,0}$. The subscaled lattice generated in L_1 by $L_{1,0} \cup \{x\}$ (more precisely their $\mathcal{L}_{\mathrm{SC}}$-reducts) is finite hence by Proposition 6.4.3 there is a chain $L_{1,0} \subset L_{1,1} \subset \cdots \subset L_{1,r}$ of SC-primitive extensions of subscaled lattices such that $L_{1,0} \cup \{x\} \subseteq L_{1,r}$. Endow each $L_{1,i}$ with the $\mathcal{L}_{\mathrm{ASC}}$-structure induced by L_1. It suffices to prove that φ extends to an $\mathcal{L}_{\mathrm{ASC}}$-embedding $\varphi_1 : L_{1,1} \to L_2$. Indeed, repeating the argument will give an $\mathcal{L}_{\mathrm{ASC}}$-embedding $\varphi_r : L_{1,r} \to L_2$ extending φ, and by symmetry the conclusion will then follow by a back and forth argument.

Identifying $L_{1,0}$ with its image by φ we can replace $L_{1,0}$ and $L_{2,0}$ by a common $\mathcal{L}_{\mathrm{ASC}}$-structure L_0 of L_1 and L_2. Now $L_{1,1}$ is generated over L_0 by an SC-primitive tuple (x_1, x_2) with signature $\sigma_{\mathrm{At}} = (g, \{(h_1, k_1), (h_2, k_2)\}, q)$. In particular $q = \text{sc-dim}\, x_i$ and $k_i = \mathrm{asc}(x_i)$ for $i = 1, 2$. If $q = 0$ then for each i, $\text{sc-dim}\, x_i = 0$ hence $k_i > 0$ because L_1 is standard. In other words $q \neq 0$ or $k_1 k_2 \neq 0$ hence Lemma 9.3 gives an sc-primitive tuple (y_1, y_2) in L_2 with signature σ_{At}. Let $L_{2,1}$ be the asc-substructure of L_2 generated by $L_{1,1} \cup \{y_1, y_2\}$. By Proposition 6.4.2 φ extends to an $\mathcal{L}_{\mathrm{SC}}$-isomorphism φ_1 from $L_{1,1}$ to $L_{2,1}$ which maps each x_i to y_i. By construction $\mathrm{asc}(x_i) = \mathrm{asc}(y_i)$, and by Proposition 6.4.1 φ_1 is the identity map on L_0, so $\mathrm{asc}(\varphi_1(z)) = \mathrm{asc}(z)$ for every $z \in \mathcal{I}(L_{1,1})$. Hence φ_1 is an $\mathcal{L}_{\mathrm{ASC}}$-isomorphism by Remark 8.2, which proves the result.
∎

10 Applications to lattices of p-adic semi-algebraic sets

In this section K denotes a fixed p-adically closed field. For every semi-algebraic set X contained in K^m we let $L(X)$ denote the lattice of semi-algebraic subsets of X closed in X, endowed with its natural structure of ASC-lattice. Note that every $A \in L(X)$ of dimension 0 is finite, hence $L(X)$ is *standard*.

As already mentioned in the introduction, the results of the previous section lead us to conjecture in [3] and finally to prove in [4] the following result.

Theorem 10.1 (Theorem 3.4 in [4]) *Let X be a non-empty semi-algebraic subset of K^m without isolated points. Assume that X is open in its topological closure $\overline{X}$ and let $Y_1, \ldots, Y_s$ be a collection of closed semi-algebraic subsets of $\partial X = \overline{X} \setminus X$ such that $Y_1 \cup \cdots \cup Y_s = \partial X$. Then there is a partition of X in non-empty semi-algebraic sets $X_1, \ldots, X_s$ such that $\partial X_i = Y_i$ for $1 \leq i \leq s$.*

We can now combine this theorem with the results of Section 10 in order to get the following applications.

Theorem 10.2 *Let X be any semi-algebraic subset of K^m. Then $L(X)$ is a super ASC-lattice. In particular its complete theory is decidable and eliminates quantifiers in $\mathcal{L}_{\mathrm{ASC}}$.*

Proof: By construction $L(X)$ is an ASC-lattice satisfying the Atomicity property. The Catenarity Property will be proved in the appendix in much more general settings (Proposition 11.8). We focus here to the Splitting Property. So let $A, B_1, B_2 \in L(X)$ such that $B_1 \cup B_2 \ll A$ and A has no isolated point.

