Semi-algebraic triangulation over p-adically closed fields

Luck Darnière

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Abstract

We prove a triangulation theorem for semi-algebraic sets over a p-adically closed field, quite similar to its real counterpart. We derive from it several applications like the existence of flexible retractions and splitting for semi-algebraic sets.

Contents

1 Introduction

Our knowledge of geometric objects in affine spaces over p-adic fields, that is the field $\mathbf{Q}_p$ of p-adic numbers or a finite extension of it, has grown tremendously in the past decades. Several remarkable analogies have emerged with real geometry, in spite of the striking differences between the real and the p-adic metrics. The present paper raises a new such analogy: we prove a triangulation theorem over p-adically closed fields, quite similar to its real counterpart. Let us first recall the classical results in p-adic geometry which will be used here.

Semi-algebraic sets over a field F are the finite unions of sets defined by finitely many conditions "$f(x) = 0$" or "$f(x)$ has a non-zero N-th root in F", where f is a polynomial function with m variables. Of course if F is real closed we can restrict the last conditions to $N = 2$ (that is to "$f(x) > 0$") and if F is algebraically closed to $N = 1$ (that is to $f(x) \neq 0$). It is shown in [Mac76] that semi-algebraic sets over $\mathbf{Q}_p$ are stable under the taking of boolean combinations *and* projections (from $\mathbf{Q}_p^m$ to $\mathbf{Q}_p^{m-1}$, for every m). This is a p-adic version of Tarski's theorem for real closed fields (and of Chevalley's theorem for algebraically closed fields). Prestel and Roquette (see [PR84]) have generalized it to arbitrary p-adically closed fields (a p-adic version of real closed fields).

Denef has proved in [Den84] a Cell Decomposition Theorem for p-adic semi-algebraic sets very similar to its real counterpart, and derived from it the rationality of Poincaré series. Another major application of cell decomposition is that it provides a good dimension theory for semi-algebraic sets (see [SvdD88]). Nowadays a cell over $\mathbf{Q}_p$ is generally defined as the set of $(x,t) \in \mathbf{Q}_p^m \times \mathbf{Q}_p$ such that

$$|\nu(x)|\square_1|t-c(x)|\square_2|\mu(x)| \quad \text{and} \quad t-c(x) \in H \tag{1}$$

where c, μ, ν are semi-algebraic functions (that is functions whose graph is semi-algebraic), $\square_1$ and $\square_2$ are $\leq$, $<$ or no relation, and H is $\{0\}$ or a coset of some semi-algebraic subgroup $\mathbf{G}$ of $\mathbf{Q}_p^\times = \mathbf{Q}_p \setminus \{0\}$. We call it a cell mod $\mathbf{G}$. Denef worked with cells mod $\mathbf{Q}_p^\times$ and implicitly with cells mod $P_N^\times$, the multiplicative group of non-zero N-th powers. This use of cells mod $P_N^\times$ was then made more explicit by Cluckers. Gradually, people began to replace them by cells mod

$$Q_{N,M}^\times = \bigcup_{k\in\mathbf{Z}} p^{kN}(1+p^M\mathbf{Z}_p)$$

where $\mathbf{Z}_p$ denotes the ring of p-adic integers (and $\mathbf{Z}$ the ring of integers). Indeed the Cell Decomposition Theorem only asserts that every semi-algebraic set $S \subseteq \mathbf{Q}_p^{m+1}$ has a finite partition in cells mod $P_N^\times$ for some N. But it usually comes with a Cell Preparation Theorem (similar to Weierstrass preparation) which says that, given semi-algebraic functions $\theta_1,\dots,\theta_r$ from S to $\mathbf{Q}_p$, for some positive integers N, M, e there is a partition of S in finitely many cells mod $Q_{N,M}^\times$ on each of which

$$|f(x,t)|^e = |h(x)| \cdot |t-c(x)|^\alpha$$

where h is a semi-algebraic function, $\alpha \in \mathbf{Z}$ and c is as in (1). Using such a preparation, Cluckers has proved in [Clu01] that for every two infinite semi-algebraic sets A, B over a p-adically closed field, there is a semi-algebraic bijection from A to B if and only if A and B have the same dimension.

Note that these semi-algebraic bijections are not continuous in general: for example Clucker's theorem applies to the valuation ring $\mathbf{Z}_p$, which is compact, and to $\mathbf{Z}_p \setminus \{0\}$, which is not. This lack of global continuity conditions is due to the fact that the cell decomposition techniques treat the various cells of the partition independently, without giving any information on how their frontiers touch each other. This is where triangulations come into the picture[1].

The real Triangulation Theorem says that every semi-algebraic set over $\mathbf{R}$ is semi-algebraically homeomorphic to the union of a simplicial complex, that is (informally) a finite family of simplexes which touch each other along their faces. We have introduced in [Dar17] a notion of polytopes and simplexes adapted to $\mathbf{Q}_p$. We delay precise definitions to Section 2 but give here some intuition on it. The p-adic polytopes share many structural properties with their real counterpart:

- As inverse images by the valuation (in $Q_{1,M}^r$) of subsets of $\mathbf{Z}^r$, they are defined by very special (simple) systems of $\mathbf{Q}$-linear inequalities.

[1] A different improvement of cell decompositions facing this question is given by stratifications. Such stratifications have been recently introduced in p-adically closed field [CCL12], and in more general non-standard Henselian valued fields [Hal14]. However their relationship with the p-adic triangulation is quite unclear at the moment, due to the very peculiar conditions involved in the definition of p-adic simplexes.

- There is a notion of "faces" attached to them with good properties: every face of a polytope S is itself a polytope; if S'' is a face of S' and S' a face of S then S'' is a face of S; the union of the proper faces of S is a partition of its frontier.
- Among the p-adic polytopes, the simplexes are those whose number of facets is minimal[2] in a very strong sense: a simplex has at most one facet, which is itself a simplex.
- Last but not least, every p-adic polytope can be divided in simplexes by a certain uniform process of "Monotopic Division" which offers a good control both on their shapes and their faces.

Just as in the real case, we can then define a simplicial complex over $\mathbf{Q}_p$ essentially as a finite family of simplexes in $Q_{1,M}^r$, for some positive integer M, which touch each other along their faces (again, see Section 2 for a more precise definition). A simplified version of our main result, the Triangulation Theorem 2.20, can then be stated as follows.

Theorem (Triangulation). *For every semi-algebraic set $S \subseteq \mathbf{Q}_p^m$ there is a semi-algebraic homeomorphism $\varphi : T \to S$ whose domain T is the union of a simplicial complex $\mathcal{T}$.*

Moreover, given semi-algebraic functions $\theta_1, \dots, \theta_r$ from S to $\mathbf{Q}_p$, $(\mathcal{T}, \varphi)$ can be chosen so that on every $T \in \mathcal{T}$ the valuation of each $\theta_i \circ \varphi(y)$ is a $\mathbf{Z}$-linear function of the valuations of the coordinates of $y \in T$.

Remark 1.1. The simplexes in the above complex $\mathcal{T}$ are not contained in $\mathbf{Q}_p^m$ but in finitely many copies of $\mathbf{Q}_p^q$ for various q, usually much larger than m. This is the main, but harmless, difference with the triangulation in the real case.

In the real case, cell decomposition and triangulation hold not only for semi-algebraic sets over $\mathbf{R}$ but also over any real-closed fields, and more generally for definable sets in o-minimal expansions of such fields. In the p-adic case, Denef's Cell Decomposition Theorem holds in arbitrary p-adically closed fields. Several variants of it, sometimes weaker, have been proved to hold in some, if not all, P-minimal expansions of such fields (see [DvdD88] and [Clu04] for subanalytic sets, [HM97], [CKDL17], [CKL16], [CCKL17] and [DH17] for definable sets in P-minimal and p-optimal structures).

In the present paper we do not restrict ourselves to $\mathbf{Q}_p$ and its finite extensions, but work in an arbitrary p-adically closed field K fixed once and for all. Apart of the p-adic fields there are many natural examples of such: the algebraic closure of $\mathbf{Q}$ inside $\mathbf{Q}_p$ (which is not complete), the t-adic completion of the field $\bigcup_{n\geq 1} \mathbf{Q}_p((t^{1/n}))$ of Puiseux series over $\mathbf{Q}_p$ (whose value group is not $\mathbf{Z}$, but $\mathbf{Z} \times \mathbf{Q}$ lexicographically ordered), and many others (every ultraproduct of p-adically closed fields is still p-adically closed). We let v denote the (unique) p-adic valuation of K, R its valuation ring and $\mathcal{Z}$ its value group. As usual $\mathcal{Z}$ is augmented by an element $+\infty$ for $v(0)$, and we let $\Gamma = \mathcal{Z} \cup \{0\}$.

Almost all the arguments in our proofs remain valid for definable sets in p-optimal structures on K satisfying the Extreme Value Property (see [DH17]).

[2] Real simplexes can be characterised, among the polytopes of a given dimension d, as those whose number is minimal (namely $d + 1$).

Unfortunately there is one single exception: the proof of the Good Direction Lemma 4.5, which involves polynomial functions, does not generalize to the more general "basic functions" involved in the definable sets in p-optimal structures. Thus we will stick to the semi-algebraic framework in all this paper.

It is organised as follows. All the needed prerequisites, in particular those concerning p-adic simplexes, are recalled in Section 2, which culminates with the final statement of the Triangulation Theorem for semi-algebraic sets and functions in m variables (Theorem 2.20). We denote it $\mathbf{T}_m$. All the applications presented below are then derived from $\mathbf{T}_m$ in Section 3. By means of these applications and Denef's Cell Preparation Theorem we prove in Section 4 a "largely continuous cell preparation up to a small deformation" for semi-algebraic functions in $m+1$ variables (Theorem 4.7). Sections 5 to 7 are then devoted to our main constructions, which are summarized in Lemma 6.1 and Lemma 7.11 (see also Remark 1.2 below). In Section 8, we finally derive $\mathbf{T}_{m+1}$ from $\mathbf{T}_m$ by means of these two technical lemmas, which finishes the proof of our p-adic triangulation theorem for every m.

Remark 1.2. Denef's Cell Decomposition Theorem gives a partition of any semi-algebraic set $S \subseteq K^{m+1}$ in finitely many cells, but we do not control how these cells touch each other. On the other hand, if a cell C is defined by functions c, μ, ν which extend to continuous functions $\bar{c}$, $\bar{\mu}$, $\bar{\nu}$ on the closure of $\widehat{C}$, the frontier of C naturally decomposes in cells, each of which is defined by means of $\bar{c}$, $\bar{\mu}$, $\bar{\nu}$. These auxiliary cells can be seen as "faces" of C. It allows us to speak of "complexes of cells", in a sense which will be made precise in sections 5 and 6. The main results of these sections prove that after only a linear deformation of S, which can be chosen arbitrarily "small" (that is close to the identity), it is possible to decompose the image of S in a *complex* of cells. Moreover one can require this complex to be a tree with respect to the specialisation order.

We now present several other applications of the Triangulation Theorem, all of which will be proved in Section 3.

Theorem (Lifting)**.** *For every semi-algebraic function $f : X \subseteq K^m \to K$ such that $|f|$ is continuous, there is a continuous semi-algebraic function $h : X \to K$ such that $|f| = |h|$.*

In the above theorem $|x| = p^{-v(x)}$ is the usual p-adic norm if $\mathcal{Z} = \mathbf{Z}$. Otherwise this p-adic norm is not defined but can be replaced without inconvenience by the following generalization: we let $|a| = aR^\times = \{au : u \in R^\times\}$ for every $a \in K$, and $|K| = \{|a| : a \in K\}$. The latter is naturally ordered by inclusion, and isomorphic to Γ with the reverse order : $|a| \leq |b|$ if and only if $v(a) \geq v(b)$. So $|a|$ is just a multiplicative notation for $v(a)$: we have $|ab| = |a|.|b|$ and $|a+b| \leq \max(|a|, |b|)$, and of course $|a| = 0$ if and only if $a = 0$.

The real counterpart of the above result is quite obvious. On the contrary, the next two results do not hold in real geometry. In the same vein as Clucker's result on classification of semi-algebraic sets up to semi-algebraic bijection [Clu01], they confirm the intuition that the lack of connectedness and of "holes" (in the sense of algebraic topology, see below) makes semi-algebraic sets over p-adically closed fields much more flexible than over real closed fields.

Recall that a **retraction** of a topological space X onto a subspace Y is a continuous map $\sigma : X \to Y$ such that $\sigma(y) = y$ for every $y \in Y$. When such a retraction exists on a Hausdorff space X, then necessarily Y is closed in X.

Over the reals, the main obstruction for the existence of retractions is the existence of "holes" which are detected by homotopy. This does not work over p-adic fields. Indeed, replacing the unit interval $[0, 1]$ in the reals by the unit ball in K, that is the ring R of the p-adic valuation of K, we may say that a non-empty semi-algebraic set $X \subseteq K^m$ is "semi-algebraically contractible" if there is a continuous semi-algebraic function $H : X \times R \to X$ and $a \in X$ such that $H(x, 1) = x$ and $H(x, 0) = a$ for every $x \in X$. But this is always true: given any $a \in X$ the function $H(x, s) = x$ if s is invertible in R and $H(x, s) = a$ otherwise, has all the required properties. However it is another story to prove that retractions actually exist.

Theorem (Retraction). *For every non-empty semi-algebraic sets $Y \subseteq X \subseteq K^m$, there is a semi-algebraic retraction of X onto Y if and only if Y is closed in X.*

It is worth mentioning that it is the next Splitting Theorem, already conjectured in [Dar06], which was the main motivation for the research presented in this paper. Here ∂X denotes the topological frontier of X, see Section 2.

Theorem (Splitting). *Let X be a relatively open non-empty semi-algebraic subset of K^m without isolated points, and $Y_1, \cdots, Y_s$ a collection of closed semi-algebraic subsets of ∂X such that[3] $Y_1 \cup \cdots \cup Y_s = \partial X$. Then there is a partition of X into non-empty[4] semi-algebraic sets $X_1, \ldots, X_s$ such that $\partial X_i = Y_i$ for $1 \leq i \leq s$.*

The trivial remark that every ball $B \subseteq K^m$ is disconnected can be seen as a very special case of the above Splitting Property (applied to $X = B$ with $Y_1 = Y_2 = \emptyset$). This property is actually (in a sense which can be made precise, see [Dar]) the strongest possible form of disconnectedness that can be observed in a finite dimensional topological space whose points are closed. It is a versatile property which we encountered in different contexts with minor changes (see [Dar], [DJ18]). In the present paper, it plays a key role in the induction step.

A **limit value** for a function $f : X \subseteq K^m \to K$ at a point x adherent to X, is a value $l \in K$ such that (x, l) is adherent to the graph of f. Of course f tends to l at x if and only if l is the unique limit value of f at x. Let us say that f is **largely continuous** on a subset A of X if the restriction of f to A has a unique limit value at every point adherent to A, that is if f extends to a continuous function on the topological closure of A. If A is not mentioned it simply means that f is largely continuous on its domain X. Finally f is **piecewise largely continuous** if there exists a finite partition of X in semi-algebraic pieces on which f is largely continuous. Of course in that case f has finitely many limit values at every point adherent to X.

