Triangulation of p-adic semi-algebraic sets

Luck Darnière

Thursday, November 2^{nd}

1. Introduction
   - Semi-algebraic sets
   - *p*-adically closed fields
   - Quantifiers elimination
   - Which triangulation?
2. Simplicial complexes
3. Main result and applications

1.1 - Semi-algebraic sets

K is any field.

$$\mathbf{P_N} := \{y^N \;/\; y \in K\} \qquad \left(\mathbf{P_N^\times} := P_N \setminus \{0\}\right).$$

$A \subseteq K^m$ is **semi-algebraic** if it is a finite union of sets defined by:

$$f_1 = \cdots = f_r = 0 \;\text{ and } g_1 \in P_{N_1}^\times \text{ and } \;\cdots\; \text{ and } g_s \in P_{N_s}^\times.$$

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Remarks

- If K is algebraically closed, $g_i \in P_N^\times \iff g_i \neq 0$.
- If K is real closed:

$$g_i \in P_{2n}^\times \iff g_i > 0.$$

$$g_i \in P_{2n+1}^\times \iff g_i \neq 0,$$

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K is **p-adically closed** if $\mathbf{Q} \subseteq K$ and there is a valuation v on K such that:

1. (K, v) is Henselian.
2. The residue field of (K, v) is finite, with characteristic p.
3. The value group $\mathcal{Z} = v(K^\times)$ is a **Z-group**:
   i) $\mathcal{Z}$ has a smallest element > 0 ;
   ii) $\mathcal{Z}/n\mathcal{Z} \simeq \mathbf{Z}/n\mathbf{Z}$ for every $n \geq 1$.

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Theorem (Chevalley (19??), Tarski (1948), Macintyre (1976))

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This means that for every such field K:

- By stabilizing algebraic sets (defined by $f = 0$ with f pol.) projections and boolean combinations we obtain *exactly* the semi-algebraic sets.
- $A \subseteq K^m$ is semi-algebraic $\iff$ A is definable (in the language of rings).

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Tools

- Cell decomposition.
- "Good Direction" Lemma.
- Simplexes (faces, splitting...).

2.1 - The real case

A **real polytope** A is the *strict* convex hull of a finite set $A_0 \subseteq \mathbf{R}^q$ (the points of its frontier ∂A are excluded).

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Properties

Let $A \subseteq \mathbf{R}^q$ be a real polytope.

1. *A is relatively open and precompact.*
2. *A can be defined by finitely many inequalities on linear maps.*
3. *Every face of A is a polytope.*
4. *The faces of A form a complex and a partition of $\overline{A}$.*

The **specialisation order** on the subsets of a topological space is defined by

$$B \leq A \iff B \subseteq \overline{A}.$$

The **facets** of a polytope are its proper faces which are maximal (with respect to the specialization order).

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Proposition

Let $A \subseteq \mathbf{R}^q$ be a real polytope.

1. *A has at least $\geq \dim(A) + 1$ facets.*
2. *Equality holds $\iff$ A is a simplex.*

2.2 - Topological complexes

Let X be a topological space, and $\mathcal{A}$ a *finite* family of subsets of X. $\mathcal{A}$ is a **complex** of subsets of X if:

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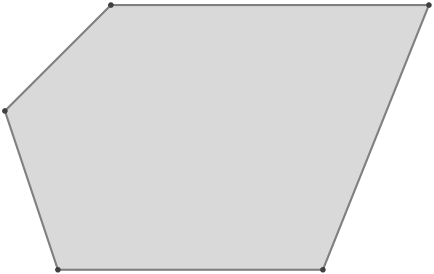
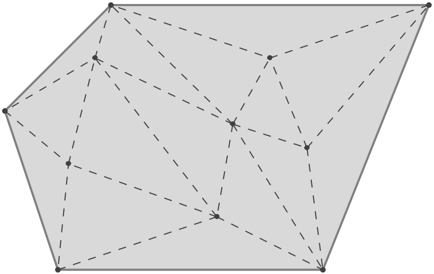
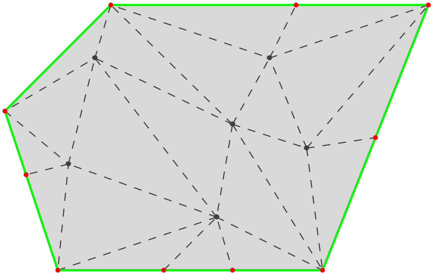
NB: $\overline{A}_1 \cap \overline{A}_2 = \bigcup\{B \in \mathcal{A} \,/\, B \leq A_1 \text{ and } B \leq A_2\}$.