The same holds true for their closures in K^m, denoted $\overline{A}$, $\overline{B}_1$, $\overline{B}_2$. Indeed $\overline{A} \setminus A \ll \overline{A}$ and

$$\overline{A} \setminus (\overline{B}_1 \cup \overline{B}_2) \subseteq \big(\overline{A} \setminus A\big) \cup \big(A \setminus (B_1 \cup B_2)\big).$$

Apply Theorem 10.1 to $W = \overline{A} \setminus (\overline{B}_1 \cup \overline{B}_2)$, $Y_1 = \overline{B}_1$ and $Y_2 = \overline{B}_2$. It gives a partition of W in non-empty semi-algebraic sets W_1, W_2 whose frontiers are respectively $\overline{B}_1$, $\overline{B}_2$. Then $\overline{W}_1 \cup \overline{W}_2 = \overline{A}$,

$\overline{W}_1 \cap \overline{W}_2 = \overline{B}_1 \cap \overline{B}_2$ and each $\overline{W}_i = W_i \cup \overline{B}_i$. Let $A_1 = \overline{W}_1 \cap A = (W_1 \cap A) \cup B_1$ and define A_2 accordingly. We have to check that A_1, A_2 split A along B_1, B_2.

A is dense in $\overline{A} = \overline{W}$ and $W_1 = \overline{W} \setminus (W_2 \cup \overline{B}_1 \cup \overline{B}_2) = \overline{W} \setminus (\overline{W}_2 \cup \overline{B}_1)$ is open in $\overline{W}$, hence $A \cap W_1$ is dense in W_1. In particular $A \cap W_1 \neq \emptyset$, and symmetrically $A \cap W_2 \neq \emptyset$. Clearly $A_1 \cup A_2 = A$, $A_1 \cap A_2 = B_1 \cap B_2$ and each $A_i \supseteq B_i$ by construction. So it only remains to check that $A - A_1 = A_2$ in $L(X)$, that is that the closure of $A \setminus A_1$ in X (hence in A) is A_2. Note that $A \setminus A_1 = A \setminus \overline{W}_1$ and

$$\overline{A} \setminus \overline{W}_1 = \big(W_1 \cup W_2 \cup \overline{B}_1 \cup \overline{B}_2\big) \setminus \big(W_1 \cup \overline{B}_1\big) = W_2 \cup \big(\overline{B}_2 \setminus \overline{B}_1\big).$$

In particular $A \setminus A_1 = A \setminus \overline{W}_1 = (\overline{A} \setminus \overline{W}_1) \cap A$ contains $W_2 \cap A$ and is contained in $(W_2 \cup \overline{B}_2) \cap A = \overline{W}_2 \cap A = A_2$. The conclusion will follow, if we can prove that $W_2 \cap A$ is dense in A_2. Since $A_2 = (W_2 \cap A) \cup B_2$ it suffices to check that $B_2 \subseteq \overline{W_2 \cap A}$. But this is clear since $W_2 \cap A$ is dense in W_2, hence in $\overline{W}_2 = W_2 \cup \overline{B}_2$.

■

Corollary 10.3 *Let F be a q-adically closed field (for some prime q not necessarily equal to p). Let $X \subseteq K^m$ and $Y \subseteq F^n$ be two semi-algebraic sets.*

1. *If $m = n$, $K \preccurlyeq F$ and $X = Y \cap K^n$ then $L(X) \preccurlyeq L(Y)$.*
2. *$L(X) \equiv L(Y) \iff$ their prime $\mathcal{L}_{\text{ASC}}$-substructures are isomorphic.*
   In particular $L(K^m) \equiv L(F^n)$ if and only if $m = n$.
3. *If K and F are countable then $L(X) \equiv L(Y) \iff L(X) \simeq L(Y)$.*

Proof: The two first points follow immediately from Theorem 10.2. Note that $L(K^m) \equiv L(F^m)$ is a special case because their prime sublattice is just the two-element lattice with the same $\mathcal{L}_{\text{ASC}}$-structure, because K^m and F^m both have pure dimension m. The last point follows from Proposition 9.5 since both $L(X)$ and $L(Y)$ are standard and countable.

■

Given a pair of semi-algebraic sets $X \subseteq K^m$ and $Y \subseteq F^n$, we say that a homeomorphism $\psi : X \to Y$ is **pre-algebraic** if for every semi-algebraic sets $A \subseteq X$ and $B \subseteq Y$ defined over K and F respectively, $\psi(A)$ and $\psi^{-1}(B)$ are still semi-algebraic sets defined over K and F. It is obviously sufficient to check this for semi-algebraic sets A, B closed in X, Y respectively. In other words, a bijection $\psi : X \to Y$ is a pre-algebraic if and only if taking direct images by ψ defines an $\mathcal{L}_{\text{ASC}}$-isomorphism from $L(X)$ to $L(Y)$ (which also ensures that ψ is a homeomorphism). When $K = F$, semi-algebraic homeomorphisms are obviously pre-algebraic. The converse is false, as the following example shows.