[3]Note that $Y_1, \ldots, Y_s$ are not assumed to be disjoint. All of them can be equal to ∂X, for example.

[4]A **partition** of a set X is for us a family of two-by-two disjoint subsets of X covering X. We do not assume that the pieces must be non-empty. So when it happens by exception, like here, that this property is required and does not follow from the context, we explicitly mention it.

Theorem (Largely Continuous Splitting)**.** *Let $f : X \subseteq K^m \to K$ be a semi-algebraic function whose graph has bounded*[5] *domain. If f has finitely many limit values at every point adherent to X then f is piecewise largely continuous.*

The real counterpart of this result is easily seen to be true, by means of a triangulation and the trivial remark that every real simplex is connected (see Section 3). This last argument is no longer valid in the p-adic case but, as we will see, the existence of retractions allows us to bypass this problem and recover the full result in the p-adic context.

We can also mention two other applications of the Triangulation Theorem, to p-adic semi-algebraic geometry and to model theory, which are outside of the scope of this paper.

(i) One of the main advantages of proving the Triangulation Theorem for every p-adically closed field, not only for p-adic fields, is that it allows us to combine it with the very powerful model theoretic compactness theorem. This in turn provides "uniform" triangulations, which *almost* give us for free a p-adic analogue of Hardt's Theorem[6]. Some difficulties still remain because it is much less easy to construct homeomorphisms between p-adic simplexes than between real simplexes (see Problem 2). Hopefully this will be addressed in a further paper.

(ii) By [Dar06] the Splitting Property for p-adic sets (which was only conjectural at this time) ensures that the complete theory of the lattice $L(K^m)$ of closed semi-algebraic subsets of K^m is decidable. This is in contrasts with the real case, since we know from [Grz51] that the complete theory of $L(\mathbf{R}^m)$ is undecidable for every $m \geq 2$. Moreover the theory of $L(K^m)$ only depends on m, not on the p-adically closed field involved and not even on p, hence it is the same for $L(K^m)$ and $L(\mathbf{Q}_2^m)$ (see [Dar]).

Finally let us present a few open problems tightly connected with the present work.

Problem 1. *Extend the Triangulation Theorem to p-adic subanalytics sets, and more generally to definable sets in some p-optimal expansions of K.*

Problem 2. *By giving reasonable sufficient conditions for different p-adic simplexes to be homeomorphic, classify p-adic semi-algebraic sets up to semi-algebraic homeomorphisms.*

Problem 3. *For any semi-algebraic set $S \subseteq K^m$, construct a triangulation $(\varphi, \mathcal{T})$ such that the image of $\mathcal{T}$ by φ is a stratification of S. Or conversely use existing stratifications of S (see footnote 1) to construct a better (or a more general) triangulation.*

[5] This boundedness assumption could easily be removed. It suffices to add to K a point at infinity and require that f has finitely many limit values in $\tilde{K} = K \cup \{\infty\}$ at every point of the closure of X in $\tilde{K}^m$, using the same construction as in the preparation of the proof of Lemma 3.3.

[6] Hardt's Theorem in real geometry says that the fibers of a semi-algebraic projection have finitely many homeomorphism types.

2 Prerequisites and notation

We let $\mathbf{N}$ denote the set of positive integers and $\mathbf{N}^* = \mathbf{N} \setminus \{0\}$. For all integers k, l we let $[\![k, l]\!]$ be the set of integers i such that $k \le i \le l$ (hence an empty set if $k > l$).

Recall that we have fixed once and for all a p-adically closed field K. Following [PR84], this is the fraction field of a (unique) Henselian valuation ring R such that the residue field of R is finite, the value group $\mathcal{Z}$ of R has a least strictly positive element, and $\mathcal{Z}/n\mathcal{Z}$ has exactly n elements for every integer $n \ge 1$. We fix once and for all a generator π of the maximal ideal of R, and let $R^\times = R \setminus \pi R$ denote the multiplicative group of invertible elements of R.

Let $\mathcal{Q}$ denote the divisible hull of $\mathcal{Z}$ and $\Omega = \mathcal{Q} \cup \{+\infty\}$. As an ordered group, $\mathbf{Z}$ identifies naturally to the smallest non-trivial convex subgroup of $\mathcal{Z}$. We consider $\mathbf{Z}$ and $\mathbf{Q}$ as embedded into $\mathcal{Q}$ *via* this identification.

For every subset X of K we let $X^* = X \setminus \{0\}$. However, if X^* is a subgroup of the multiplicative group of K, we denote it $X^\times$ in order to highlight this property (so $R^* = R \setminus \{0\} \neq R^\times$ but $K^* = K \setminus \{0\} = K^\times$). For every subgroup G of $K^\times$ we let $xG = \{xg : g \in G\}$ for every $x \in K$, and $K/G = \{xG : x \in K\}$. For example $xR^\times = |x|$ and $K/R^\times = |K|$. Abusing the notation, $0G = \{0\}$ will be denoted 0 whenever the context makes it unambiguous.

In order to ease the notation, given $a \in K^m$, $A \subseteq K^m$ and $f : X \to K^m$ we will often write va for $v(a)$, vA for the direct image $v(A)$, vf for the composite $v \circ f$, and similarly for $|A|$ and $|f|$.

At some rare places it will be convenient to add to K a new element ∞ (and to Γ and $|K|$ new elements $-\infty$ and $+\infty$ respectively) with the natural convention that $0^{-1} = \infty$, $\infty^{-1} = 0$, $v(\infty) = -\infty$, $|\infty| = +\infty$, and $a.\infty = \infty$ for every $a \in K^\times$. We also let $0.\infty = 1$ and $0^0 = 1$ when needed.

2.a Topology and coordinate projections

When an m-tuple a is given, it is understood that $(a_1, \dots, a_m)$ are its coordinates, except if otherwise specified. For every $a \in K^m$ we let

$$va = \big(va_1, \dots, va_m\big) \quad \text{and} \quad |a| = \big(|a_1|, \dots, |a_m|\big).$$

This should not be confused with $\|a\| = \max(|a_1|, \dots, |a_m|)$. For $r \in K^\times$ the (clopen) **ball** of center a and radius r is defined as

$$B(a, r) = \big\{x \in K^m : \|x - a\| \le |r|\big\}.$$

The valuation induces a topology on K, which is inherited by $|K|$ and Γ. The topology generated on Ω by the open intervals and the intervals $]a, +\infty]$ for $a \in \mathcal{Q}$, extends the topology of Γ. The direct products of these topological spaces are endowed with the product topology. For every subset X of any of these spaces, $\overline{X}$ denotes the topological closure of X. In particular $\overline{\mathcal{Z}} = \Gamma$ and $\overline{\mathcal{Q}} = \Omega$. Note that Γ is closed in Ω. The **specialisation preorder** on the subsets of X is defined by $B \le A$ iff $B \subseteq \overline{A}$.

We let $\partial X = \overline{\overline{X} \setminus X}$ denote the **frontier** of X. We say that X is **relatively open** if it is open in $\overline{X}$, that is if $\partial X = \overline{X} \setminus X$.

When a function f is largely continuous (see Section 1) we usually denote $\overline{f}$ the continuous extension of f to the closure of its domain. On the contrary, the restriction of f to some subset A of its domain is denoted $f_{|A}$.

The **support** of an element a of K^m (or $|K|^m$), denoted $\operatorname{Supp} a$, is the set of indexes k such that $a_k \neq 0$. The support of an element b of Γ^m, denoted $\operatorname{Supp} b$, is the set of indexes k such that $b_k = +\infty$. Note that with this definition, one has that for every $a \in K^m$

$$\operatorname{Supp} a = \operatorname{Supp} |a| = \operatorname{Supp} v(a).$$

For every subset S of K^m (resp. $|K|^{m+1}$, resp. Γ^{m+1}) and every $I \subseteq \{1, \dots, m\}$ we let

$$F_I(S) = \big\{a \in \overline{S} : \operatorname{Supp} a = I\big\}.$$

When $F_I(S) \neq \emptyset$ we call it the **face of S with support I**. The **coordinate projection** of K^m (resp. $|K|^m$, Γ^m) onto its face with support I will be denoted π_I. So $\pi_I(a)$ is the unique point b with support I such that $b_i = a_i$ for every $i \in I$.

For every $a \in K^{m+1}$ (resp. $|K|^{m+1}$, resp. Γ^{m+1}) we let $\widehat{a}$ denote the tuple of the first m coordinates of a, so that $a = (\widehat{a}, a_{m+1})$. If A is a set of $(m+1)$-tuples we let $\widehat{A} = \{\widehat{a} : a \in A\}$, and if $\mathcal{A}$ is a family of such sets we let

$$\widehat{\mathcal{A}} = \big\{\widehat{A} : A \in \mathcal{A}\big\}.$$

We call $\widehat{A}$ (resp. $\widehat{\mathcal{A}}$) the **socle** of A (resp. $\mathcal{A}$).

Given two families $\mathcal{H}$, $\mathcal{A}$ of subsets of K^{m+1} we say that $\mathcal{H}$ is **finer** than $\mathcal{A}$ if every $H \in \mathcal{H}$ which meets a set $A \in \mathcal{A}$ is contained in A. If moreover $\mathcal{H}$ is a partition of $\bigcup \mathcal{A}$ we say that $\mathcal{H}$ **refines** $\mathcal{A}$. We will often distinguish between "*horizontal refinements*" for which $\widehat{\mathcal{H}} = \widehat{\mathcal{A}}$, and "*vertical refinement*" for which $\mathcal{H}$ is the family of $A \cap (X \times K)$ where A ranges over $\mathcal{A}$ and X over a refinement of the socle of $\mathcal{A}$.

2.b Semi-algebraic sets

For every integer $N \geq 1$ let $P_N^\times = P_N \setminus \{0\}$ with[7]

$$P_N = \big\{a \in K : \exists x \in K,\ a = x^N\big\}.$$

$P_N^\times$ is a clopen subgroup of $K^\times$ with finite index, and $P_1 = K^\times$. Hence a subset K^m is a **semi-algebraic set** if it is a boolean combination of finitely many sets S_i defined by conditions

$$f_i(x) \in P_{N_i} \tag{2}$$

where the f_i's are m-ary polynomial functions. A **semi-algebraic map** is a function whose graph is semi-algebraic. Rational functions, root functions and monomial functions (see below) are semi-algebraic, among many others.

Abusing a little bit the terminology, we also say that a subset S of $K^m \times |K|^n$ is semi-algebraic if $\{(x,t) \in K^{m+n} : (x, |t|) \in S\}$ is semi-algebraic. Similarly a function $f : X \subseteq K^m \to |K|^n$ is semi-algebraic if its graph is. When a map φ is defined on the disjoint union of finitely many semi-algebraic sets A_i living in different copies of K^m, we say that φ is semi-algebraic if its restriction to each A_i is semi-algebraic in the classical sense.

[7]The notation P_N is sometimes used for the set of *non-zero* N-th powers. The conventions used here leads to denote it $P_N^\times$, so as to highlight its multiplicative group structure.

Remark 2.1. If N' divides N then $P_N^\times$ is a clopen subgroup of $P_{N'}^\times$ with finite index. For this reason, all the integers N_i appearing in (2) can be replaced by any common multiple N. Note also that $0 \in P_N$ is an empty condition, equivalent to $1 \in P_N$, hence all the f_i's can be assumed to be non-zero polynomials.

Theorem 2.2 (Macintyre). *If $S \subseteq K^{m+1}$ is semi-algebraic then $\widehat{K}$ is semi-algebraic.*

This fundamental result has many consequences. The most prominent one is that a subset S of K^m is semi-algebraic if and only if there is a first-order formula[8] $\varphi(x)$ in $\mathcal{L}_{\rm ring} = \{0, 1, +, -, \times\}$ (the language of rings), possibly with parameters in K, such that

$$S = \{a \in K^n : K \models \varphi(a)\}.$$

Remark 2.3. Given m-ary definable functions f, g, the set of points in K^m satisfying the condition "$|f(x)| \leq |g(x)|$" is known to be semi-algebraic[9]. Thus we will consider these expressions as abbreviations for some first order formulas in the language of rings stating the same property). Similarly, if $\varphi(x, y)$ is a formula with $m + n$ variables and $S \subseteq K^n$ is definable by a formula $\psi(y)$ then we will consider $\exists y \in S, \varphi(x, y)$ as a formula since it is an abbreviation for the genuine formula $\exists y, \psi(y) \wedge \varphi(x, y)$.

Another important consequence of Macintyre's theorem is that every p-adically closed field is elementarily equivalent to a finite extension of $\mathbf{Q}_p$ (see [PR84]). In other words, there is a finite extension L of $\mathbf{Q}_p$ such that K and L satisfy exactly the same parameter-free formulas in $\mathcal{L}_{\rm ring}$. The following property transfers from L to K by means of this elementary equivalence. Recall that a family $(C_a)_{a\in A}$ of semi-algebraic subsets of K^n is **uniformly semi-algebraic** if $A \subseteq K^m$ is definable and there is a formula $\varphi(x, y)$ with $m + n$ free variables such that $C_a = \{b \in K^m : K \models \varphi(a, b)\}$ for every $a \in A$.

Theorem 2.4. *Let $(C_\alpha)_{\alpha\in R^*}$ be a uniformly definable family of non-empty, closed and bounded subsets of K^n, such that $|\beta| \leq |\alpha|$ implies that $C_\beta \subseteq C_\alpha$. Then $\bigcap_{\alpha\in R^*} C_\alpha$ is non-empty.*

The next classical properties can easily be derived from this theorem (or transfered from L to K by elementary equivalence).

Theorem 2.5. *For every continuous semi-algebraic function $f : X \subseteq K^m \to K^n$ whose domain X is closed and bounded, $f(X)$ is closed and bounded. As a consequence:*

1. *$\|f\|$ is bounded and attains its bounds.*
2. *If f is injective then it is a homeomorphism from X to $f(X)$.*

Corollary 2.6. *For every bounded semi-algebraic subset X of K^m which is non-empty, there is an element $x \in X$ such that $\|x\|$ is maximal on X.*

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[8]For the notion of first order formula, we refer the reader to any introductory book of model-theory, such as [Hod97] for example.

[9]This follows from the non-trivial fact that R is definable by means of the Kochen operator (see [PR84]).

Another crucial property of the semi-algebraic structure on a p-adically closed fields is the existence of so called "built-in Skolem functions" (see [vdD84], or the appendix of [DvdD88] for a more constructive proof). Basically, this property says that for every semi-algebraic subset A of K^{m+n}, the coordinate projection of A onto K^m has a *semi-algebraic* section.

Theorem 2.7 (Skolem functions)**.** *Let $X \subseteq K^m$ be semi-algebraic set and $\varphi(x,t)$ a formula with $m+n$ free variables. If, for every $a \in X$ there is $b \in K^n$ such that $K \models \varphi(a,b)$, then there exists a semi-algebraic function $\xi : X \to K^n$ (called a Skolem function) such that $K \models \varphi(x, \xi(x))$ for every $x \in X$.*

For example, if a semi-algebraic function $f : X \to K$ takes values in P_N, then Theorem 2.7 applied to the formula $\varphi(x,t)$ saying that "$f(x) = t^N$" gives a semi-algebraic function $\xi : X \to K$ such that $f = \xi^N$.