The proper faces of a real polytope A form a complex.



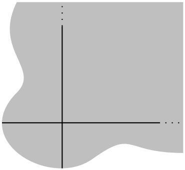
Every polytope is the union of a simplicial complex.



Any given simplicial complex
refining the complex of proper faces of A
can be extended by "Barycentric Division"
to a simplicial complex partitionning $\overline{A}$.

2.3 - The discrete case

For this talk we will take $\mathcal{Z} = \mathbf{Z}$, but any other $\mathbf{Z}$-group will be all right. We let $\Gamma := \mathbf{Z} \cup \{+\infty\}$.



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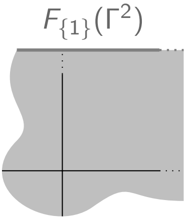

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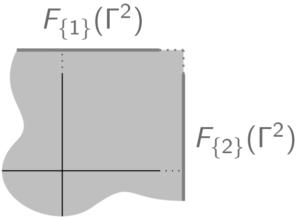

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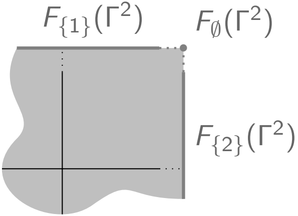

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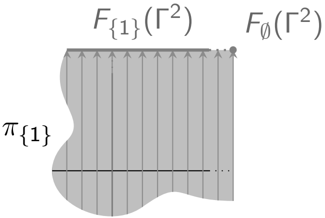

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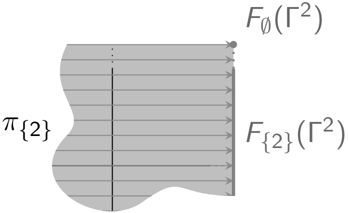

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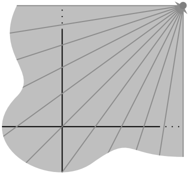

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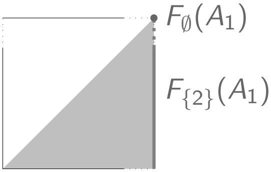
$$\mathbf{F_I(A)} := \left\{ a \in \overline{A} \,/\, \text{ Supp}\, a = I \right\} = \overline{A} \cap F_I(\Gamma^q).$$

If non-empty, $F_I(A)$ is the **face** of A with support I.



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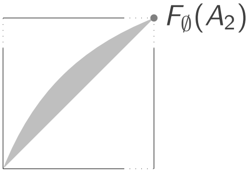

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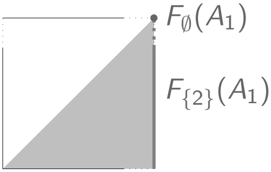

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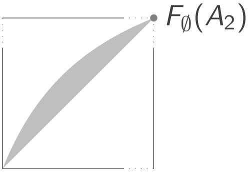

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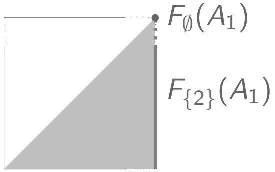

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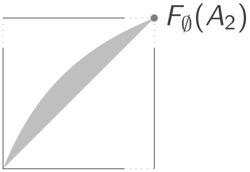

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NB$_1$: Every subset of Γ^q which is bounded below is precompact.
NB$_2$: The set of faces of $A \subseteq \mathbf{Z}^3$ is not always a complex!

$A \subseteq \mathbf{Z}^q$ is **semi-linear mod N** if it is defined by

$$f_1(x) \geq 0 \text{ and } \cdots \text{ and } f_r(x) \geq 0 \text{ and } g_1(x) \in N\mathbf{Z} \text{ and } g_s(x) \in N\mathbf{Z}$$

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The following conditions:

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However $F_{\{3\}}(A) = \{+\infty\} \times \{+\infty\} \times 2\mathbf{N}$ is only semi-linear *mod 2*.