Example 10.4 Assume that the p-valuation of K has value group $\mathbf{Z}$, and let R be its valuation ring. Applying Theorem 10.5 below to $X = K$ and $Y = R$ gives a pre-algebraic homeomorphism $\varphi : K \to R$. Since its value group is $\mathbf{Z}$, the p-valuation defines a metric on K and its completion K' is known to be an elementary extension of K. If φ would be semi-algebraic, it would then uniquely extend to a semi-algebraic homeomorphism from K' to its p-valuation ring R'. But this is not possible because R' is compact and K' is not. Thus φ is not semi-algebraic.

Theorem 10.5 *Let K, F be* countable *p-adically closed fields, and $X \subseteq K^m$, $Y \subseteq F^n$ be two semi-algebraic sets. Let $L^0(X)$ and $L^0(Y)$ be the prime $\mathcal{L}_{\text{ASC}}$-substructures of $L(X)$ and $L(Y)$ respectively. Then X and Y are pre-algebraically homeomorphic if and only if $L^0(X)$ and $L^0(Y)$ are $\mathcal{L}_{\text{ASC}}$-isomorphic. In particular, any two semi-algebraic sets over K and F with the same pure dimension $d \geq 1$ are pre-algebraically homeomorphic.*

Proof: One direction is obvious: every pre-algebraic homeomorphism $\psi : X \to Y$ induces an $\mathcal{L}_{\text{ASC}}$-isomorphism from $L(X)$ to $L(Y)$, which maps their respective prime $\mathcal{L}_{\text{ASC}}$-substructures one to each other. Conversely, assume that an $\mathcal{L}_{\text{ASC}}$-isomorphism is given from $L^0(X)$ to $L^0(Y)$. By Proposition 9.5 it extends to an $\mathcal{L}_{\text{ASC}}$-isomorphism $\varphi : L(X) \to L(Y)$. For every $t \in X$, φ maps $\{t\}$ to an atom $\{t'\}$ of $L(Y)$. Let $\psi(t) = t'$, this defines a bijection $\psi : X \to Y$ such that $\psi(A) = \varphi(A)$ for every $A \in L(X)$, hence ψ is a pre-algebraic homeomorphism. The last statement follows.

■

11 Appendix: scaled lattices in tame topological structures

We have claimed that $\mathrm{L}_{\mathrm{def}}(X)$ in Example 1.1 is a scaled lattice. In order to prove this, we first need a simpler axiomatisation of scaled lattices.

Fact 11.1 *Let L be a co-Heyting algebra and $a \in L$ an i-pure element. For every $b \in L$, if $\dim b < i$ then $a - b = a$.*

Proof: Let $b' = a - b$, and assume for a contradiction that $b' \neq a$. Then $\dim a - b' = i$ because a is i-pure. But $a = (a \wedge b) \vee (a - b)$ by TC_1, so $a - b' = (a \wedge b) - (a - b)$ by TC_2. In particular $a - b' \leq a \wedge b \leq b$, so $\dim a - b' \leq \dim b < i$, a contradiction.
■

Given a co-Heyting algebra L, let us say that an element $a \in L$ has a **pure decomposition** in L if for some integer k, $a = \bigvee_{0 \leq i \leq k} a_i$ with each a_i an i-pure element of L and $\dim a_i \wedge a_j < \min(i, j)$ for every $i \neq j$. Of course in that case $\dim a$ is the largest integer i such that $a_i \neq \mathbf{0}$.

Proposition 11.2 *If an element a in a co-Heyting algebra L has a pure decomposition $a = \bigvee_{0 \leq i \leq d} a_i$ then a_d is the largest d-pure element in L smaller than a, and $a - a_d = \bigvee_{0 \leq i < d} a_i$. In particular, such a pure decomposition (with fixed d) is unique.*

Proof: Assume that $b \in L$ is d-pure and $b \leq a$, so that $b - a = \mathbf{0}$. For every $i < d$, $\dim a_i < d$ hence $b - a_i = b$ by Fact 11.1. So $b - a = b - a_d$ by TC_4, hence $b - a_d = \mathbf{0}$ that is $b \leq a_d$. This determines a_d as the largest d-pure element in L smaller than a. Moreover $a - a_d = \bigvee_{i<d} (a_i - a_d)$ by TC_2, and each $a_i - a_d = a_i$ by Fact 11.1 (because a_i is i-pure and $\dim a_i \wedge a_d < i$ by assumption). The uniqueness of the pure decomposition follows by decreasing induction.
■

Proposition 11.3 *Let L be a $\mathcal{L}_{\mathrm{SC}}$-expansion of a co-Heyting algebra. L is a d-scaled lattice if and only if, for every $a \in L$:*

$\mathbf{SC_1^d}$*:* $a = \bigvee_{0 \leq i \leq d} \mathrm{C}^i(a)$ *and* $\forall i > d$, $\mathrm{C}^i(a) = \mathbf{0}$.