There is a good dimension theory for semi-algebraic sets over p-adically closed fields, see [SvdD88] and [vdD89]. We will repeatedly use the following properties of this dimension, for every semi-algebraic sets A, B and semi-algebraic map f defined on A. By convention $\dim \emptyset = -\infty$.

1. $\dim A = 0$ if and only if A is finite non-empty.
2. $\dim A \cup B = \max(\dim A, \dim B)$.
3. If $A \neq \emptyset$, $\dim \partial A < \dim A$.
4. $\dim f(A) \leq \dim A$.

The **local dimension** of a semi-algebraic set $A \subseteq K^m$ at a point $a \in A$ is the minimum of $\dim U$, for every semi-algebraic neighbourhood U of a in A (with respect to the relative topology, induced by K^m on A). A is **pure dimensional** if it has the same local dimension at every point. Note that if a semi-algebraic set B is open in A and A is pure dimensional then so is B, and that a cell is pure dimensional if and only if its socle is. This last point, combined with Denef's Cell Decomposition Theorem 4.1 and a straightforward induction, shows that every semi-algebraic set A is the union of finitely many pure dimensional ones.

2.c Root functions and monomial functions

Following Lemma 1.3 in [CL12] there is for each integer $M > 0$ a unique group homomorphism $\overline{ac}_M$ from $K^\times$ to $(R/\pi^M R)^\times$ such that $\overline{ac}_M(\pi) = 1$ and $\overline{ac}_M(u) = u + \pi^M R$ for every $u \in R^\times$. The construction of $\overline{ac}_M$ given in [CL12] shows that for each integer $N > 0$ the set

$$Q_{N,M} = \{0\} \cup \left\{x \in P_N^\times \cdot (1+\pi^M R) : \overline{ac}_M(x) = 1\right\}$$

is semi-algebraic. $Q_{N,M}^\times = Q_{N,M} \setminus \{0\}$ is a clopen subgroup of $K^\times$ with finite index. When $v(K^\times) = \mathbf{Z}$ then $Q_{N,M}^\times = \bigcup_{k \in \mathbf{Z}} \pi^{kN}(1+\pi^M R)$ so the above definition of $Q_{N,M}$ is compatible with the notation of the introduction.

If $M > 2v(N)$, Hensel's Lemma implies that $1 + \pi^M R \subseteq P_N$, hence $Q_{N,M}$ is contained in P_N. The importance of $Q_{N,M}$ comes from the following property, which also follows from Hensel's lemma (see for example lemma 1 and corollary 1 in [Clu01]).

Lemma 2.8. *The function $x \mapsto x^e$ is a group endomorphism of $Q^\times_{N,M}$. If $M > v(e)$ this endomorphism is injective and its image is $Q^\times_{eN,v(e)+M}$.*

In particular $x \mapsto x^e$ defines a continuous bijection from $Q_{1,v(e)+1}$ to $Q_{e,2v(e)+1}$. We let $x \mapsto x^{1/e}$ denote the reverse continuous bijection. In particular it is defined on $Q_{N,M}$ for every N, M such that e divides N and $M > 2v(e)$.

For all positive integers e, n we let

$$U_e = \{x \in K : x^e = 1\} \quad \text{and} \quad U_{e,n} = U_e \cdot (1 + \pi^n R).$$

Analogously to Landau's notation $\mathcal{O}(x^n)$ of calculus, we let $\mathcal{U}_{e,n}(x)$ denote *any* semi-algebraic function in the multi-variable x with values in $U_{e,n}$. Any such function is the product of two semi-algebraic functions, with values in U_e and $1 + \pi^n R$ respectively. So, given a family of functions f_i, g_i on the same domain X, we write that $f_i = \mathcal{U}_{e,n} g_i$ for every i, when there are semi-algebraic functions $\omega_i : X \to R$ and $\chi_i : X \to U_e$ such that for every x in X

$$f_i(x) = \chi_i(x)\big(1 + \pi^n \omega_i(x)\big) g_i(x).$$

$\mathcal{U}_{1,n}(x)$ is simply denoted $\mathcal{U}_n(x)$.

Remark 2.9. If $f(x) = \mathcal{U}_n(x)$ for some $n > 2v(e)$ then $f^{1/e}$ is well defined and takes values in $1 + \pi^{n-v(e)} R$. Therefore we can write $\mathcal{U}_n(x) = (\mathcal{U}_{n-v(e)}(x))^e$.

A function g is **N-monomial** on $S \subseteq K^q$ if either it is constantly equal to ∞ or there exists $\xi \in K$ and $\beta_1, \dots, \beta_q \in \mathbf{Z}$ such that

$$\forall x = (x_1, \dots, x_q) \in S,\ g(x) = \xi \prod_{i=1}^{q} x_i^{N\beta_i}.$$

In this definition we use when necessary our convention that $0^0 = 1$. A function f is **N-monomial mod $U_{e,n}$** if $f = \mathcal{U}_{e,n} g$ with g an N-monomial function.

2.d Discrete and p-adic simplexes

We say that $f : S \subseteq F_I(\Gamma^q) \to \Omega$ is **affine** if either it is constantly equal to $+\infty$, or there are elements $\alpha_0 \in \mathcal{Q}$ and $\alpha_i \in \mathbf{Q}$ for $i \in I$ such that

$$\forall x \in S,\ f(x) = \alpha_0 + \sum_{i \in I} \alpha_i x_i.$$

Polytopes[10] in Γ^q are defined by induction on q. The only polytope in Γ^0 is Γ^0 itself (which is a one-point set). For every $I \subseteq [\![1, q+1]\!]$, a subset A of $F_I(\Gamma^{q+1})$ is a **discrete polytope** of Γ^{q+1} if $\widehat{A}$ is a discrete polytope of Γ^q and if there is a pair (μ, ν) of *largely continuous* affine maps from $\widehat{A}$ to Ω, called a **presentation** of A, such that $0 \le \mu \le \nu$ and

$$A = \Big\{ a \in F_I(\Gamma^{q+1}) : \widehat{a} \in \widehat{A} \text{ and } \mu(\widehat{a}) \le a_{q+1} \le \nu(\widehat{a}) \Big\}.$$

Example 2.10.

[10] In [Dar17] we introduced discrete polytopes in Γ^q as "largely continuous precells mod N", for an arbitrary q-tuple N of positive integers. In the present paper $N = (1, \dots, 1)$ will not play any role so we remove it from the definition.

- $A = \mathbf{N} \times \mathbf{N}$ is a discrete polytope with two facets $F_{\{1\}}(A) = \mathbf{N} \times \{+\infty\}$ and $F_{\{2\}}(A) = \{+\infty\} \times \mathbf{N}$.
- $B = \{(x, y) \in \mathbf{Z}^2 : 0 \leq y \leq x\}$ is a discrete simplex, with proper faces $F_{\{2\}}(B) = \{+\infty\} \times \mathbf{N}$ and $F_\emptyset(B) = \{(+\infty, +\infty)\}$.
- $C = \{(x, y, z) \in \mathbf{Z}^3 : (x, y) \in B \text{ and } z = 2y - 2x\}$ is a subset of $\mathbf{Z}^3$ defined by linear inequalities, whose proper faces $F_{\{3\}}(C)$ and $F_\emptyset(C)$ are linearly ordered by specialization. However the linear map $\nu(x, y) = 2y - 2x$ defining C is not largely continuous on B: it has no limit when (x, y) tends to $(+\infty, +\infty)$ in B. Note that $F_{\{3\}}(C) = \{+\infty\}^2 \times 2\mathbf{N}$ can not be defined by linear inequalities. Thus $F_{\{3\}}(C)$ is definitely not a polytope, and so neither is C.

All the references in the next proposition are taken from [Dar17].

Proposition 2.11. *Let $q \geq 1$ and $A \subseteq F_I(\Gamma^q)$ be a discrete polytope. Let (μ, ν) be a largely continuous presentation of A, let J be a subset of I, and $\widehat{J} = J \setminus \{q\}$. Finally let $Y = F_{\widehat{J}}(\widehat{A})$. Then $F_J(A) \neq \emptyset$ if and only if either $q \in J$ and $\bar{\mu} < +\infty$ on Y, or $q \notin J$ and $\bar{\nu} = +\infty$ on Y (Proposition 3.11). When this happens:*

1. *$F_J(A) = \pi_J(A)$ (Proposition 3.3).*
2. *The socle of $F_J(A)$ is a face of $\widehat{A}$: $\widehat{F_J(A)} = F_{\widehat{J}}(\widehat{A}) = Y$ (Proposition 3.7).*
3. *$F_J(A)$ is a discrete polytope and $(\bar{\mu}_{|Y}, \bar{\nu}_{|Y})$ is a presentation of it (Proposition 3.11):*

$$F_J(A) = \big\{b \in F_J(\Gamma^q) : \widehat{b} \in Y, \text{ and } \bar{\mu}(\widehat{b}) \leq b_q \leq \bar{\nu}(\widehat{b})\big\}.$$

We will also use the next result (Proposition 3.5 in [Dar17]).

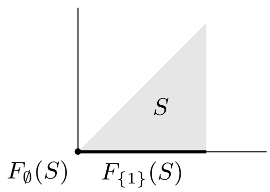
Proposition 2.12. *Let $A \subseteq \Gamma^q$ be a discrete polytope, $f : A \to \Omega$ be an affine map and $B = F_J(A) = \pi_J(A)$ a face of A. Assume that f extends to a continuous map $f^* : A \cup B \to \Omega$. Then f^* is affine and if $f^* \neq +\infty$ then $f = f^*_{|B} \circ \pi_{J|A}$. In particular if $f^* \neq +\infty$ then $f(A) = f^*(B)$.*

A **discrete simplex** is a discrete polytope whose faces are linearly ordered by specialization. This is a "monohedral largely continuous precell mod $(1, \ldots, 1)$" in [Dar17]. Of course every face of a simplex is a simplex (see Remark 3.12 of [Dar17]).

For every $M \geq 1$ we let $D^M R^q = (R \cap Q_{1,M})^q$ and define **p-adic simplexes of index M** as the inverse images of discrete simplexes by the restriction of the valuation to $D^M R^q$. The faces of a simplex S of index M are obviously the pre-images in $D^M R^q$ of the faces of vS. In particular they are linearly ordered by specialization. S is closed if and only if vS is a singleton in Γ^q. If S is not closed, its largest proper face T is called its **facet** and $\partial S = \overline{T}$.

Remark 2.13. With the notation of Proposition 2.12, if $S = v^{-1}(A) \cap D^M R^q$ and $T = v^{-1}(B) \cap D^M R^q$ then $T = F_J(S)$, and so by Proposition 2.12 $T = \pi_J(S)$. We will sometimes refer to the restriction of π_J to A (resp. S) as to "the coordinate projection of A onto B (resp. of S onto T)".

Example 2.14. $S = \{(x, y) \in D^1R^2 : 0 < |x| \leq |y| \leq 1\}$ is a simplex of index 1 (it is the inverse image in D^1R^2 of the discrete simplex B in example 2.10). Intuitively we can visualise it (more exactly its image in $|K| \times |K|$) in the next figure, with its faces $F_{\{1\}}(S) = D^1R \times \{0\}$ and $F_\emptyset(S) = \{(0, 0)\}$. More general simplexes will be defined by triangular systems of inequalities between norms of largely continuous monomial functions with rational exponents, hence their intuitive representations will usually have curved shapes.



2.e Simplicial complexes

We will have to consider complexes of sets, of cells and of simplexes. All of them are finite families of subsets of a topological space X, organised in a such a way that one controls how the closures of these sets intersect.

Recall first that an ordered set $\mathcal{A}$ is a **tree** if for every A in $\mathcal{A}$, the set of elements in $\mathcal{A}$ smaller than A is linearly ordered. It is a **rooted tree** if it has one smallest element. A **lower subset** of $\mathcal{A}$ is a subset $\mathcal{B}$ of $\mathcal{A}$ such that whenever an element of $\mathcal{A}$ is smaller than an element of $\mathcal{B}$, it belongs to $\mathcal{B}$.

Now, given a finite family $\mathcal{A}$ of pairwise disjoint subsets of X, we call $\mathcal{A}$ a **closed complex** if every $A \in \mathcal{A}$ is relatively open and if its frontier ∂A is a union of elements of $\mathcal{A}$. The specialization preorder is then an order on $\mathcal{A}$. If $\mathcal{A}$, ordered by specialization, is a tree (resp. a rooted tree) we call it a **closed monoplex** (resp. **rooted closed monoplex**). A **complex** (resp. **monoplex**) is then an arbitrary subfamily of a closed complex (resp. closed monoplex). Of course a complex $\mathcal{A}$ is a closed complex if and only if $\bigcup \mathcal{A}$ is closed.

Remark 2.15. Using that every semi-algebraic set is the disjoint union of finitely many pure dimensional ones, and that $\dim \partial A < \dim A$ for every semi-algebraic set A, a straightforward induction shows that every finite family of semi-algebraic subsets of K^m can be refined by a complex of pure dimensional semi-algebraic sets.

A **simplicial complex** $\mathcal{S}$ in $D^M R^q$ (resp. in Γ^q) is a complex of simplexes in $D^M R^q$ (resp. in Γ^q).

Remark 2.16. We do not require in our definition of a simplicial complex $\mathcal{S}$ in $D^M R^q$ that different simplexes must have different supports. However it will follow from our construction that the simplicial complexes produced by $\mathbf{T}_m$ do have this additional property and more: for every $S, S' \in \overline{\mathcal{S}}$, $S' \leq S$ if and only if $\operatorname{Supp} S' \subseteq \operatorname{Supp} S$ (see Remark 7.10). So the tree $\mathcal{S}$, ordered by specialisation, is isomorphic to the set $\{\operatorname{Supp} S : S \in \mathcal{S}\}$ ordered by inclusion.

Let $\mathcal{S}$ be a finite family of simplexes in $D^M R^q$ (or Γ^q). Then $\mathcal{S}$ is a simplicial complex if and only if for every $S, T \in \mathcal{S}$, $\overline{S} \cap \overline{T}$ is the union of the common faces of S and T. When this happens:

1. $\mathcal{S}$ is a monoplex;
2. every subset $\mathcal{S}_0$ of $\mathcal{S}$ in $D^M R^q$ is again a simplicial complex;
3. $\bigcup \mathcal{S}_0$ is closed in $\bigcup \mathcal{S}$ if and only if $\mathcal{S}_0$ is a lower subset of $\mathcal{S}$.

Let $\overline{\mathcal{S}}$ denote the family of all the faces of the elements of $\mathcal{S}$. We call it the **closure** of $\mathcal{S}$, and say that $\mathcal{S}$ is **closed** if $\mathcal{S} = \overline{\mathcal{S}}$. Note that $\mathcal{S}$ is a complex (resp. a closed complex) if and only if $\mathcal{S} \subseteq \overline{\mathcal{S}}$ (resp. $\mathcal{S} = \overline{\mathcal{S}}$) and the elements of $\overline{\mathcal{S}}$ are pairwise disjoint.