Proposition

Let $A \subseteq \mathbf{Z}^q$ *be semi-linear set mod* N*. Let* $I, J \subseteq \{1, \ldots, q\}$ *be such that* $F_I(A)$ *and* $F_J(A)$ *are non-empty.*

1. $F_I(A) = \pi_I(A)$ *is the projection of* A *over* $F_I(\Gamma^q)$*.*
2. $F_J(A) \leq F_I(A) \iff J \subseteq I$*. When this happens* $F_J(A) = F_J(F_I(A))$*.*
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Problems

- The faces of a semi-linear set (mod N) *aren't* semi-linear (mod N') in general.

Proposition

Let $A \subseteq \mathbf{Z}^q$ be semi-linear set mod N. Let $I, J \subseteq \{1, \ldots, q\}$ be such that $F_I(A)$ and $F_J(A)$ are non-empty.

1. $F_I(A) = \pi_I(A)$ *is the projection of* A *over* $F_I(\Gamma^q)$.
2. $F_J(A) \leq F_I(A) \iff J \subseteq I$. *When this happens* $F_J(A) = F_J(F_I(A))$.
3. $F_{I \cap J}(A) \neq \emptyset$.

It follows that the set of proper faces of A *is a distributive lower semi-lattice which partitions* ∂A.

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- The faces of a semi-linear set (mod N) *aren't* semi-linear (mod N') in general. They are **Presburger sets** (= finite union of semi-linear sets mod N').

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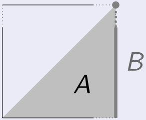
Problems

- The faces of a semi-linear set (mod N) *aren't* semi-linear (mod N') in general. They are **Presburger sets** (= finite union of semi-linear sets mod N').
- If $A \subseteq \mathbf{Z}^q$ is a Presburger set, the proposition is no longer true.

Proposition (Dichotomy)

Let $A \subseteq F_I(\Gamma^q)$ be a semi-linear set mod N. Let B be a proper face of A, and $f : A \cup B \to \Gamma$ be a function which is continuous on $A \cup B$ and **Q***-linear on A.*



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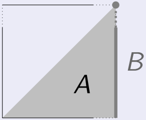

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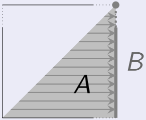

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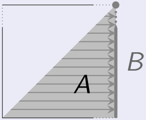

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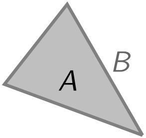

- If $\varepsilon(b) = 0$ at some point $b \in B$ then $\varepsilon_{|B} = 0$.
- Otherwise $\varepsilon(a) = \varepsilon(b)$ for every $a \in A$ and $b \in B$.

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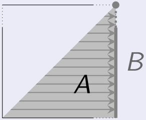

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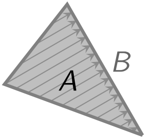

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The distance $\delta : \Gamma \to \mathbf{R}_+$ extends to $\boldsymbol{\Omega} := \mathbf{Q} \cup \{+\infty\}$.

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Example

On $X = \mathbf{Z}^2$ the function $f(x, y) = x - y$ is continuous but not largely continuous: it has no limit at $(+\infty, +\infty)$.

The **basement** of $A \subseteq \Gamma^{q+1}$ is its projection $\widehat{\mathbf{A}}$ onto Γ^q.

$A \subseteq \mathbf{Z}^q$ is **discrete polytope** if $A = \mathbf{Z}^0$ or $q \geq 1$ and

$$(x, t) \in A \iff x \in \widehat{A} \text{ and } \mu(x) \leq t \leq \nu(x),$$

where $\widehat{A}$ is a discrete polytope, $\mu, \nu : \widehat{A} \to \Omega$ are **Q**-linear maps (or $+\infty$), *largely continuous* and *non-negative*. Such a couple (μ, ν) is a **presentation** of A.

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NB: Every discrete polytope is precompact and semi-linear.
In particular, for every face $B = F_J(A)$ we have $B = \pi_J(A)$.
We then denote by $\pi_{\mathbf{B}} := \pi_J$ the **projection of A onto B**.