$\mathbf{SC_2}$*:* $\forall i$, $\mathrm{C}^i(a)$ *is i-pure.*

$\mathbf{SC_3}$*:* $\forall i \neq j$, $\dim \mathrm{C}^i(a) \wedge \mathrm{C}^j(a) < \min(i, j)$.

Proof: Clearly $\mathrm{SS}_1^{\mathrm{d}}$ is $\mathrm{SC}_1^{\mathrm{d}}$. Moreover SC_0 implies that $\mathrm{SS}_4 \Leftrightarrow \mathrm{SC}_3$ and $\mathrm{SS}_{13} \Leftrightarrow \mathrm{SC}_2$. So every d-scaled lattice satisfies conditions $\mathrm{SC}_1^{\mathrm{d}}$ to SC_3. Reciprocally, assume that L satisfies these conditions. Then it satisfies SC_0 (by SC_2 and $\mathrm{SC}_1^{\mathrm{d}}$) hence also $\mathrm{SS}_1^{\mathrm{d}}$, SS_4 and SS_6. The uniqueness of the pure decomposition of a implies that L satisfies also $\mathrm{SS}_2^{\mathrm{d}}$.

For every $b \in L$ and every $k \geq \dim_L b$, we have $b = \bigvee_{i \leq k} \mathrm{C}^i(b)$ by $\mathrm{SC}_1^{\mathrm{d}}$, and $\mathrm{C}^k(a) - \mathrm{C}^i(b) = \mathrm{C}^k(a)$ by Fact 11.1 and SC_2). So $\mathrm{C}^k(a) - b = \mathrm{C}^k(a) - \mathrm{C}^k(b)$ by TC_2, which proves SC_5.

It remains to check SS_3, for every $a, b \in L$ of dimension $\leq k$. Clearly $\mathrm{C}^k(a) \vee \mathrm{C}^k(b)$ is smaller than $a \vee b$ and k-pure, hence smaller than $\mathrm{C}^k(a \vee b)$ by Proposition 11.2. On the other hand by $\mathrm{SC}_1^{\mathrm{d}}$ and TC_2

$$(a \vee b) - (\mathrm{C}^k(a) \vee \mathrm{C}^k(b)) = \bigvee_{i \leq k} (\mathrm{C}^i(a) \vee \mathrm{C}^i(b)) - (\mathrm{C}^k(a) \vee \mathrm{C}^k(b)) \leq \bigvee_{i<k} \mathrm{C}^i(a) \vee \mathrm{C}^i(b). \tag{15}$$

Actually we have equality, by $\mathrm{SC}_1^{\mathrm{d}}$ and Fact 11.1. Anyway $(a \vee b) - (\mathrm{C}^k(a) \vee \mathrm{C}^k(b))$ has dimension $< k$ by (15). On the other hand, by $\mathrm{SC}_1^{\mathrm{d}}$ and TC_2, $(a \vee b) - (\mathrm{C}^k(a) \vee \mathrm{C}^k(b))$ is the join of $\mathrm{C}^i(a \vee b) - (\mathrm{C}^k(a) \vee \mathrm{C}^k(b))$ for $i \leq k$. Since $\dim(a \vee b) - (\mathrm{C}^k(a) \vee \mathrm{C}^k(b)) < k$ this implies that $\mathrm{C}^k(a \vee b) - (\mathrm{C}^k(a) \vee \mathrm{C}^k(b)) = \mathbf{0}$ hence $\mathrm{C}^k(a \vee b) \leq \mathrm{C}^k(a) \vee \mathrm{C}^k(b)$. The conclusion follows.
■

From now on, let $\mathcal{K} = (K, \dots)$ be a first-order structure defining a topology on K. Endow K^m with the product topology, and define the **dimension** of a non-empty definable set $X \subseteq K^m$ as the largest integer $r \geq 0$ such that for some coordinate projection[15] $\pi : K^m \to K^r$, $\pi(X)$ has non-empty

[15] A **coordinate projection** $\pi : K^m \to K^r$ is a function defined by $\pi(x_1, \dots, x_m) = (x_{i_1}, \dots, x_{i_r})$ for some fixed $i_1 < \dots < i_r$ in $\{1, \dots, m\}$.

interior. By convention $\dim \emptyset = -\infty$. Recall that for every $x \in X$ the local dimension $\dim(X, x)$ is the minimum of $\dim U \cap X$ as U ranges over the definable neighbourhood of x. Let $W_k(X)$ denote the set of $x \in X$ such that there is a definable neighbourhood B of x and a coordinate projection $\pi : K^m \to K^k$ which induces by restriction a homeomorphism between $B \cap X$ and an open subset of K^k. We say that $\mathcal{K}$ is a **tame topological structure** if it satisfies the following properties, for every definable sets $X, Y \subseteq K^m$ and every definable function $f : X \to K^n$.