If $\mathcal{S}$ is a simplicial complex, we say $\mathcal{T}$ is a **simplicial subcomplex** of $\mathcal{S}$ of if $\mathcal{T}$ is a simplicial complex such that $\overline{\mathcal{T}}$ *refines* a lower subset of $\overline{\mathcal{S}}$, and $\bigcup \mathcal{T}$ is a closed subset of $\bigcup \mathcal{S}$.

The following results are respectively Theorem 6.3 and Proposition 6.4 of [Dar17].

Theorem 2.17 (Monotopic Division)**.** *Let S be a simplex in $D^M R^q$ and $\mathcal{T}$ a simplicial complex in $D^M R^q$ which is a partition of ∂S. Let $\varepsilon : \partial S \to K^\times$ be a definable function such that the restriction of $|\varepsilon|$ to every proper face of S is continuous. Then there exists a finite partition $\mathcal{U}$ of S such that $\mathcal{U} \cup \mathcal{T}$ is a simplicial complex in $D^M R^q$, $\mathcal{U}$ contains for every $T \in \mathcal{T}$ a unique simplex U with facet T, and moreover $\big\|u - \pi_J(u)\big\| \leq \big|\varepsilon(\pi_J(u))\big|$ for every $u \in U$, where $J = \mathrm{Supp}(T)$.*

Proposition 2.18. *Let $A \subseteq D^M R^q$ be a relatively open set. Assume that A is the union of a simplicial complex $\mathcal{A}$ in $D^M R^q$. Then for every integer $n \geq 1$ there exists a finite partition of A in semi-algebraic sets $A_1, \dots, A_n$ such that $\partial A_k = \partial A$ for every k.*

Finally, a **simplicial complex of index** M is a collection $\mathcal{S} = \{\mathcal{S}_i\}_{i \in I}$ of finitely many[11] rooted simplicial complexes $\mathcal{S}_i$ in $D^M R^{q_i}$, for various integers q_i. The **closure** of $\mathcal{S}$ is the collection of the closures of the $\mathcal{S}_i$'s. It has **separated supports** if each $\mathcal{S}_i$ is. If $\mathcal{T} = (\mathcal{T}_i)_{i \in \mathcal{I}}$ is a collection of families $\mathcal{T}_i$ of subsets of $D^M R_i^q$ we let $\biguplus \mathcal{T}$ denote the disjoint union of the $\bigcup \mathcal{T}_i$'s. We say that $\mathcal{T}$ is a **simplicial subcomplex** of $\mathcal{S}$ if each $\mathcal{T}_i$ is a simplicial subcomplex of $\mathcal{S}_i$.

Given a semi-algebraic homeomorphism φ from $\biguplus \mathcal{S}$ to a subset X of K^m, we will let

$$\varphi(\mathcal{S}) = \big\{\varphi(S) : S \in \mathcal{S}\big\}.$$

If $\mathcal{S}$ is closed, $\varphi(\mathcal{S})$ is obviously a closed monoplex of pure dimensional semi-algebraic sets partitioning X.

Remark 2.19. With φ as above, $\mathcal{S}$ is closed if and only if X is closed and bounded. Indeed, each $\bigcup \mathcal{S}_i$ is clopen in $\biguplus \mathcal{S}$, hence its homeomorphic image X_i by φ is clopen in X. In particular X is closed and bounded in K^m if and only if so is each X_i. Let φ_i be the semi-algebraic homeomorphism from $\bigcup \mathcal{S}_i$ to X_i induced by restriction of φ. Note that $\bigcup \mathcal{S}_i$ is bounded (it is contained in R^{q_i}). By Theorem 2.5 applied to φ_i and φ_i^{-1} it follows that $\bigcup \mathcal{S}_i$ is closed in K^{q_i}, that is $\mathcal{S}_i$ is closed, if and only if X_i is closed and bounded in K^m.

We can now state precisely our main result.

[11] Possibly zero if the index set I is empty.

Theorem 2.20 (Triangulation $\mathbf{T}_m$)**.** *Given a finite family $(\theta_i : A_i \subseteq K^m \to K)_{i\in I}$ of semi-algebraic functions and integers $n, N \geq 1$, for some integers e, M which can be made arbitrarily large*[12]*, there exists a simplicial complex $\mathcal{T}$ of index M and a semi-algebraic homeomorphism φ from the disjoint union of the simplexes in $\mathcal{T}$ to $\bigcup_{i\in I} A_i$ such that for every i in I:*

1. *$\{\varphi(T) : T \in \mathcal{T}$ and $\varphi(T) \subseteq A_i\}$ is a partition of A_i.*
2. *$\forall T \in \mathcal{T}$ such that $\varphi(T) \subseteq A_i$, $\theta_i \circ \varphi_{|T}$ is N-monomial mod $U_{e,n}$.*

We call the pair $(\mathcal{T}, \varphi)$ given by $\mathbf{T}_m$ a **triangulation** of the θ_i's with **parameters** (n, N, e, M). When a finite family $(A_i)_{i\in I}$ of semi-algebraic sets is given, the result of the application of $\mathbf{T}_m$ to the indicator functions of the A_i's is called a triangulation of $(A_i)_{i\in I}$.

3 Applications

In all this section we assume $\mathbf{T}_m$ and derive some applications. The proof of the Triangulation Theorem goes by induction on m, and most of the following applications are actually needed in the induction step. So it is important to emphasize that throughout this section, the integer m will be fixed.

Theorem 3.1. *If $f : X \subseteq K^m \to K$ is semi-algebraic and $|f|$ is continuous, then there exists a function $h : X \to K$ semi-algebraic* and *continuous such that $|f| = |h|$ on X.*

Proof: $\mathbf{T}_m$ gives a triangulation $(\mathcal{T}, \varphi)$ of f with parameters $(1, 1, e, M)$. On every $S \in \mathcal{T}$, $f \circ \varphi_{|S} = \mathcal{U}_{e,1}\psi$ with $\psi : S \to K$ a 1-monomial function. Thus for some q_S such that S is contained in $D^M R^{q_S}$, there are λ_S in K and $\alpha_{1,S}, \ldots, \alpha_{q_S,S}$ in $\mathbf{Z}$ such that:

$$\forall x \in S,\ |f \circ \varphi(x)| = \left| \lambda_S \prod_{i=1}^{q_S} x_i^{\alpha_{i,S}} \right| \tag{3}$$

Let $\alpha_{0,S} = v\lambda_S$ and $\xi_S : vS \to \Gamma$ be defined by:

$$\forall a \in vS,\ \xi_S(a) = \alpha_{0,S} + \sum_{i=1}^{q_S} \alpha_{i,S} a_i$$

By construction $\xi_S(vx) = vf(\varphi(x))$ for every $x \in S$ (in particular ξ_S only depends on $f \circ \varphi$, even if the coefficients $\alpha_{i,S}$ in (3) are not uniquely determined by $f \circ \varphi$ on S). By assumption vf is continuous on X hence so is $vf \circ \varphi$ on $\biguplus \mathcal{T}$. In particular ξ_S extends continuously to vT for every face T of S in $\mathcal{T}$, and the restriction to vT of such an extension $\overline{\xi}_S$ is precisely ξ_T. By proposition 2.12 it follows that if $\xi_T \neq +\infty$ (that is if $f \circ \varphi \neq 0$ on T) then $\xi_S = \xi_T \circ \pi_T$ where π_T denotes the coordinate projection of vS to vT (see Remark 2.13).

Now, for every S in $\mathcal{T}$ let $g_S : S \to K$ be defined (by induction on $\mathcal{T}$ ordered by specialization) as follows:

[12] The exact meaning of "e, M can be made arbitrarily large" is a bit special here: it says that for any given integers $e_* \geq 1$ and $M_* \geq 1$, the integers e, M can be chosen so that e_* divides e and $M_* \leq M$.

1. If $f \circ \varphi = 0$ on S, $g_S = 0$.

2. If S is minimal (with respect to the specialisation preorder) among the simplexes in $\mathcal{T}$ on which $f \circ \varphi \neq 0$ then for every $x \in S$:

$$g_S(x) = \pi^{\alpha_{0,S}} \prod_{i=1}^{q_S} x_i^{\alpha_{i,S}}$$

3. Otherwise $g_S = g_T \circ \pi_T$ where π_T is the coordinate projection (see Remark 2.13) of S onto its smallest proper face T in $\mathcal{T}$ on which $f \circ \varphi \neq 0$.

By construction $vg_S(x) = \xi_S(vx)$ for every $x \in S$ hence $|g_S| = |f \circ \varphi|$ on S. Moreover for every face T of S in $\mathcal{T}$ and every $y \in T$, $g_S(x)$ tends to $g_T(y)$ as x tends to y in S (because $g_S(x) = g_T(\pi_T(x))$ if $g_T \neq 0$, and otherwise because $|g_S| = |f \circ \varphi|$ on S, $|g_T| = |f \circ \varphi| = 0$ on T and $|f \circ \varphi| = |f| \circ \varphi$ is continuous by assumption).

The function $h : X \to K$ defined by $h = g_S \circ \varphi^{-1}$ on every $\varphi(S)$ with S in $\mathcal{T}$, is clearly semi-algebraic. By construction $|f| = |h|$ on X, and by the above argument h is continuous on X.

■

Theorem 3.2. *For all non-empty semi-algebraic sets $Y \subseteq X \subseteq K^m$, there is a semi-algebraic retraction of X onto Y if and only if Y is closed in X.*

Proof: One direction is general. For the converse we assume that Y is closed in X. Let $(\mathcal{S}, \varphi)$ be a triangulation of X, Y given by $\mathbf{T}_m$, and let $\mathcal{T}$ be the family of simplexes T in $\mathcal{S}$ such that $\varphi(T) \subseteq Y$. It suffices to construct a continuous retraction of $\biguplus \mathcal{S}$ onto $\biguplus \mathcal{T}$.

Let $\mathcal{S}_0 = \mathcal{T}$ and σ_0 be the identity map on $\biguplus \mathcal{T}$. Because Y is closed in X, $\mathcal{T}$ is a lower subset of $\mathcal{S}$. Let k be a positive integer and assume that there is a lower subset $\mathcal{S}_{k-1}$ of $\mathcal{S}$ containing $\mathcal{T}$, and a retraction σ_{k-1} of $\biguplus \mathcal{S}_{k-1}$ to $\mathcal{T}$. If $\mathcal{S}_{k-1} = \mathcal{S}$ we are done. Otherwise let S be a minimal element (with respect to the specialisation order) in $\mathcal{S} \setminus \mathcal{S}_{k-1}$, and let[13] $\mathcal{S}_k = \mathcal{S}_{k-1} \cup \{S\}$. It only remains to build a retraction τ of $\biguplus \mathcal{S}_k$ onto $\biguplus \mathcal{S}_{k-1}$. Indeed $\sigma_{k-1} \circ \tau$ will then be a continuous retraction of $\mathcal{S}_k$ onto $\mathcal{T}$, and the result will follow by induction.

If S has no proper face in $\mathcal{S}$ then it is clopen in $\biguplus \mathcal{S}_k$. So the map τ which is the identity map on $\biguplus \mathcal{S}_k$ and which sends every point of S to an arbitrary given point of $\biguplus \mathcal{S}_{k-1}$ is continuous on $\biguplus \mathcal{S}_k$, and a retraction of $\biguplus \mathcal{S}_k$ onto $\biguplus \mathcal{S}_{k-1}$.

Otherwise let T be the largest proper face of S in $\mathcal{S}$. By minimality of S, T belongs to $\mathcal{S}_{k-1}$. Let π_T be the coordinate projection of S onto T. The frontier of S inside $\biguplus \mathcal{S}_k$ is the closure of T in $\biguplus \mathcal{S}_k$, hence the function τ which coincides with the identity map on $\biguplus \mathcal{S}_{k-1}$ and with π_T on S is continuous. It is then a retraction $\biguplus \mathcal{S}_k$ onto $\biguplus \mathcal{S}_{k-1}$, which finishes the proof.

■

The Splitting Theorem 3.4 is a strengthening of the next lemma using retractions.

[13] We are abusing the notation here: $\mathcal{S}$ is a finite collection of simplicial simplexes $\mathcal{S}^{(i)}$ in $D^M R^{q_i}$ for various q_i, $\mathcal{S}_{k-1}$ is a collection of lower subsets $\mathcal{S}_{k-1}^{(i)}$ of $\mathcal{S}^{(i)}$, there is an index i_0 such that S belongs to $\mathcal{S}^{(i_0)}$, and what we have denoted abusively $\mathcal{S}_{k-1} \cup \{S\}$ is actually the collection of all the $\mathcal{S}_{k-1}^{(i)}$'s for $i \neq i_0$ and of $\mathcal{S}_{k-1}^{(i_0)} \cup \{S\}$.

Lemma 3.3. *Let $X \subseteq K^m$ be a relatively open semi-algebraic set without isolated points and $n \geq 1$ an integer. Then there exists a partition of X in semi-algebraic sets X_k for $1 \leq k \leq n$ such that $\partial X_k = \partial X$ for every k.*

We are going to prove Lemma 3.3 by using a triangulation $(\mathcal{U}, \varphi)$ of $(X, \partial X)$ and applying Proposition 2.18 to $\varphi^{-1}(X)$. In order to ensure that this set is still relatively open, we first reduce to the case where X is bounded by means of the following construction.

Let $\tilde{K} = K \cup \{\infty\}$ and for every $I \subseteq \{1, \ldots, m\}$ let $K_I^m = \tilde{K}_I^m \cap K^m$ where

$$\tilde{K}_I^m = \{x \in \tilde{K}^m : x_k \in R \iff k \in I\}.$$

Let $R_I^m = \{x \in R^m : \forall k \notin I,\ x_k \neq 0\}$, and for every $x \in R_I^m$ let $\psi_I(x) = (y_k)_{1 \leq k \leq m}$ be defined by $y_k = x_k$ if $k \in I$, and $y_k = 1/(\pi x_k)$ otherwise. Clearly ψ_I is semi-algebraic homeomorphism from R_I^m to K_I^m which extends uniquely to a homeomorphism $\tilde{\psi}_I$ from R^m to $\tilde{K}_I^m$.

Proof: Note first given a partition of X in finitely many semi-algebraic pieces $U_1, \ldots, U_r$ which are clopen in X, it suffices to prove the result separately for each U_j. Indeed, each U_j will then be relatively open with $\partial U_j \subseteq \partial X$ (because U_j is clopen in X), and $\bigcup_{j \leq r} \partial U_j = \partial X$ (because $\partial U_j = \overline{U}_j \setminus X$ and $\bigcup_{j \leq r} \overline{U}_j = \overline{X}$). So, if a partition of each U_j in semi-algebraic pieces $(U_{j,k})_{1 \leq k \leq n}$ is found such that $\partial U_{j,k} = \partial U_j$ for every k, then the union X_k of $U_{j,k}$ for $1 \leq j \leq r$ defines a partition of X in semi-algebraic pieces and we have $\partial X_k = \bigcup_{j \leq r} \partial U_{j,k}$ (same argument as above) hence $\partial X_k = \bigcup_{j \leq r} \partial U_j = \partial X$.

Now, as I ranges over the subsets of $\{1, \ldots, m\}$, the sets $X \cap K_I^m$ form a partition of X in semi-algebraic sets clopen in X. By the argument above we can deal separately with each of these sets, hence we can reduce to the case where $X \subseteq K_I^m$ for some I.