Proposition

Let $A \subseteq F_I(\Gamma^{q+1})$ *be a polytope and* $B = F_J(A)$ *be a face of* A*.*

1. $\widehat{B} = F_{\widehat{J}}(\widehat{A})$ *with* $\widehat{J} := J \setminus \{q+1\}$*.*

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1. $\widehat{B} = F_{\widehat{J}}(\widehat{A})$ *with* $\widehat{J} := J \setminus \{q+1\}$.
2. *Let* (μ, ν) *be a presentation of* A. *Then* $(x, t) \in F_J(\Gamma^{q+1})$ *belongs to* B *iff:*

$$x \in \widehat{B} \quad \text{and} \quad \bar{\mu}(x) \leq t \leq \bar{\nu}(x).$$

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Thus B is a polytope and $(\bar{\mu}, \bar{\nu})_{|\widehat{B}}$ *is a presentation of B.*

Reminder

Real simplexes are, among the polytopes of any given dimension, those whose number of facets is minimal ($= \dim A + 1$).

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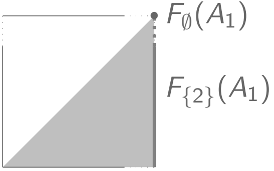
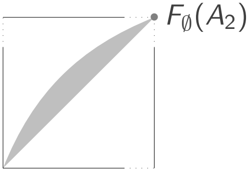
A discrete polytope is a **simplex** if is has got *at most one* facet, which is a simplex. Hence it is a simplex iff its faces form a *chain*.

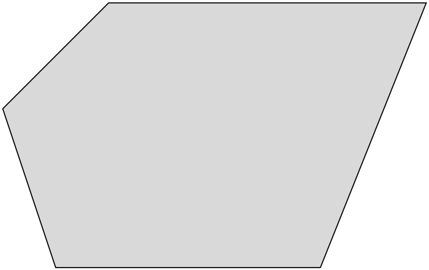


$A_1 \ : \ 0 \le y \le x$



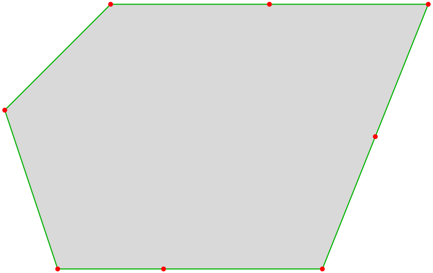
$A_2 \ : \ 0 \le x \le y \le 2x$

2.4 - Division

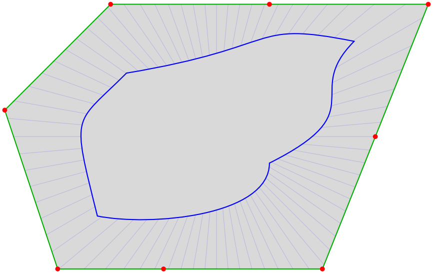


A is a polytope.

2.4 - Division



$\mathcal{T}$ is a simplicial complex refining the complex of proper faces of A.



$\varepsilon : \partial A \to K^\times$ controls the distance to the boundary:
$\forall T \in \mathcal{T}, \quad V_T(\varepsilon) := \{a \in A \,/\, \|a - \pi_T(a)\| \leq \|\varepsilon(\pi_T(a))\|\}$
is a "neighborhood of T inside A".

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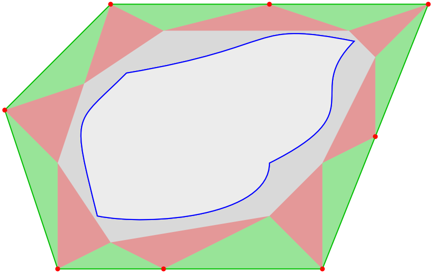
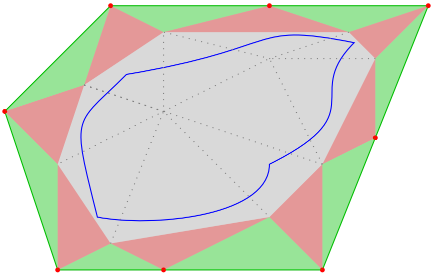


$T \in \mathcal{T}$ can be "inflated" *inside* $V_T(\varepsilon)$ to a simplex S_T whose facet is T.