Dim1: $\dim(f(X)) \leq \dim(X)$.

Dim2: $\dim X \cup Y = \max(\dim X, \dim Y)$.

Dim3: $\dim(X) = \dim(\overline{X})$ and if $X \neq \emptyset$, then $\dim(\overline{X} \setminus X) < \dim(X)$.

Dim4: If $\dim(X) = d \geq 0$ then $\dim(X \setminus W_d(X)) < d$.

Example 11.4 Ever o-minimal, C-minimal or P-minimal expansion of a field K is tame (see [16], [11], [2]). More generally, every dp-minimal expansion of a field K which is not strongly minimal is tame (see [14]). Following [8] we may also consider the models of visceral theories having finite definable choice and no space-filling function: all of them are tame. This applies in particular, with the interval topology, to every divisible ordered Abelian group whose theory is weakly o-minimal.

Note that by (Dim1), $\dim f(X) = \dim X$ if f is bijective. For every integer $k \geq 0$ we let

$$\Delta_k(X) = \big\{x \in X \mid \dim(X, x) = k\big\}.$$

In particular, X has pure dimensional if and only if $X = \Delta_d(X)$ with $d = \dim X$. The sets $\Delta_k(X)$ form a partition of X. For every $k \geq 0$, $\bigcup_{l \geq k} \Delta_l(X)$ is closed in X (for every k), while $W_k(X)$ is open in X.

Proposition 11.5 *With the above notation and assumptions, $W_k(X)$ is a dense subset of $\Delta_k(X)$. If non-empty, they have dimension k. In particular, X has pure dimension k if and only if $W_k(X)$ is non-empty and dense in X.*

Proof: If $x \in W_k(X)$, there is a definable neighborhood U of x in X, a coordinate projection $\pi : K^m \to K^k$ and an open subset V of K^k such that π induces by restriction a homeomorphism between U and V. In particular $\dim U = k$ by (dim1), hence $\dim W_k(X) \geq k$. For every sufficiently small neighbourhood U' of x in X we have $U' \subseteq U$, hence π induces by restriction a homeomorphism between U' and an open subset of K^k, so $\dim U' = k$. This proves that $\dim(X, x) = k$ hence $W_k(X) \subseteq \Delta_k(X)$.

We turn now to density. Pick $x \in \Delta_k(X)$ and a neighbourhood U of x in X. By shrinking U if necessary we may assume that $\dim U = k$. From (Dim4) we know that $W_k(U) \neq \emptyset$. On the other hand, $W_k(U) \subseteq W_k(X)$ because U is open in X. Consequently $W_k(U) \subseteq B \cap W_k(X)$ and so $B \cap W_k(X) \neq \emptyset$. This proves density.

By (Dim3) we have $\dim \Delta_k(A) = \dim W_k(A)$, so it only remains to check that $W_k(X)$ has dimension k, provided it is not empty. Clearly $\dim W_k(X) \geq k$. If $\dim W_k(X) = l > k$ then by (Dim4) $W_l(W_k(X))$ is non-empty. But $W_k(X)$ is open in X, hence $W_l(W_k(X))$ is contained in $W_l(X)$. So $W_l(W_k(X))$ is contained both in $W_l(X)$ and in $W_k(X)$, a contradiction since $W_l(X)$ and $W_k(X)$ are disjoint (they are contained in $\Delta_k(X)$ and $\Delta_l(X)$ respectively).

The last point follows, since X has pure dimension k if and only if $X = \Delta_k(X) \neq \emptyset$.

■

Recall that $\mathrm{L_{def}}(X)$ denotes the co-Heyting algebra of all the definable sets $A \subseteq X$ closed in X, expanded by the functions $(\mathrm{C}^i)_{i \in \mathbf{N}}$ defined by $\mathrm{C}^i(A) = \overline{\Delta_i(A)} \cap A$.

Proposition 11.6 *Let $\mathcal{K} = (K, \dots)$ be a tame topological structure, and $X \subseteq K^m$ be a definable set.*

1. *For every $A \in L$, $\dim_{\mathrm{L_{def}}(X)} A = \dim A$.*
2. *$\mathrm{L_{def}}(X)$ is a d-scaled lattice, with $d = \dim X$.*

Remark 11.7 The first item ensures that $A \in L$ is k-pure in $\mathrm{L_{def}}(A)$ if and only if it is so in the geometric sense, that is $A = \Delta_k(A)$ or equivalently (by Proposition 11.5) $W_k(A)$ is dense in A.