Let $Y = \psi_I^{-1}(X)$ and $\hat{X}$ be the closure of X in $\hat{K}_I^m$. Note that $\hat{\psi}_I(\overline{Y}) = \hat{X}$. The fact that $\overline{X} \setminus X$ is closed in K^m, hence in K_I^m, implies that $\hat{X} \setminus X$ is closed in $\hat{K}_I^m$. It follows that its image under $\hat{\psi}_I^{-1}$, which is precisely $\overline{Y} \setminus Y$, is closed in R^m, hence in K^m. Thus Y is relatively open. It then suffices to prove the result for Y, that is we can assume that $X = Y$ is bounded. Of course we can assume as well that ∂X is non-empty (otherwise $X_1 = X$ and $X_k = \emptyset$ for $2 \leq k \leq n$ is obviously a solution).

$\mathbf{T}_m$ gives a triangulation $(\mathcal{U}, \varphi)$ of $(X, \partial X)$. $\mathcal{U}$ is the disjoint union of finitely many simplicial complexes $\mathcal{U}_j$ in $D^M R^{q_j}$ for $1 \leq j \leq r$. Let $U_j = \varphi(\bigcup \mathcal{U}_j) \cap X$ for every j, this defines a partition of X in semi-algebraic sets clopen in X. By using again the initial remark in this proof, it suffices to check the result for each U_j separately. So we can assume that $\mathcal{U}$ itself is a simplicial complex in $D^M R^q$ for some q.

By construction $\overline{X}$ is semi-algebraic, closed and bounded, and φ^{-1} is semi-algebraic and continuous, so $\varphi^{-1}(\overline{Y}) = \overline{\varphi^{-1}(Y)}$ for every semi-algebraic[14] $Y \subseteq \overline{X}$. Let $A = \varphi^{-1}(X)$, we have $\overline{A} = \varphi(\overline{X})$ hence $\overline{A} \setminus A = \varphi(\overline{X} \setminus X)$ is closed, that is A is relatively open. Proposition 2.18 then applies to A and gives a partition of A in semi-algebraic sets $A_1, \ldots, A_n$ such that $\partial A_k = \partial A$ for every k.

[14]For every continuous map $f : X \subseteq K^q \to K^r$ and every $Y \subseteq X$, if X is closed then $f(\overline{Y}) \subseteq \overline{f(Y)}$. The reverse inclusion holds if X is compact, or if f, Y, X are semi-algebraic and X is closed and bounded (see Theorem 2.5).

For $1 \leq k \leq n$ let $X_k = \varphi(A_k)$. These semi-algebraic sets form a partition of X, because $A_1, \dots, A_n$ form a partition of A. Moreover, since $\bigcup \mathcal{U}$ is semi-algebraic, closed and bounded, we have $\varphi(\overline{B}) = \overline{\varphi(B)}$ for every semi-algebraic[15] set B contained in $\bigcup \mathcal{U}$. It follows that for $1 \leq k \leq n$ we have $\partial X_k = \varphi(\partial A_k) = \varphi(\partial A) = \partial X$, which proves the result.
■

Theorem 3.4. *Let X be a relatively open non-empty semi-algebraic subset of K^m without isolated points, and $Y_1, \cdots, Y_s$ a collection of closed semi-algebraic subsets of ∂X such that $Y_1 \cup \cdots \cup Y_s = \partial X$. Then there is a partition of X in non-empty semi-algebraic sets $X_1, \dots, X_s$ such that $\partial X_i = Y_i$ for $1 \leq i \leq s$.*

Proof: X is non-empty and has no isolated point, hence is infinite. The result is obvious for $s = 0$ (there is nothing to prove) and $s = 1$ (take $X_1 = X$). By induction it suffices to prove it for $s = 2$. Indeed, if $s \geq 3$ and the result is proved for $s-1$, then the result for $s = 2$ applied to X with $Z_1 = Y_1 \cup \cdots \cup Y_{s-1}$ and $Z_2 = B_s$ gives a partition in two pieces X'_1, X'_2 such that $\partial X'_l = Z_l$ for $l = 1, 2$, and the induction hypothesis applied to X'_1 with $Y_1, \dots, Y_{s-1}$ gives a partition of X'_1 in pieces $X_1, \dots, X_{s-1}$ such that $\partial X_i = Y_i$ for $1 \leq i \leq s$. The conclusion follows, by taking $X_s = X'_2$. So from now on we assume that $s = 2$.

It suffices to prove the weaker result that a partition (X'_1, X'_2) exists with all the required properties for (X_1, X_2) except possibly the condition that they are non-empty. Indeed, if such a partition is found and for example $X'_2 = \emptyset$ then necessarily $Y_2 = \partial X'_2 = \emptyset$. In that case pick any $x \in X$, and choose a clopen neighbourhood V of x such that $V \cap \partial X$ is empty (this is possible because X is relatively open). Then $X_1 = X \setminus V$ and $X_2 = X \cap V$ give the conclusion.

Let $\rho : \overline{X} \to \partial X$ be a continuous retraction of $\overline{X}$ onto ∂X given by Theorem 3.2. Let $V \subseteq \partial X$ be any semi-algebraic set open in ∂X, Z its closure and $A = \rho^{-1}(Z) \cap X$. We are claiming that $\partial A = Z$. Note that A is closed in X by continuity of ρ, because A is the inverse image of the closed set Z by $\rho_{|X}$. So it suffices to prove that $\overline{A} \cap \partial X = Z$, or equivalently that $\overline{A} \cap \partial X$ contains V and is contained in Z. For the first inclusion let y be any element of V, and W any neighbourhood of y. We have to prove that $W \cap A \neq \emptyset$. By continuity of ρ at $y = \rho(y)$ there is a neighbourhood U of y such that $U \cap \overline{X}$ is contained in $\rho^{-1}(W \cap V)$. In particular

$$U \cap W \cap X \subseteq U \cap \overline{X} \subseteq \rho^{-1}(W \cap V) \subseteq \rho^{-1}(V) = A$$

so $U \cap W \cap A = U \cap W \cap X$. On the other hand, $U \cap W \cap X \neq \emptyset$ because $U \cap W$ is a neighbourhood of y and $y \in V \subseteq \overline{X}$. *A fortiori* $W \cap A$ is non-empty. This proves that $y \in \overline{A}$, hence that $V \subseteq \overline{A} \cap \partial X$. Conversely, if y' is any element of $\partial X \setminus Z$, there is a neighbourhood W' of y' such that $W' \cap \partial X$ is disjoint from Z. By continuity of ρ, $\rho^{-1}(W)$ is then a neighbourhood of y' in $\overline{X}$. It is disjoint from $\rho^{-1}(Z) = A$ hence $y' \notin \overline{A}$. So $\overline{A}$ is disjoint from $\partial X \setminus Z$. That is $\overline{A} \cap \partial X \subseteq Z$, which proves our claim.

Let $Z_1 = \overline{Y_1 \setminus Y_2}$ and $Z_2 = \overline{Y_2 \setminus Y_1}$. For $k = 1, 2$ let $A_k = \rho^{-1}(Z_k)$. Let Z_0 be the closure of $\partial X \setminus (Z_1 \cup Z_2)$ and $A_0 = \rho^{-1}(Z_0)$. The above claim gives that $\partial A_k = Z_k$ for $0 \leq k \leq 2$. Let B_0 be the set of non-isolated points of A_0. Clearly $\partial B_0 = \partial A_0 = Z_0$ since $A_0 \setminus B_0$ is finite. In particular B_0 is relatively open,

[15] See footnote 14

and Lemma 3.3 gives two semi-algebraic sets B_1, B_2 partitioning B_0 such that $\partial B_1 = \partial B_2 = Z_0$. So if we set $X_1 = A_1 \cup B_1$ and $X_2 = A_2 \cup B_2 \cup (A_0 \setminus B_0)$ we get the conclusion.

■

Theorem 3.5. *Let $f : X \subseteq K^m \to K$ be a semi-algebraic function with bounded graph (that is f is a bounded function on a bounded domain). If it has finitely many limit values at every point of $\overline{X}$ then f is piecewise largely continuous.*

Note that the counterpart of Theorem 3.5 for real-closed fields holds. Indeed, by triangulation we can reduce to the case of a continuous function f on a simplex $S \subseteq K^m$. The assumption that f has finitely many limit values at every point of $\overline{S}$ then implies directly that f is largely continuous. Indeed, this follows easily from the fact that over real-closed fields the direct image by a continuous semi-algebraic map of any semi-algebraically connected set (such as $S \cap B$ with B a ball centered at any point of $\overline{S}$) is again semi-algebraically connected.

On the contrary, p-adic simplexes are not at all semi-algebraically connected and it can happen that a function satisfying all these assumptions on a p-adic simplex is not largely continuous. For example on the simplex $S = D^1R^*$ the semi-algebraic function f defined by $f(x) = 0$ if $v(x) \in 2\mathcal{Z}$ and $f(x) = 1$ otherwise is a continuous, bounded function having two distinct limit values at 0. Thus f is not largely continuous. It is obviously piecewise largely continuous, though.

Proof: Every semi-algebraic function is piecewise continuous (see for example [Mou09]). So, replacing f by its restriction to the pieces of an appropriate partition of X if necessary, we can assume that f is continuous. Removing $X \cap \partial X$ if necessary (using a straightforward induction on $\dim X$ and the fact that $\dim \partial X < \dim X$) we can even assume that X is relatively open. The proof then goes by induction on the lexicographically ordered tuples (e, e') where $e = \dim X$ and $e' = \dim \partial X$. If ∂X is empty, that is X is closed, then f is largely continuous and the result is obvious. So let us assume that $e' \geq 0$ (hence $e \geq 1$) and the result is proved for smaller tuples (e, e').

Let $D = (\partial X \times K) \cap \overline{\mathrm{Gr}\, f}$. The projection of D onto ∂X has finite fibers hence D is a union of cells of type 0. The number of these cells, say N, then bounds the cardinality of these fibers, that is the number of limit values of f at every point of ∂X. For every $a \in \partial X$ let $D_a = \{t \in K : (a, t) \in D\}$. We first show that $\widehat{D} = \partial X$, that is $D_a \neq \emptyset$ for every $a \in \partial X$. For every $\varepsilon \in R^*$ let $C_\varepsilon = (B(a, \varepsilon) \times K) \cap \overline{\mathrm{Gr}\, f}$. This is a uniformly semi-algebraic family of closed and bounded semi-algebraic subsets of K^n. Each of them is non-empty because C_ε contains $(x, f(x))$ for any x in $B(a, \varepsilon) \cap X$ (which is non-empty since $a \in \partial X$). Obviously $C_{\varepsilon_1} \subseteq C_{\varepsilon_2}$ whenever $|\varepsilon_1| \leq |\varepsilon_2|$, so $\bigcap_{\varepsilon \in R^*} C_\varepsilon$ is non-empty by Theorem 2.4. This last set is equal to D_a, which proves our claim.

For $1 \leq i \leq N$ let W_i be the set of $a \in \partial X$ such that D_a has exactly i elements. These sets W_i form a partition of D in semi-algebraic pieces. By Theorem 2.7 (and a straightforward induction) there are semi-algebraic functions $f_{i,j} : W_i \to K$ such that $D_a = \{f_{i,j}(a)\}_{1 \leq j \leq i}$ for every $a \in D_a$. Since $\dim \partial X < \dim X$, by the induction hypothesis these functions $f_{i,j}$ are piecewise largely continuous. This gives a partition of ∂X in semi-algebraic pieces V_k for $1 \leq k \leq r$, and a family of largely continuous semi-algebraic functions

$g_{k,l} : V_k \to K$ for $1 \le l \le s_k$ such that $V_k \subseteq W_{s_k}$ and D is the union of the graphs of all these functions $g_{k,l}$.

Theorem 3.4 applied to X and the sets $\overline{V_k}$ for $1 \le k \le r$ gives a partition of X in semi-algebraic pieces X_k such that $\partial X_k = \overline{V_k}$. It suffices to prove that the restrictions of f to each X_k is piecewise largely continuous. So we can assume that $r = 1$ and $X = X_1$. That is, we have a semi-algebraic set $V = V_1$ dense in ∂X and largely continuous functions $g_l = g_{1,l} : V \to K$ for $1 \le l \le s = s_1$ such that $D_a = \{g_l(a)\}_{1 \le l \le s}$ has s elements for every $a \in D_a$. Replacing V by $V \setminus \partial V$ if necessary we can assume that V is relatively open.

Let $\rho : \overline{X} \to \overline{V}$ be a continuous retraction given by Theorem 3.2. For $1 \le l \le s$ let

$$U_l = \big\{ x \in X : \forall k \neq l,\ \big|f(x) - \bar{g}_l\big(\rho(x)\big)\big| < \big|f(x) - \bar{g}_k\big(\rho(x)\big)\big| \big\}.$$

Each U_l is open in X by continuity of f, ρ and the $\bar{g}_k$'s. Their complements $X' = X \setminus \bigcup_{l=1}^{s} U_l$ are closed in X, hence $\partial X' \subseteq \partial X$. Moreover, for every $a \in V$, the limit values of f at a being by construction the pairwise distinct $g_l(a)$ for $1 \le l \le s$, there exists $\varepsilon \in R^*$ such that every point of $B(a, \varepsilon) \cap X$ belongs to one of the U_l's. In other words $B(a, \varepsilon) \cap X' = \emptyset$ hence a does not belong to the closure of X'. So $\partial X \subseteq \partial X \setminus V = \partial V$, in particular $\dim \partial X' < \dim V = \dim \partial X$ hence the induction hypothesis applies to the restriction of f to X'.

It only remains to check that the restrictions of f to each U_l are piecewise largely continuous. We are claiming that f has only one limit value at every point a of $\overline{U}_l \setminus \partial V$. Note that $\overline{U}_l$ is the disjoint union of $\overline{U}_l \cap X$ and $\overline{U}_l \cap \partial X$, and that $\partial X = V \cup \partial V$. Obviously, if $a \in \overline{U}_l \cap X$ then by continuity of f, $f(x)$ tends to $f(a)$ as x tends to a in U_l. Now if $a \in (\overline{U}_l \setminus \partial V) \setminus X$ then $a \in V$, $\rho(a) = a$ and $\bar{g}_k(\rho(a)) = g_k(a)$ for every k. Hence by definition of U_l, $f(x)$ is closer to $g_l(a)$ than to every other $g_k(a)$, so $g_l(a)$ is the only possible limit value of $f(x)$ as x tends to a in U_l, which proves our claim. So the semi-algebraic function g which coincides with f on $\overline{U}_l \cap X$ and with g_l on V is continuous. The frontier of its domain is contained in $\overline{U_l} \cap \partial X \subseteq \partial X = V \cup \partial V$ and is disjoint from V, hence is contained in ∂V. By the induction hypothesis, g is then piecewise largely continuous, hence so is $f_{|U_l}$ since f and g coincide on U_l.

∎

4 Largely continuous cell decomposition

This section recalls the main theorem of [Den84] in order to emphasize some details which appear only in its proof. These details are important for us because they ensure that the functions defining the cells involved in the conclusions inherit certain properties, defined below, from the functions in the assumptions. Using them we are going to derive from $\mathbf{T}_m$ a new preparation theorem for semi-algebraic functions "up to a small deformation" (Theorem 4.7). The point is that after such a deformation, we get a Cell Preparation Theorem involving only cells defined by largely continuous functions.