2.4 - Division



The remaining of A splits in (clopen?) simplexes.

Proposition (Monotopic division with constraint)

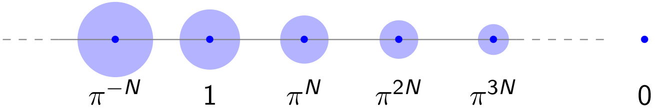
Let $A \subseteq \Gamma^q$ be a polytope and $\mathcal{T}$ a simplicial complex refining the complex of proper faces of A. Let $\varepsilon : \partial A \to \mathbf{Z}$ be a linear function.
Then there exists a simplicial complex $\mathcal{S}$ in Γ^q such that:

1. $\mathcal{T} \subseteq \mathcal{S}$ *and* $\bigcup \mathcal{S} = \overline{A}$;
2. $\forall T \in \mathcal{T}$, *there is a unique* $S_T \in \mathcal{S}$ *with facet* T ;
3. $\forall a \in S_T,\ \delta(a, \pi_T(a)) \leq 2^{-\varepsilon(\pi_T(a))}$;
4. *every other* $S \in \mathcal{S}$ *is clopen.*

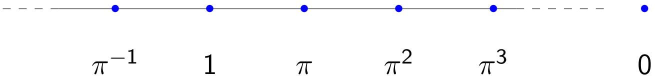
2.5 - The p-adic case

From now we let K be a p-adically closed field.
For sake of simplicity we assume that $v(K) = \Gamma = \mathbf{Z} \cup \{+\infty\}$.

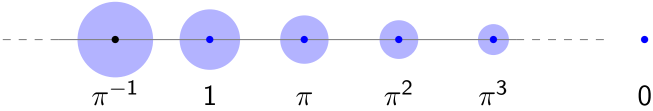
- **R**$:=$ the p-adic valuation ring.
- $\pi :=$ a generator of the maximal ideal of R.
- For every $x \in K^q$, $\|\mathbf{x}\| := \max\limits_{1 \leq i \leq q} 2^{-v(x_i)}$.
- $\mathbf{B(x,r)} := \{y \in K^q \;/\; \|x - y\| \leq \|r\|\}$.
- $\mathbf{Q_{N,M}} := \bigcup_{k \in \Gamma} \pi^{Nk}(1 + \pi^M R) = \bigcup_{k \in \Gamma} B(\pi^{Nk}, \pi^{Nk+M})$.



$\{\pi^k\}_{k \in \mathbf{Z}}$ is not a semi-algebraic set.



$\{\pi^k\}_{k\in\mathbf{Z}}$ is not a semi-algebraic set. But $Q_{1,M}^\times$ is a semi-algebraic neighborhood of $\{\pi^k\}_{k\in\mathbf{Z}}$ (and a sub-group of $K^\times$ with finite index).



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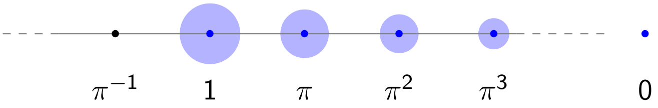

$\mathbf{D^MR}$ $:= Q_{1,M} \cap R$.

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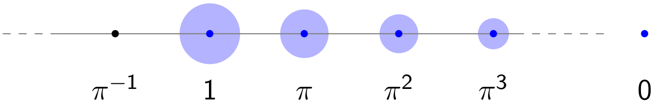

$\mathbf{D^M R} := Q_{1,M} \cap R$.

A ***p*-adic polytope** is the pre-image, by the *p*-adic valuation restricted to $D^M R^q$, of a discrete polytope (in Γ^q). Same thing for ***p*-adic simplexes**.

NB: *p*-adic polytopes inherit from discrete polytopes all their nice properties regarding faces, projections, présentations... and monotopic division!

A **simplicial complex of index M** is a finite family $\mathcal{T} = (\mathcal{T}_i)_{1 \leq i \leq n}$ where each $\mathcal{T}_i$ is a simplicial complex in $D^M R^{q_i}$.

3.1 - Triangulation and monomialisation

A **simplicial complex of index M** is a finite family $\mathcal{T} = (\mathcal{T}_i)_{1\leq i\leq n}$ where each $\mathcal{T}_i$ is a simplicial complex in $D^M R^{q_i}$.