Proof: In order to ease the notation let $L = \mathrm{L_{def}}(X)$.

(1) We can assume that $A \neq \emptyset$. By Fact 2.3, $\dim_L A$ is then the foundation of rank of A in $L \setminus \{\mathbf{0}\}$ for the strong order $\ll$. It suffices to prove, by induction on k, that $\dim_L A \geq k$ if and only if $\dim A \geq k$. This is clear for $k = 0$ so let us assume that $k \geq 1$ and the result is proved for $k-1$.

If $\dim A \geq k$ there is a coordinate projection $\pi : K^m \to K^k$ and a non-empty definable open set $U \subseteq K^k$ contained in $\pi(K)$. Let Y be any hyperplane of K^k intersecting U, and $B = \pi^{-1}(Y \cap U) \cap A$. Clearly $Y \subseteq \overline{U \setminus Y}$ hence $B \subseteq \overline{A \setminus B}$, that is $B \ll A$. Since $\dim Y = k-1$ we have $\dim B \geq k-1$ by (1), hence $\dim_L B \geq k-1$ by induction hypothesis, and finally $\dim_L A \geq k$ since $B \ll A$.

Reciprocally, if $\dim_L A \geq k$ by Fact 2.3 there is $B \in L$ such that $B \ll A$ and $\dim_L B \geq k-1$. By induction hypothesis $\dim B \geq k-1$. We have $B \subseteq \overline{A \setminus B} \cap B \subseteq \overline{A \setminus B} \setminus (A \setminus B)$, so $\dim B < \dim A \setminus B$ by (Dim3). *A fortiori* $\dim B < \dim A$ hence $\dim A \geq k$.

(2) For every $i \leq m$ and every $A \in \mathrm{L_{def}}(X)$, $\mathrm{C}^i(A) = \overline{\Delta_i(A)} \cap A = \overline{W_i(A)} \cap A$ by Proposition 11.5. The scheme $\mathrm{SC}_1^{\mathrm{d}}$ then follows from (Dim4) by a straightforward induction. Moreover each $\mathrm{C}^i(A)$ is i-pure in L by Remark 11.7, hence SC_2 holds true. Finally, for every $i < j$, since $W_i(A)$ is open in A and disjoint from $W_j(A)$, it is also disjoint from $\overline{W_j(A)} \cap A$ hence

$$\mathrm{C}^i(A) \cap \mathrm{C}^j(A) = \overline{W_i(A)} \cap \overline{W_j(A)} \cap A \subseteq \overline{W}_i \setminus W_i.$$

So $\dim \mathrm{C}^i(A) \cap \mathrm{C}^j(A) < i$ by (Dim3), which proves SC_3. So $\mathrm{L_{def}}(X)$ is a scaled lattice by Proposition 11.3.

■

We turn now to the Catenarity Property. We do not expect it to be completely general. This property is well known over for *o*-minimal fields (it follows immediately from the triangulation theorem). We are going to prove it for every dp-minimal expansion $\mathcal{K} = (K, v, \dots)$ of a non-trivially valued field having definable Skolem functions. This assumption on Skolem function is somewhat restrictive but it includes the case of any *p*-adic field with its semi-algebraic structure (or even its subanalytic structure), which is sufficient for our needs. We will use Proposition 3.7 in [14], which says that:

Dim5: Every definable function $f : X \subseteq K^k \to K^l$ is continuous on a definable set X' dense in X.

Proposition 11.8 *Let $\mathcal{K} = (K, v, \dots)$ be a dp-minimal expansion of a non-trivially valued field (K, v) having definable Skolem functions. For every non-negative integers $0 \leq r < q < p \leq m$ and every definable sets $C \subseteq A \subseteq K^m$, if A is p-pure and $\dim C \leq r$, there exists a q-pure definable set $B \subseteq A$ such that $C \subseteq \overline{B}$.*

The catenarity of $\mathrm{L_{def}}(X)$, for every definable set $X \subseteq K^m$, follows immediately.

Proof: We are going to simplify the problem several times, using repeatedly the obvious facts that: (i) every open subset of a p-pure set is p-pure, and so is its closure; (ii) the union of finitely many p-pure sets is p-pure, and; (iii) if $T \subseteq \overline{S} \subseteq K^m$ then $\pi(T) \subseteq \pi(\overline{S})\overline{\pi(S)}$ for every coordinate projection $\pi : K^m \to K^k$.