In order to do so, it is crucial for us to control the boundary of any cell C we are dealing with. Ideally, we would like it to decompose naturally in cells defined by functions obtained for the functions defining C by passing to the limits, just as it is done for the faces of discrete polytopes (Item 3 of Proposition 2.11).

With this aim in mind, we now introduce a sharper notion of cell mod $\mathbf{G}$, for any clopen semi-algebraic subgroup $\mathbf{G}$ of $K^\times$ with finite index.

A **presented cell** A **mod** $\mathbf{G}$ in K^{m+1} is a tuple (c_A, ν_A, μ_A, G_A) with c_A a semi-algebraic function on a non-empty domain $X \subseteq K^m$ with values in K (called the **center** of A), ν_A and μ_A either semi-algebraic functions on X with values in $K^\times$ or constant functions on X with values 0 or ∞ (called the **bounds** of A), and G_A an element of K/G (called the **coset** of A), having the property that for every $x \in X$ there is $t \in K$ such that:

$$|\nu_A(x)| \leq |t - c_A(x)| \leq |\mu_A(x)| \quad \text{and} \quad t - c_A(x) \in G_A \tag{4}$$

We say that A is **largely continuous** if its center and bounds are. In any case the set of tuples $(x,t) \in X \times K$ satisfying (4) is a cell, in the general sense given in the introduction. When we want to distinguish this set from the presented cell A we call it the **cellular set underlying** A. Nevertheless, abusing the notation, we will also denote it A most often. The conditions enumerated above (4) ensure that the domain X of c_A, μ_A, ν_A is exactly the socle $\widehat{A}$ of A. When two presented cells A and B have the same underlying cellular set we write it $A \simeq B$.

From now onwards we will use the word "**cell**" mostly for *presented cells* but also very often for the *underlying cellular sets*, the difference being clear from the context. For instance we will freely talk of disjoint (presented) cells, of bounded (presented) cells, of (presented) cells partitioning some set and so on, meaning that the corresponding cellular sets have these properties. Also for any $Z \subseteq \widehat{A}$ we will write $A \cap (Z \times K)$ both for this (cellular) set and for the presented cell $(c_{A|Z}, \nu_{A|Z}, \mu_{A|Z}, G_A)$. The latter will also be denoted $(c_A, \nu_A, \mu_A, G_A)_{|Z}$. Similarly $\operatorname{Gr} c_A$ both denotes the graph of c_A and the presented cell $(c_A, 0, 0, \{0\})$.

A presented cell A is of **type** 0 if $G_A = \{0\}$, of type 1 otherwise. The type of A is denoted $\operatorname{tp} A$. We say that A is **well presented** if either $v\nu_A - v\mu_A$ is unbounded or $\nu_A = \mu_A$. We call A a **fitting cell** if it has **fitting bounds**, that is, for every $x \in \widehat{A}$:

$$|\mu_A(x)| = \sup\{|t - c_A(x)| : (x,t) \in A\}$$

$$|\nu_A(x)| = \inf\{|t - c_A(x)| : (x,t) \in A\}$$

Sometimes it will be convenient to write $G_A = \lambda_A \mathbf{G}$ for some $\lambda_A \in G_A$. We will always do this uniformly, so that $\lambda_A = \lambda_B$ whenever $G_A = G_B$. To that end a set $\Lambda_{\mathbf{G}}$ of representatives of $K/\mathbf{G}$ is fixed once and for all, and when we consider a presented cell A mod $\mathbf{G}$ it is understood that λ_A is the unique element of $G_A \cap \Lambda_{\mathbf{G}}$. In addition, we require from this set of representatives that every $\lambda \in \Lambda_{\mathbf{G}}$ has the smallest possible positive valuation. In particular if $\mathbf{G} = P_N^\times$ or $Q_{N,M}^\times$ and A is a cell mod $\mathbf{G}$ of type 1 then $0 \leq v\lambda_A < N$.

For every family $\mathcal{A}$ of presented cells in K^{m+1} we let[16] $\operatorname{CB}(\mathcal{A})$ denote the family of all the functions c_A, μ_A, ν_A for $A \in \mathcal{A}$. Given another family $\mathcal{D}$ of presented cells in K^{m+1} we say that:

1. $\mathcal{D}$ belongs to $\operatorname{Vect} \mathcal{A}$ if for every $D \in \mathcal{D}$, c_D, ν_D are K-linear combinations of functions $f_{|\widehat{D}}$ for $f \in \operatorname{CB}(\{A \in \mathcal{A} : \widehat{D} \subseteq \widehat{A}\})$, and either μ_D is such a linear combination as well or $\mu_D = \infty$.

[16]Here the letters CB stand for "center and boundaries".

2. $\mathcal{D}$ belongs to $\operatorname{Alg}_n \mathcal{A}$ if $\mathcal{D}$ is finer than $\mathcal{A}$ and for every $A \in \mathcal{A}$, every $D \in \mathcal{D}$ contained in A and every $(x,t) \in D$:

   (a) either $t - c_A(x) = \mathcal{U}_n(x,t)(t - c_D(x))$;

   (b) or $t - c_A(x) = \mathcal{U}_n(x,t)h_{D,A}(x)$ where $h_{D,A} : \widehat{D} \to R$ is the product of (finitely many) linear combinations of functions $c_{B|\widehat{D}}$ such that $B \in \mathcal{A}$ and $\widehat{D} \subseteq \widehat{B}$.

These somewhat cumbersome definitions help us to express Denef's Cell Decomposition Theorem in a slightly more precise way than in [Den84].

Theorem 4.1 (Denef)**.** *Given a semi-algebraic subgroup* $\mathbf{G}$ *of* $K^\times$ *with finite index, let* $\mathcal{A}$ *be a finite family of presented cells mod* $\mathbf{G}$ *in* K^{m+1}*. Then for every positive integer* n *there exists a finite family* $\mathcal{D}$ *of fitting cells mod* $\mathbf{G}$ *refining* $\mathcal{A}$ *such that* $\widehat{\mathcal{D}}$ *is a partition of* $\bigcup \widehat{\mathcal{A}}$ *and* $\mathcal{D}$ *belongs to* $\operatorname{Vect} \mathcal{A}$ *and to* $\operatorname{Alg}_n \mathcal{A}$*.*

This is essentially theorem 7.3 of [Den84]. Indeed, for any given integer N, if n is large enough then $1 + \pi^n R \subseteq P_N \cap R^\times$. Hence $\mathcal{U}_n(x,t)$ in conditions (2a), (2b) of the definition of $\operatorname{Alg}_n \mathcal{A}$ can be written $u(x,t)^N$ with u a semi-algebraic function from A to $R^\times$ (thanks to Theorem 2.7). This is how the above result is stated in [Den84] with $\mathbf{G} = K^\times$. Our slightly more precise form, as well as the additional properties involving $\operatorname{Vect} \mathcal{A}$ and $\operatorname{Alg}_n \mathcal{A}$, appear only in the proof of theorem 7.3 in [Den84] (still with $\mathbf{G} = K^\times$). The generalization to fitting cells mod an arbitrary clopen semi-algebraic group $\mathbf{G}$ with finite index in $K^\times$ is straightforward[17].

Given a polynomial function f, we say that a function $h : X \subseteq K^m \to K$ belongs to $\operatorname{coalg}(f)$ if there exists a finite partition of X into definable pieces H, on each of which the degree in t of $f(x,t)$ is constant, say e_H, and such that the following holds. If $e_H \leq 0$ then $h(x)$ is identically equal to 0 on H. Otherwise there is a family $(\xi_1, \dots, \xi_{r_H})$ of K-linearly independent elements in an algebraic closure of K and a family of definable functions $b_{i,j} : H \to K$ for $1 \leq i \leq e_H$ and $1 \leq j \leq r_H$, and $a_{e_H} : H \to K^*$ such that for every x in H

$$f(x,T) = a_{e_H}(x) \prod_{1 \leq i \leq e_H} \left(T - \sum_{1 \leq j \leq r_H} b_{i,j}(x)\xi_j \right)$$

and

$$h(x) = \sum_{1 \leq i \leq e_H} \sum_{1 \leq j \leq r_H} \alpha_{i,j} b_{i,j}(x)$$

with the $\alpha_{i,j}$'s in K. If $\mathcal{F}$ is any family of polynomial functions we let $\operatorname{coalg}(\mathcal{F})$ denote the set of linear combinations of functions in $\operatorname{coalg}(f)$ for f in $\mathcal{F}$.

Theorem 4.2 (Denef)**.** *Let* $\mathcal{F} \subseteq K[X,T]$ *be a finite family of polynomials, with* X *an* m*-tuple of variables and* T *one more variable. Let* $N \geq 1$ *be an integer and* $\mathcal{A}$ *a family of boolean combinations of subsets of the form* $f^{-1}(P_N)$ *with*

[17] Here is a sketchy proof. For each $A \in \mathcal{A}$ let B_A be the cell mod $K^\times$ with the same center of bounds as A. Denef's construction applied to the family $\mathcal{B}$ of all these cells B_A gives a family $\mathcal{C}$ of cells mod $K^\times$ refining $\mathcal{B}$. Each C in $\mathcal{C}$ is the union of a finite family $\mathcal{D}_C$ of cells mod $\mathbf{G}$ with the same center and bounds as C, each of which is clopen in C (because $\mathbf{G}$ is clopen in $K^\times$ with finite index). For each $A \in \mathcal{A}$ let $\mathcal{D}_A$ be the family all the cells in $\bigcup\{D_C : C \in \mathcal{C}\}$ contained in A. The family $\mathcal{D} = \bigcup \mathcal{D}_A : A \in \mathcal{A}\}$ gives the conclusion.

$f \in \mathcal{F}$. For every integer $n \geq 1$ there is a finite family of fitting cells mod $P_N^\times$ refining $\mathcal{A}$, with center and bounds in coalg$(\mathcal{F})$*, and for every such cell H a positive integer $\alpha_{f,H}$ and a semi-algebraic function $h_{f,H} : \widehat{H} \to K$ such that for every $(x,t) \in H$:*

$$f(x,t) = \mathcal{U}_n(x,t)h_{f,H}(x)(t - c_H(x))^{\alpha_{f,H}}.$$

Proof: W.l.o.g. we can assume that every f in $\mathcal{F}$ is non constant and that n is large enough so that $1 + \pi^n R \subseteq P_N^\times$. Theorem 7.3 in [Den84] gives a finite family of cells B mod $K^\times$ partitioning K^m, and for each of them a positive integer $\alpha_{f,B}$ and semi-algebraic functions $u_{f,B} : B \to R^\times$ and $h_{f,B} : \widehat{B} \to K$ such that:

$$\forall (x,t) \in B,\ f(x,t) = u_{f,B}(x,t)^N h_{f,B}(x)(t - c_B(x))^{\alpha_{f,B}} \tag{5}$$

Moreover the functions $u_{f,B}^N$ constructed in the proofs of lemma 7.2 and theorem 7.3 in [Den84] are precisely of the form $1 + \pi^n \omega_{f,B}$ for some semi-algebraic function $\omega_{f,B}$ on B, and the functions c_B, μ_B, ν_B constructed there belong to coalg$(\mathcal{F})$. Refining the socle of B if necessary we can ensure that $h_{f,B}(x)P_N^\times$ is constant as (x,t) ranges over B. On the other hand B splits into finitely many cells mod $P_N^\times$, with the same center and bounds as B, because $P_N^\times$ has finite index in $K^\times$. On each of these cells H, $f(x,t)P_N^\times$ is constant by (5). Hence H is either contained or disjoint from A, for every $A \in \mathcal{A}$. So the family of all these cells H which are contained in $\bigcup \mathcal{A}$ gives the conclusion.
■

Using that every semi-algebraic function is piecewise continuous, the cells mod $P_N^\times$ given by Theorem 4.2 can easily be chosen with continuous center and bounds. However it is not possible to ensure that they are largely continuous (think of the case where $\mathcal{A}$ consists of a single semi-algebraic set which is itself the graph of a semi-algebraic function which is not largely continuous). Our aim, in the remainder of this section, is to find a work-around. We are going to prove that it can be done, not exactly for θ but for a function $\theta \circ u_\eta$ where $\eta \in K^m$ can be chosen arbitrarily small and u_η is the linear automorphism of K^{m+1} defined by:

$$\forall (x,t) \in K^m \times K, \quad u_\eta(x,t) = (x + t\eta, t). \tag{6}$$

Remark 4.3. The smaller η is, the closer u_η is to the identity map since $\|\eta\|$ is also the norm (in the usual sense for linear maps) of $u_\eta - \mathrm{Id}$. So the functions $\theta \circ u_\eta$ can be considered as "arbitrarily small deformations" of θ.

In [vdD98] a good direction for a subset S of K^{m+1} is defined as a non-zero vector $x = (x_1, \dots, x_{m+1}) \in K^{m+1}$ such that every line directed by x has finite intersection with S. It is more convenient to identify such collinear vectors hence we redefine **good directions** for S as the points $x = [x_1, \dots, x_{m+1}]$ in the projective space $\mathbf{P}^m(K)$ such that every affine line in K^{m+1} directed by x has finite intersection with S.

Analogously we call $x \in \mathbf{P}^m(K)$ a **geometrically good direction** for a family $\mathcal{F}$ of polynomials in $K[X,T]$ if for every algebraic extension F of K and every $f \in \mathcal{F}$, x is a good direction for the zero set of f in F^{m+1}.

Remark 4.4. With the above notation, $[\eta, 1]$ is a good direction for S if and only if the projection of $u_\eta^{-1}(S)$ onto K^m has finite fibers. Indeed for every $a \in K^m$ and every $t \in K$ we have:

$$(a, 0) + t(\eta, 1) \in S \iff (a + t\eta, t) \in S \iff (a, t) \in u_\eta^{-1}(S)$$

Therefore $[\eta, 1]$ is a geometrically good direction for $\mathcal{F}$ if and only if for every algebraic extension F of K and every $f \in \mathcal{F}$, the projection onto F^m of the zero set of $f \circ u_\eta$ in F^{m+1} has finite fibers.

Lemma 4.5 (Good Direction)**.** *For every finite family $\mathcal{F}$ of non-zero polynomials in $K[X, T]$, the set of geometrically good directions for $\mathcal{F}$ contains a non-empty Zariski open subset of $\mathbf{P}^m(K)$. In particular, for every non-zero $\varepsilon \in R$ there is $\eta \in R^m$ such that $\|\eta\| \leq |\varepsilon|$ and $[\eta, 1]$ is a good direction for $\mathcal{F}$.*

Proof: Let $p_\mathcal{F}$ be the product of the polynomials in $\mathcal{F}$, and d its total degree. Then $p_\mathcal{F}$ can be written as $p_\mathcal{F} = p_\mathcal{F}^\circ - q_\mathcal{F}$ with $p_\mathcal{F}^\circ$ a non zero homogeneous polynomial of degree d and $q_\mathcal{F}$ a polynomial of total degree $< d$.