Theorem (Triangulation of sets)

For every semi-algebraic $A \subseteq K^m$, there exists a simplicial complex $\mathcal{T}$ of index M and a semi-algebraic homeomorphism $\varphi : \biguplus \mathcal{T} \to A$. Moreover M can be taken arbitrarily large.

Here $\biguplus \mathcal{T}$ denotes the disjoint union of the $\bigcup \mathcal{T}_i$'s.

NB: This can be done simultaneously for a finite family $(A_i)_{i\in I}$ of semi-algebraic sets. We call $(\mathcal{T}, \varphi)$ a **triangulation** of $(A_i)_{i\in I}$.

- $\mathbf{U_e} := \{x \in K \ / \ x^e = 1\}$.

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- $\mathbf{U_{e,n}} := U_e \cdot (1 + \pi^n R) = \bigcup_{e \in U_e} B(e, \pi^n)$

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NB: $U_{e,n}$ is a sub-group of $K^\times$ and a neighborhood of U_e.

f is **N-monomial mod $U_{e,n}$** on a domain $S \subseteq K^q$ if there exists a semi-algebraic $u : S \to U_{e,n}$, $\xi \in K$ and $\beta_1, \ldots, \beta_q \in \mathbf{Z}$ such that

$$\forall x \in S, \quad f(x) = u(x) \cdot \xi \cdot \prod_{i=1}^{q} x_i^{N\beta_i}$$

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$$\forall x \in S, \quad f(x) = u(x) \cdot \underbrace{\xi \cdot \prod_{i=1}^{q} x_i^{N\beta_i}}_{g(x)}$$

This is equivalent to say that $f = \chi \cdot (1 + \pi^n \omega) \cdot g$ with $\chi : S \to U_e$, $\omega : S \to R$ and g N-monomial (all semi-algebraic). With other words:

$$\left\| \frac{f}{g\chi} - 1 \right\| \leq \|\pi^n\|.$$

Theorem (Triangulation/monomialisation of functions)

Let $(\theta_i : A_i \subseteq K^m \to K)_{i \in I}$ be a finite family of semi-algebraic functions and n, N be positive integers. Them there exists a semi-algebraic triangulation $(\mathcal{T}, \varphi)$ of $(A_i)_{i \in I}$ of index M such that:

> *each $\theta_i \circ \varphi_{|T}$ is N-monomial mod $U_{e,n}$*
> *(for every $i \in I$ and $T \in \mathcal{T}$, provided $\varphi(T) \subseteq A_i$).*

Moreover e, M can be taken arbitrarily large.

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Moreover e, M can be taken arbitrarily large.

We let $\mathbf{T}_m$ denote this statement.
$(\mathcal{T}, \varphi)$ is an ***N*-monomialisation** (**mod** $U_{e,n}$ of **index** M) of the θ_i's.

3.2 - Lifting

Theorem

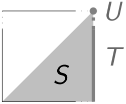
Let $\eta : A \subseteq K^m \to K$ be a semi-algebraic function such that $\|\eta\|$ is continuous. Then there exists a semi-algebraic continuous function $h : A \subseteq K^m \to K$ such that $\|h\| = \|\eta\|$.

Sketchy proof

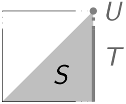
$\mathbf{T}_m$ reduces to the case where:

- $A = \overline{S}$ with S a simplex in $D^M R^q$;
- $\eta : \overline{S} \to K$ is N-monomial mod $U_{e,n}$ on every face of S.

Note that $v \circ \eta$ then defines a $\mathbf{Z}$-linear map on every face of $v(S)$.



- $v(\eta(x, y)) = \alpha_0 + \alpha_1 v(x) + \alpha_2 v(y)$ on S ;
- $v(\eta(+\infty, y)) = \beta_0 + \beta_2 v(y)$ on T ;
- $v(\eta(+\infty, +\infty)) = +\infty$ on U.