Step 1. For every $I \subseteq \{1, \dots, m\}$ with p elements let $\pi_I : (x_i)_{1 \leq i \leq m} \mapsto (x_i)_{i \in I}$ be the corresponding coordinate projection. Let A_I be the set of $a \in A$ such that π_I induces by restriction a homeomorphism between a neighbourhood of a in A and an open subset of K^p, and let $C_I = C \cap \overline{A_I}$. Each A_I is p-pure, and by Proposition 11.5 their union is dense in A, hence C is the union of the C_I's. So it suffices to find for each I a q-pure definable set $B_I \subseteq I$ such that $C_I \subseteq \overline{B_I}$, and let B be their union. This reduces to the case where $A = A_I$ and $C = C_I$ for some I.

Step 2. Observe that that $\pi_I(C)$ is contained in the closure of $\pi_I(A)$. By the previous step $\pi_I(A)$ is open in K^p, hence p-pure. Assume that we can find a q-pure subset Y of $\pi_I(A)$ whose closure contains $\pi_I(C)$. Let $B = \pi_I^{-1}(Y) \cap A$, this is a p-pure subset of A (because the restriction of π_I to A is a local homeomorphism) and $C \subseteq \overline{B}$. So it suffices to solve the problem for $\pi_I(A)$ and $\pi_I(C)$. With other words, we can assume that A is an open subset of K^m, hence $p = m$.

Step 3. For every $J \subseteq \{1, \dots, m\}$ with $s \leq r$ elements, let C_J be the set of $c \in C$ such that π_J (defined as in the first reduction) induces by restriction a homeomorphism between a neighbourhood of c in C and an open subset of K^s. If we can find for each J a p-pure definable set $B_J \subseteq A$ such that C_J is contained in the closure of B_J, then we are done by letting B be the union of the B_J's. This (and a decreasing induction on r) reduces to the case where $C = C_J$ for some J, hence π_J is a local homeomorphism from C to an open subset of K^r. Reordering the coordinates if necessary we can then assume that $J = \{1, \dots, r\}$.

Step 4. Let $Z = \pi_J(C)$ and $X = \pi_J(A)$. We have $Z \subseteq \overline{X}$ hence the dimension of $Z \setminus X$ is $< r$ by (Dim3). So is the dimension of $\pi_J^{-1}(Z \setminus X) \cap C$ (because π_J is now a local homeomorphism). Since C is the union of $\pi_J^{-1}(Z \setminus X) \cap C$ and $\pi_J^{-1}(Z \cap X) \cap C$, by a straightforward induction on the dimension of C this reduces to the case where $C = \pi_J^{-1}(Z \cap X) \cap C$, that is $Z \cap X = Z$, or equivalently $\pi_J(C) \subseteq \pi_J(A)$.

Step 5. Since π_J is a local homeomorphism on C, over any point $z \in \pi_J(C)$ the fibers $C_z = \pi_J^{-1}(z) \cap C$ are discrete, hence finite by Proposition 1.1 in [14]. The same holds true in every elementary extension of $\mathcal{K}$ so, by the model-theoretic compactness theorem, their cardinality must be uniformly bounded by some integer N. For every $k \leq N$ let C_k be the set of $c \in C$ such that the fibers of π_J over $\pi_J(c)$ has cardinality k. This is a finite partition of C in definable set. It suffices to solve the problem separately for A and each C_k, which reduces to the case where $C = C_k$ for some k.

Step 6. We can find definable Skolem functions $f_1, \dots, f_k$ from $\pi_J(C)$ to K^m such that for each $z \in \pi_J(C)$, the fiber $\pi_J^{-1}(z) \cap C = \{f_1(z), \dots, f_k(z)\}$. For each $l \leq k$ let $C_l = f_l(\pi_J(C))$. This is again a finite partition of C in definable sets. So the problem boils down to the case where $C = C_l$ for some l, that is π_J induces a bijection from C to $Z = \pi_J(C)$, and $f = f_1$ is the reciprocal bijection. After this reduction we cannot assume anymore that $Z = \pi_J(C)$ is open in K^r. However, the complement in Z of the interior of Z in K^r has dimension $< r$ by (Dim4). By (Dim5) the set of discontinuities of f also has dimension $< r$. Hence, by a straightforward induction on the dimension of C, we can reduce to the case where Z is open, and $f : Z \to C$ and the restriction of π_J are reciprocal homeomorphisms.

Step 7. One can easily check that C is contained in the closure of $A' = \pi_J^{-1}(Z) \cap A$. The latter is open. This reduces to the case where $A = A'$, that is $\pi_J(A) = \pi_J(C)$. In particular, the restriction ρ of $f \circ \pi_J$ to $A \cup C$ then defines a continuous retraction onto C (that is ρ is continuous on $A \cup C$ and $\rho(c) = c$ for every $c \in C$).