Let $b \in K^{m+1}$ be non-zero and x the corresponding point in $P^m(K)$. It is not a geometrically good direction for $\mathcal{F}$ if and only if for some algebraic extension F of K and some $a \in F^m$ the line $a + F.b$ is contained in the zero set of $p_\mathcal{F}$ in F^{m+1}, that is $p_\mathcal{F}(a + tb) = 0$ for every $t \in F$ or equivalently $p_\mathcal{F}^\circ(a + Tb) = q_\mathcal{F}(a + Tb)$. This implies that the degree in T of $p_\mathcal{F}^\circ(a + Tb)$ is $< d$. In particular the coefficient of T^d in $p_\mathcal{F}^\circ(a+Tb)$ is zero. A straightforward computation shows that this coefficient is just $p_\mathcal{F}^\circ(b)$.

So every element in $\mathbf{P}^m(K)$ which is outside the zero set of $p_\mathcal{F}^\circ$ is a geometrically good direction for $\mathcal{F}$. This proves the main point. Now if K^m is identified with its image in $\mathbf{P}^m(K)$ by the mapping $a \mapsto [a, 1]$ then every ball in K^m is Zariski dense in $\mathbf{P}^m(K)$, so the last claim of the lemma holds.
■

Lemma 4.6. *Assume $\mathbf{T}_m$. Let $\eta \in K^m$ be such that $[\eta, 1]$ is a geometrically good direction for $\mathcal{F}$. Let u_η be as in (6) and $\mathcal{F}_\eta = \{f \circ u_\eta : f \in \mathcal{F}\}$. Then every function in* coalg$(\mathcal{F}_\eta)$ *whose graph is bounded is piecewise largely continuous.*

Proof: The functions in coalg$(\mathcal{F}_\eta)$ are linear combinations of functions in coalg(f_η) for $f \in \mathcal{F}$, hence it suffices to fix any f in $\mathcal{F}$ and prove the result for coalg(f_η). Let d be the degree in T of f, and F a Galois extension of K in which every polynomial in $K[T]$ of degree $\leq d$ factors. Given a basis $\mathcal{B} = (\xi_1, \ldots, \xi_r)$ of F over K, for each integer $e \leq d$ let $a_e \in K[X]$ be the coefficient of T^e in f_η, let $A_e \subseteq K^m$ be the set of elements $x \in K^m$ such that $f_\eta(x, T)$ has degree e in T, and choose a family of semi-algebraic functions $b_{i,j} : A_e \to K$ such that for every $x \in A_e$

$$f_\eta(x, T) = a_e(x) \prod_{i \leq e} \Big(T - \sum_{j \leq r} b_{i,j}(x) \xi_j \Big). \tag{7}$$

Let $Z_F(f_\eta)$ denote the zero set of f_η in F, and $\sigma_1, \ldots, \sigma_r$ be the list of K-automorphisms of F. Fix an integer $i \leq e$, and for every $x \in A_e$ let

$$\lambda_i(x) = \sum_{j \leq r} b_{i,j}(x) \xi_j.$$

For every $k \leq r$ we have

$$\sigma_k(\lambda_i(x)) = \sum_{j \leq r} b_{i,j}(x)\sigma_k(\xi_j).$$

Inverting the matrix $(\sigma_k(\xi_j))_{j\leq r,k\leq r}$ gives for every $j \leq r$ the function $b_{i,j}$ as a linear combination of $\sigma_k \circ \lambda_i$ for $k \leq r$. By construction $\mathrm{Gr}\,\sigma_k \circ \lambda_i$ is contained in $Z_F(f_\eta)$. This set is closed, hence $\overline{\mathrm{Gr}\,\sigma_k \circ \lambda_i}$ is contained in $Z_F(f_\eta)$ too.

The projection of $Z_F(f_\eta)$ onto F^m has finite fibers since η is a good direction for $\mathcal{F}$ (see Remark 4.4). So the same holds for the closure of the graph of $\sigma_k \circ \lambda_i$. This means that each $\sigma_k \circ \lambda_i$ has finitely many different limit values at every point of $\overline{A_e}$. Obviously each $b_{i,j}$ inherits this property, hence so does every $h \in \mathrm{coalg}\, f_\eta$. If moreover the graph of h is bounded, it then follows from Theorem 3.5 (using $\mathbf{T}_m$) that h is piecewise largely continuous.
∎

Now we can turn to the "largely continuous cell preparation up to small deformation" which was the aim of this section. We obtain it by combining the above construction based on good directions and the classical cell preparation theorem for semi-algebraic functions from Denef (Corollary 6.5 in [Den84]) revisited by Cluckers (Lemma 4 in [Clu01]).

Theorem 4.7. *Assume $\mathbf{T}_m$. Let $(\theta_i : A_i \subseteq K^{m+1} \mapsto K)_{i\in I}$ be a finite family of semi-algebraic functions whose domains A_i are bounded. Then for some integer $e \geq 1$ and all integers $n, N \geq 1$ there exists a tuple $\eta \in K^m$, an integer $M_0 > 2v(e)$, an integer N_0 divisible by eN, and a finite family $\mathcal{D}$ of largely continuous fitting cells mod $Q^\times_{N_0,M_0}$, such that $\mathcal{D}$ refines $\{u_\eta^{-1}(A_i) : i \in I\}$ and such that for every $i \in I$, every $D \in \mathcal{D}$ contained in $u_\eta^{-1}(A_i)$ and every $(x, t) \in D$*

$$\theta_i \circ u_\eta(x,t) = \mathcal{U}_{e,n}(x,t)h_{i,D}(x)\big[\lambda_D^{-1}\big(t - c_D(x)\big)\big]^{\frac{\alpha_{i,D}}{e}}$$

where u_η is as in (6), $h_{i,D} : \widehat{D} \to K$ is a semi-algebraic function and $\alpha_{i,D} \in \mathbf{Z}$.

Moreover the set of $\eta \in K^m$ having this property is Zariski dense (in particular η can be chosen arbitrarily small), and the integers e, M can be chosen arbitrarily large (in the sense of footnote 12).

Remark 4.8. The above expression of $\theta_i \circ u_\eta$ is well defined because e divides N_0, $M_0 > 2v(e)$ and $\lambda_D^{-1}(t - c_D(x))$ belongs to Q_{N_0,M_0} for every $(x, t) \in D$ (see the definition of $x \mapsto x^{\frac{1}{e}}$ on Q_{N_0,M_0} after Lemma 2.8). Of course if D is of type 0, then $\lambda_D = t - c_D(x) = 0$ and we use our conventions that $0^{-1} = \infty$ and $\infty.0 = 1$.

If we were only interested in the existence of such a preparation theorem with largely continuous cells for $\theta_i \circ u_\eta$, the integer N would be of no use and could be taken equal to 1. However it will be convenient to allow different values of N when we will use Theorem 4.7 in the proof the Triangulation Theorem.

Proof: Let $e_*, M_* \geq 1$ be arbitrary integers. Corollary 6.5 in [Den84] applied to each θ_i gives an integer $e_i \geq 1$ and a family $\mathcal{A}_i$ of semi-algebraic sets partitioning A_i such that for every every A in $\mathcal{A}_i$ and every (x, t) in A:

$$\theta_i^{e_i}(x,t) = u_{i,A}(x,t)\frac{f_{i,A}(x,t)}{g_{i,A}(x,t)} \tag{8}$$

where $u_{i,A}$ is a semi-algebraic function from A to $R^\times$ and $f_{i,A}$, $g_{i,A}$ are polynomial functions such that $g_{i,A}(x,t) \neq 0$ on A. Replacing if necessary each e_i by a common multiple e of them and of e_*, we can assume that $e_i = e$ for every i and e is divisible by e_*. Let $\mathcal{A}$ be a refinement of $\bigcup_{i\in I} \mathcal{A}_i$.

Fix any two integers $n, N \geq 1$ and any integer n_0 such that $n_0 \geq n + v(e)$ and $n_0 > 2v(e)$. Since $D^{n_0}R^\times$ is a subgroup of $R^\times$ with finite index, every $A \in \mathcal{A}$ splits into finitely many semi-algebraic pieces on each of which $u_{i,A}$ is constant modulo $D^{n_0}R^\times$ (for every $i \in I$ such that $A \subseteq A_i$). Thus, refining $\mathcal{A}$ if necessary, (8) can be replaced, for every A in $\mathcal{A}$ contained in A_i and every (x,t) in A, by

$$\theta_i^e(x,t) = \mathcal{U}_{n_0}(x,t)\tilde{u}_{i,A}\frac{f_{i,A}(x,t)}{g_{i,A}(x,t)} \tag{9}$$

with $\tilde{u}_{i,A} \in R^\times$.

Each A in $\mathcal{A}$ is semi-algebraic. So there is a finite family $\mathcal{B}$ of semi-algebraic sets refining $\mathcal{A}$, an integer $N_0 \geq 1$ and a finite list $\mathcal{F}$ of non-zero polynomials in $m+1$ variables such that every element of $\mathcal{B}$ is a boolean combinations of sets $f^{-1}(P_{N_0})$ with $f \in \mathcal{F}$. By Remark 2.1, N_0 can be chosen divisible by eN. Expanding $\mathcal{F}$ if necessary, we can assume that all the polynomials $f_{i,A}$ and $g_{i,A}$ in (9) also belong to $\mathcal{F}$, except those which are equal to the zero polynomial.

Lemma 4.5 gives $\eta \in K^{m+1}$ such that $[\eta, 1]$ is a geometrically good direction for $\mathcal{F}_\eta$, where $\mathcal{F}_\eta = \{f \circ u_\eta : f \in \mathcal{F}\}$. Note that every set in $\mathcal{A}_\eta = \{u_\eta^{-1}(A) : A \in \mathcal{A}\}$ is a boolean combination of sets $f_\eta^{-1}(P_{N_0})$ with $f_\eta \in \mathcal{F}_\eta$. Denef's Theorem 4.2 applied to $\mathcal{F}_\eta$ gives a finite family $\mathcal{C}$ of fitting cells mod $P_{N_0}^\times$ which refines $\mathcal{A}_\eta$ and whose center and bounds belong to $\operatorname{coalg}\mathcal{F}_\eta$, such that for every $f \in \mathcal{F}$, every $C \in \mathcal{C}$ and every $(x,t) \in C$

$$f_\eta(x,t) = \mathcal{U}_{n_0}(x,t)h_{f,C}(x)\big(t - c_C(x)\big)^{\alpha_{f,C}} \tag{10}$$

where $h_{f,C} : \widehat{C} \to K$ is a semi-algebraic function and $\alpha_{f,C}$ is a positive integer. We removed the zero polynomial from $\mathcal{F}$, but obviously (10) holds for $f = 0$ as well, by taking $h_{f,C} = 0$ in that case. Each A_i is bounded hence so is their union $\bigcup \mathcal{A}$ as well as $\bigcup \mathcal{A}_\eta$. So the center and bounds of every cell in $\mathcal{C}$ must be bounded functions with bounded domain. By Lemma 4.6 (assuming $\mathbf{T}_m$) these functions are piecewise largely continuous. Refining the socle of $\mathcal{C}$ if necessary, and $\mathcal{C}$ accordingly, we can then reduce to the case where every cell in $\mathcal{C}$ is largely continuous. Note that $\mathcal{U}_{n_0} \circ u_\eta = \mathcal{U}_{n_0}$, so by combining (9) and (10) we get that for every $i \in I$, every $C \in \mathcal{C}$ contained in $u_\eta^{-1}(A_i)$ and every $(x,t) \in C$

$$\theta_{i,\eta}(x,t)^e = \mathcal{U}_{n_0}(x,t)h_{i,C}(x)\big(t - c_C(x)\big)^{\alpha_{i,C}} \tag{11}$$

where $\theta_{i,\eta} = \theta_i \circ u_\eta$, $h_{i,C} : \widehat{C} \to K$ is a semi-algebraic function and $\alpha_{i,C} \in \mathbf{Z}$. For any integer $M_0 > 2v(e)$, $Q_{N_0,M_0}^\times$ is a subgroup with finite index in $P_{N_0}^\times$ hence every such cell C mod $P_{N_0}^\times$ splits into finitely many cells D mod $Q_{N_0,M_0}^\times$ with the same center, bounds and type as C. The integer M_0 can be chosen arbitrarily large, in particular greater than M_*. Let $\mathcal{D}$ be the family of all these cells D. From (11) and Lemma 2.8 we derive that for every $i \in I$, every $D \in \mathcal{D}$ contained in $u_\eta^{-1}(A_i)$ and every $(x,t) \in D$

$$\theta_{i,\eta}(x,t)^e = \mathcal{U}_{n_0}(x,t)\tilde{h}_{i,D}(x)\Big(\big[\lambda_D^{-1}\big(t - c_D(x)\big)\big]^{\frac{\alpha_{i,D}}{e}}\Big)^e \tag{12}$$

where $\tilde{h}_{i,D} = h_{i,C}$ and $\alpha_{i,D} = \alpha_{i,C}$ with C the unique cell in $\mathcal{C}$ containing D. The factor $\mathcal{U}_{n_0}$ in (12) can be written $\mathcal{U}^e_{n_0-v(e)}$ by Remark 2.9. Thus (12) implies that $\tilde{h}_{i,D}$ takes values in P_e. So by Theorem 2.7 there is a semi-algebraic function $h_{i,D}$ such that $\tilde{h}_{i,D} = h^e_{i,D}$. As a consequence, from (12) it follows that there is a semi-algebraic function $\chi_{i,D}$ with values in U_e such that for every $(x,t) \in D$

$$\theta_{i,\eta}(x,t) = \chi_{i,D}(x,t)\mathcal{U}_{n_0-v(e)}(x,t)h_{i,D}(x)\big[\lambda_D^{-1}\big(t-c_D(x)\big)\big]^{\frac{\alpha_{i,D}}{e}} \qquad (13)$$

By construction $n_0 - v(e) \geq n$ hence the factor $\mathcal{U}_{n_0-v(e)}$ can *a fortiori* be replaced by $\mathcal{U}_n$. Then $\chi_{i,D}\mathcal{U}_n$ (which is just $\mathcal{U}_{e,n}$) replaces $\chi_{i,D}\mathcal{U}_{n_0-v(e)}$ in (13), which proves the result.
■

5 Cellular complexes

For this and the next section, let $\mathbf{G}$ be a fixed semi-algebraic clopen subgroup of $K^\times$ with finite index. Then $v\mathbf{G}$ is a subgroup of $\mathcal{Z}$ with finite index, hence $v\mathbf{G} = N_0\mathcal{Z}$ for some integer $N_0 \geq 1$. Our aim in these two sections is to prove that every finite family of bounded largely continuous fitting cells mod $\mathbf{G}$, such as the one given by Theorem 4.7, can be refined in a complex of cells mod $\mathbf{G}$ satisfying certain restrictive assumptions defined below.