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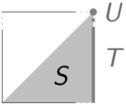
Let $\eta^* : v(\overline{S}) \to \mathbf{Z}$ be defined by:

- $\eta^*(x',y') = \alpha_0 + \alpha_1 x' + \alpha_2 y'$ on $v(S)$;
- $\eta^*(+\infty,y') = \beta_0 + \beta_2 y'$ on $v(T)$.
- $\eta^*(+\infty,+\infty) = +\infty$ on $v(U)$.

We have $\eta^*(v(x),v(y)) = v(\eta(x,y))$, and η^* is continuous on $v(\bar{S})$.



- $v(\eta(x,y)) = \alpha_0 + \alpha_1 v(x) + \alpha_2 v(y)$ on S ;
- $v(\eta(+\infty,y)) = \beta_0 + \beta_2 v(y)$ on T ;
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Let $\eta^* : v(\overline{S}) \to \mathbf{Z}$ be defined by:

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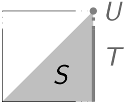
Since η^* is $\mathbf{Z}$-linear on $v(S)$ and $\eta^* \neq +\infty$ on $v(T)$, by the Dichotomy Proposition $\eta^*_{|v(S)} = \eta^*_{|v(T)} \circ \pi_{\{2\}}$.



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We have $\eta^*(v(x),v(y)) = v(\eta(x,y))$, and η^* is continuous on $v(\bar{S})$. Since η^* is $\mathbf{Z}$-linear on $v(S)$ and $\eta^* \neq +\infty$ on $v(T)$, by the Dichotomy Proposition $\eta^*_{|v(S)} = \eta^*_{|v(T)} \circ \pi_{\{2\}}$. Hence for every $(x,y) \in S \cup T$:

$$v(\eta(x,y)) = \eta^*(v(x),v(y)) = \eta^*(+\infty,v(y)) = \beta_0 + \beta_2 v(y).$$

It then suffices to let $h(x,y) = \pi^{\beta_0} y^{\beta_2}$ on $S \cup T$, and $h = 0$ on U.

3.3 - Retractions

A **retraction** of a non-empty set $A \subseteq K^m$ onto $B \subseteq A$ is a continuous map $\rho : A \to B$ such that $\rho(x) = x$ for every $x \in B$.

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NB: If such a retraction exists then B is closed in A.

Theorem

Let $B \subseteq A \subseteq K^m$ be non-empty semi-algebraic sets. There exists a semi-algebraic retraction of A onto $B \iff B$ is closed in A.

Sketchy proof

$\mathbf{T}_m$ reduces to the case where $A = \overline{S}$ and $B = \overline{T}$ with S a simplex and T a face of S. We can then take $\rho = \pi_T$.

3.4 - Splitting

Theorem

Let $A \subseteq K^m$ be a relatively open semi-algebraic without isolated points. Let $X_1, \ldots, X_r$ closed semi-algebraic sets such that $X_1 \cup \cdots \cup X_r = \partial A$. Then there exists a partition of A in semi-algebraic sets $A_1, \ldots, A_r$ such that $\partial A_k = X_k$ for $1 \leq k \leq r$.

Sketchy proof

$\mathbf{T}_m$ reduces to the case where A is simplex of $D^M R^q$. For sake of simplicity let us assume that $r = 2$ and $X_1 = X_2 = \overline{B}$ where B is the facet of A.

Let $i \in \operatorname{Supp} A \setminus \operatorname{Supp} B$. We can then take:

$$A_1 = \left\{ a \in A \,/\, v(a_i) \in 2\mathbf{N} \right\} \qquad A_2 = A \setminus A_1.$$

3.5 - Lattices of intersection

Let X be a semi-algebraic subset of K^m. Let $L(X)$:= the lattice of semi-algebraic subsets of X *closed in* X.

Theorem (Grzegorczyk 1951)

If K is algebraically closed or real closed, and if $\dim X \geq 2$ *then* $L(X)$ *is undecidable.*

NB: Crucial in the proof is the existence of irreducible or connected components.

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3. *If* $K \preceq F$ *and* X, Y *are defined by the same formula then* $L(X) \preceq L(Y)$.

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NB_2: The theory of $L(X)$ depends on $\dim X$, on the pure dimensionnal compenents of X, *etc* but *does not* depend on p.
In particular $L(Q_{p_1}^m) \equiv L(Q_{p_2}^m)$.