Using this retraction we can now finish the proof. We do it when $q = m - 1$, that is $q = p - 1$, the result for smaller values of q following immediately by decreasing induction. For every $k \in \{1, \dots, m\}$ and every $a = (a_1, \dots, a_m) \in A$ let $\rho_k(a)$ be the k-th coordinate of $\rho(a)$, so that $\rho(a) = (\rho_1(a), \dots, \rho_m(a))$. Note that $\pi_J(a) = \pi_J(\rho(a))$ by construction, hence $\rho(a) = (a_1, \dots, a_r, \rho_{r+1}(a), \dots, \rho_m(a))$. For each $k \in \{r+1, \dots, m\}$ let

$$A_k = \big\{a \in A \,|\, v\big(a_k - \rho_k(a)\big) \geq \min_{l \neq k} v\big(a_l - \rho_l(a)\big)\big\}.$$

This is the set of points $a \in A$ such that a_k is not strictly further from $\rho_k(a)$ than is a from $\rho(a)$ (see Figure 1). Clearly A_k is definable, open, and A is the union of the A_k's. In particular C is contained in the union of the $\overline{A_k}$'s.

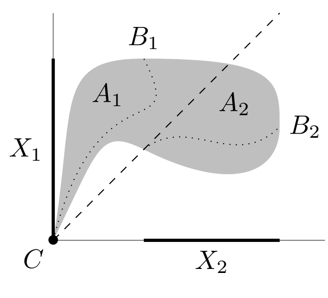

Figure 1: In K^2, the dashed line splits A (in gray) in two parts A_1, A_2.

For each $k \in \{r+1, \dots, m\}$ let $\pi_k : K^m \to K^{m-1}$ be the projection which forgets the k-th coordinate. Let θ_k be a definable section of the restriction of π_k to A_k (given by definable Skolem functions). By (Dim5) there is a definable set $X_k \subseteq \pi_k(A_k)$ dense in $\pi_k(A_k)$ such that θ_k is continuous on X_k. Finally let $B_k = \theta_k(X_k)$ (the dotted lines in Figure 1). Recall that A_k is open in K^m, hence so is $\pi_k(A_k)$ in K^{m-1}. In particular $\pi_k(A_k)$ is $(m-1)$-pure, hence so is X_k. By construction, the restriction of θ_k to X_k is a homeomorphism, so B_k is $(m-1)$-pure. Letting B be the union of the B_k's, it only remains to check that $C \subseteq \overline{B}$.

In order to do so, pick any $c = (c_1, \dots, c_m) \in C$. There is $k \in \{r+1, \dots, m\}$ such that $c \in \overline{A_k}$, hence $\pi_k(c) \in \overline{X_k}$. It suffices to prove that $\theta_k(x)$ tends to c as x tends to $\pi_k(c)$ in X_k, in order to conclude that $c \in \overline{B_k}$, and finally that $C \subseteq \overline{B}$. Let $\pi_{J,k} : K^{m-1} \to K^r$ be such that $\pi_J = \pi_{J,k} \circ \pi_k$. For every $x \in X_k$, let $a = (a_1, \dots, a_m) = \theta_k(x)$ and observe that $\pi_J(a) = \pi_{J,k}(x)$, so

$$\underbrace{f \circ \pi_J(a)}_{=\rho(a)} = f \circ \pi_{J,k}(x) \xrightarrow[x \to \pi_k(c)]{} f \circ \pi_{J,k}\big(\pi_k(c)\big) = \underbrace{f \circ \pi_J(c)}_{=\rho(c)=c}. \tag{16}$$

Consequently $\pi_k(\rho(a)) \xrightarrow[x \to \pi_k(c)]{} \pi_k(c)$, so

$$\pi_k(a) - \pi_k\big(\rho(a)\big) = x - \pi_k\big(\rho(a)\big) \xrightarrow[x \to \pi_k(c)]{} \pi_k(c) - \pi_k(c) = (0, \dots, 0),$$

that is

$$\min_{l \neq k} v\big(a_l - \rho_l(a)\big) \xrightarrow[x \to \pi_k(c)]{} +\infty. \tag{17}$$

We have $a = \theta_k(x) \in A_k$ so, by definition of A_k,

$$\min_{1 \leq l \leq m} v\big(a_l - \rho_l(a)\big) = \min_{l \neq k} v\big(a_l - \rho_l(a)\big) \tag{18}$$

By(17) and (18), we get that $a - \rho(a)$ tends to $(0, \dots, 0)$ as x tends to $\pi_k(c)$. So by (16)

$$\theta_k(x) = a = \rho(a) + \big(a - \rho(a)\big) \xrightarrow[x \to \pi_k(c)]{} c.$$

■

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