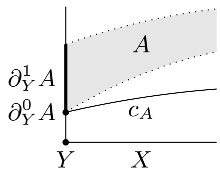
Notation. For every largely continuous fitting cell A mod $\mathbf{G}$ in K^{m+1} with socle X, recall that $A = (c_A, \mu_A, \nu_A, G_A)$ is a presented cell. For every semi-algebraic set Y contained in $\overline{X}$, $(\bar{c}_A, \bar{\nu}_A, \bar{\mu}_A, G_A)_{|Y}$ is then also a largely continuous presented cell mod $\mathbf{G}$, provided the restrictions to Y of $\bar{\nu}_A$ and $\bar{\mu}_A$ either take values in $K^\times$ or are constant, and the underlying set of tuples $(y,t) \in Y \times K$ defined by

$$|\bar{\mu}_A(y)| \leq |t - \bar{c}_A(x)| \leq |\bar{\nu}_A(x)| \text{ and } t - \bar{c}_A(x) \in G_A \qquad (14)$$

is non-empty. Similarly, the sets $\partial^0_Y A$ and $\partial^1_Y A$ defined below are (if non-empty) largely continuous fitting cells mod $\mathbf{G}$ contained in $\overline{A} \cap (Y \times K)$.

- $\partial^0_Y A = (\bar{c}_A, 0, 0, \{0\})_{|Y}$ if $\bar{\nu}_A = 0$ on Y, $\partial^0_Y A = \emptyset$ otherwise;
- $\partial^1_Y A = (\bar{c}_A, \bar{\nu}_A, \bar{\mu}_A, G_A)_{|Y}$ if $\bar{\mu}_A \neq 0$ on Y, $\partial^1_Y A = \emptyset$ otherwise.

If non empty the underlying set of $\partial^0_Y A$ is the graph of the restriction of $\bar{c}_A$ to Y, while the underlying set of ∂^A_Y is the set of $(x,t) \in Y \times K$ satisfying (14). For example, when $\bar{\nu}_A = 0 \neq \bar{\mu}_A$ on Y and $\nu_A \neq 0$ on X, we can intuitively represent these sets as follows.



Provided that on Y, $\bar{\nu}_A$ and $\bar{\mu}_A$ either take values in $K^\times$ or are constant, $\partial^0_Y A$ and $\partial^1_Y A$ are (if non-empty) largely continuous fitting cells mod $\mathbf{G}$ contained in $\overline{A} \cap (Y \times K)$.

Remark 5.1. If $\mathcal{X}$ is a partition of $\overline{X}$, the family of non-empty $\partial^i_Y A$ for $i \in \{0,1\}$ and $Y \in \mathcal{X}$ form a partition of $\overline{A}$.

Given two cells A, B in K^{m+1} and an integer $n \geq 1$, we write $B \lhd^n A$ if $B \subseteq A$ and if there exists $\alpha \in \{0,1\}$ and a semi-algebraic function $h_{B,A} : \widehat{B} \to K$ such that for every (x,t) in B:

$$t - c_A(x) = \mathcal{U}_n(x,t) h_{B,A}(x)^\alpha \big(t - c_B(x)\big)^{1-\alpha}$$

We call $h_{B,A}$ a $\lhd^n$**-transition** for (B,A). If $\mathcal{A}$, $\mathcal{B}$ are families of cells in K^{m+1} we write $\mathcal{B} \lhd^n \mathcal{A}$ if $B \lhd^n A$ for every $B \in \mathcal{B}$ and $A \in \mathcal{A}$ such that B meets A. A $\lhd^n$**-system** for $(\mathcal{B},\mathcal{A})$ is then the data of one $\lhd^n$-transition for each possible (B,A) in $\mathcal{B} \times \mathcal{A}$.

Remark 5.2. For any two finite families $\mathcal{A}$, $\mathcal{B}$ of cells mod $\mathbf{G}$, if $\mathcal{B}$ refines $\mathcal{A}$ and belongs to $\operatorname{Alg}_n \mathcal{A}$ then $\mathcal{B} \lhd^n \mathcal{A}$.

A **closed** $\lhd^n$**-complex** of cells mod $\mathbf{G}$ is a finite family $\mathcal{A}$ of largely continuous fitting cells mod $\mathbf{G}$ such that $\bigcup \mathcal{A}$ is closed, the socle of $\mathcal{A}$ is a complex of sets and for every $A, B \in \mathcal{A}$ if B meets $\overline{A}$ then for some $i \in \{0,1\}$, $\partial^i_Y A$ is a cell[18] and $B \lhd^n \partial^i_Y A$, with $Y = \widehat{B}$. If moreover $B = \partial^i_Y A$ we call $\mathcal{A}$ a **closed cellular complex** mod $\mathbf{G}$. As the terminology suggests, we are going to prove that closed $\lhd^n$- and cellular complexes are complexes of sets in the general sense of Section 2 (see Proposition 5.3). Any subset of a closed $\lhd^n$-complex (resp. closed cellular complex) is a $\lhd^n$**-complex** (resp. a **cellular complex**. As usually we call them **monoplexes** if they form a tree with respect to the specialization order.

When $\mathcal{A}$ is a $\lhd^n$-complex of cells mod $\mathbf{G}$, for all $Y \in \widehat{\mathcal{A}}$ and for all cells A, B in $\mathcal{A}$ such that B meets $\overline{A}$, there is an integer $\alpha \in \{0,1\}$ and a semi-algebraic function $h_{B,A} : \widehat{B} \to K$ such that for every (x,t) in B:

$$t - \bar{c}_A(x) = \mathcal{U}_n(x,t) h_{B,A}(x)^\alpha \big(t - c_B(x)\big)^{1-\alpha}.$$

An **inner** $\lhd^n$**-system** for $\mathcal{A}$ is the data of one function $h_{B,A}$ as above for every possible $A, B \in \mathcal{A}$.

Proposition 5.3. *Let $\mathcal{A}$ be a closed $\lhd^n$-complex of cells mod $\mathbf{G}$. Then $\mathcal{A}$ is a closed complex of sets. Moreover, for every $A, B \in \mathcal{A}$ and every $Z \in \widehat{\mathcal{A}}$ if B meets $\partial^0_Z A$ then $B = \partial^0_Z A = \operatorname{Gr} \bar{c}_{A|Z}$.*

Proof: By assumption the socle of every cell A in $\mathcal{A}$ is relatively open and pure dimensional. Thanks to the restrictions we made on the bounds in our definition of presented cells, it follows that A is also relatively open and pure dimensional.

In order to show that $\mathcal{A}$ is a partition, let A, B be two cells in $\mathcal{A}$ which are not disjoint and let $X = \widehat{A}$. Both $\widehat{B}$ and X belong to $\widehat{\mathcal{A}}$ and are not disjoint,

[18] The condition $\partial^i Y A$ is a cell means that on Y, $\bar{\mu}_A$ and $\bar{\nu}_A$ either take values in $K^\times$ or are constant.

hence $\widehat{B} = X$. Since B meets $A \subseteq \overline{A}$, by assumption B is contained in $\partial^i_X A$ with $i = \operatorname{tp} B$. But then $\partial^i_X A$ meets A, hence obviously is equal to A. So $B \subseteq A$, and equality holds by symmetry.

Now let A be any cell in $\mathcal{A}$ and $X = \widehat{A}$. Since $\mathcal{A}$ is a closed complex, every point of $\overline{A}$ belongs to a unique $B \in \mathcal{A}$. Since B meets $\overline{A}$, by assumption $B \subseteq \partial^i_Y A$ with $Y = \widehat{B}$ and $i = \operatorname{tp} B$. In particular $B \subseteq \overline{A}$, which proves that $\overline{A}$ is a union of cells in $\mathcal{A}$ (hence so is ∂A since $\mathcal{A}$ is a partition and ∂A is disjoint from A). This proves that $\mathcal{A}$ is a closed complex of sets.

The last point follows. Indeed, if B meets $\partial^0_Z A \subseteq \overline{A}$ then it is contained in $\partial^i_Y A$ for some $i \in \{0, 1\}$, with $Y = \widehat{B}$. In particular $\partial^0_Z A$ meets $\partial^i_Y A$. They are two pieces of a partition of $\overline{A}$ (see Remark 5.1) hence $\partial^0_Y A = \partial^i_Z A$. Therefore $Y = Z$ and $i = 0$, so $B \subseteq \partial^0_Z A$. That is, B is of type 0 and $c_B = \bar{c}_A$ on $\widehat{B} = Z$, so $B = \operatorname{Gr} \bar{c}_{A|Z} = \partial^0_Z A$.

■

Proposition 5.4. *Let $\mathcal{A}$ be a finite family of largely continuous fitting cells mod* $\mathbf{G}$ *and $n \geq 1$ an integer. There exists a $\lhd^n$–complex $\mathcal{D}$ of cells mod* $\mathbf{G}$ *refining $\mathcal{A}$ such that $\mathcal{D} \lhd^n \mathcal{A}$.*

In the next section we will prove that one can even require that $\mathcal{D}$ is a cellular monoplex mod G.

Proof: The proof goes by induction on $d = \dim \bigcup \widehat{\mathcal{A}}$. If a $\lhd^n$-complex $\mathcal{D}_1$ is found which proves the result for a family $\mathcal{A}_1$ of cells mod $\mathbf{G}$ containing $\mathcal{A}_1$ then obviously the family $\mathcal{D}$ of cells in $\mathcal{D}_1$ contained in $\bigcup \mathcal{A}$ proves the result for $\mathcal{A}$. Thus, enlarging $\mathcal{A}$ if necessary, we can assume that $\bigcup \mathcal{A}$ and $\bigcup \widehat{\mathcal{A}}$ are closed. By Denef's Theorem 4.1 and Remark 5.2 there is a finite family $\mathcal{B}$ of largely continuous fitting cells mod $\mathbf{G}$ refining $\mathcal{A}$ such that $\mathcal{B} \lhd^n \mathcal{A}$. Replacing $\mathcal{A}$ by this refinement if necessary we can also assume that $\mathcal{A}$ is a partition.

If $\mathcal{B}$ is any vertical refinement of $\mathcal{A}$ then obviously $\mathcal{B} \lhd^n \mathcal{A}$. Thus, by taking if necessary a finite partition $\mathcal{X}$ refining $\widehat{\mathcal{A}}$ and replacing $\mathcal{A}$ by the corresponding vertical refinement (that is the family of all cells $A \cap (X \times K)$ with $A \in \mathcal{A}$ and $X \in \mathcal{X}$ contained in $\widehat{A}$), we can assume that $\widehat{\mathcal{A}} = \mathcal{X}$ is a partition. By the same argument we can assume as well that for every $A \in \mathcal{A}$ and every $X \in \mathcal{X}$ contained in ∂X, the restrictions of $\bar{\mu}_A$ and $\bar{\nu}_A$ to X take values in $K^\times$ or are constant, hence $\partial^0_X A$ (resp. $\partial^1_X A$) is a cell with socle X whenever it is non-empty. By Remark 2.15 we can even assume that it is a complex of pure dimensional sets. Let $\mathcal{X}_d$ be the family of $X \in \mathcal{X}$ with dimension d. Note that every $X \in \mathcal{X}_d$ is open in $\bigcup \mathcal{X}$ because $\mathcal{X}$ is a complex and $\dim \bigcup \mathcal{X} = d$.

For every $X \in \mathcal{X}_d$ let $\mathcal{A}_X$ be the family of cells in $\mathcal{A}$ with socle X. For every cell $A \in \mathcal{A}_X$ of type 1 such that $\nu_A = 0$, $\operatorname{Gr} c_A$ is contained in $\overline{A}$ hence in $\bigcup \mathcal{A}_X$ since $\bigcup \mathcal{A}$ is closed and $\widehat{\mathcal{A}}$ is a partition. It may happen that $\operatorname{Gr} c_A$ does not belong to $\mathcal{A}$. With Proposition 5.3 in view we have to remedy this. Every point $(x, c_A(x))$ in $\operatorname{Gr} c_A$ belongs to some cell B in $\mathcal{A}_X$. This cell must be of type 0 otherwise the fiber $B_x = \{t \in K : (x, t) \in B\}$ would be open, hence it would contain a neighbourhood V of $c_A(x)$ and so $\{x\} \times V$ would be contained in B and meet A, which implies that $B \subseteq A$ since $\mathcal{A}_X$ is partition, in contradiction with the fact that B meets $\operatorname{Gr} c_A$. So there is a finite partition $\mathcal{Y}_A$ of X in semi-algebraic pieces Y on each of which there is a unique cell $B \in \mathcal{A}_X$ of type 0 whose center coincides with c_A on Y. Repeating the same argument for every

$A \in \mathcal{A}_X$ and every $X \in \mathcal{X}_d$ gives a finite partition $\mathcal{Y}$ of $\bigcup \mathcal{X}_d$ finer then every such $\mathcal{Y}_A$. Let $\mathcal{X}'$ be a complex of pure dimensional semi-algebraic sets refining $\mathcal{X} \cup \mathcal{Y}$. Replacing if necessary $\mathcal{A}$ by the vertical refinement defined by $\mathcal{X}'$, we can then assume from now on that for every X in $\mathcal{X}_d$ and every $A \in \mathcal{A}$ with socle X, if $\nu_A = 0$ then $\operatorname{Gr} c_A$ belongs to $\mathcal{A}$.

Let $\mathcal{A}_d = \{A \in \mathcal{A} : \widehat{A} \in \mathcal{X}_d\}$ and $\mathcal{B}$ be the union of $\mathcal{A} \setminus \mathcal{A}_d$ and of the family of non-empty $\partial^i_Y A$ for $i \in \{0,1\}$, $A \in \mathcal{A}_d$ and $Y \in \mathcal{X} \setminus \mathcal{X}_d$. Clearly $\dim \bigcup \widehat{B} < d$ so the induction hypothesis gives a $\lhd^n$–complex $\mathcal{C}$ of cells mod $\mathbf{G}$ refining $\mathcal{B}$ such that $\mathcal{C} \lhd^n \mathcal{B}$. *A fortiori* $\mathcal{C} \lhd^n (\mathcal{A} \setminus \mathcal{A}_d)$ because the latter is contained in $\mathcal{B}$. So if we let $\mathcal{D} = \mathcal{C} \cup \mathcal{A}_d$, then $\mathcal{D}$ refines $\mathcal{A}$ and $\mathcal{D} \lhd^n \mathcal{A}$. It only remains to check that $\mathcal{D}$ is a $\lhd^n$-complex, and first that $\widehat{\mathcal{D}}$ is a complex of sets.

Note that $\widehat{\mathcal{A}}_d = \mathcal{X}_d$ hence $\widehat{\mathcal{D}} = \widehat{\mathcal{C}} \cup \widehat{\mathcal{A}}_d = \widehat{\mathcal{C}} \cup \mathcal{X}_d$ is a partition, and every set in $\widehat{\mathcal{D}}$ is pure dimensional and relatively open (by induction hypothesis for $\widehat{\mathcal{C}}$ and by construction for $\mathcal{X}_d$). For every $X \in \widehat{\mathcal{D}}$, we have to prove that ∂X is a union of sets in $\widehat{\mathcal{C}} \cup \mathcal{X}_d$. If $X \in \widehat{\mathcal{C}}$ this is clear because $\widehat{\mathcal{C}}$ is a complex. Otherwise $X \in \mathcal{X}_d$ hence ∂X is a union of sets in $\mathcal{X}$ (because $\mathcal{X}$ is a complex). All these sets have dimension $< d = \dim X$ hence belong to $\mathcal{X} \setminus \mathcal{X}_d$. But $\mathcal{C}$ refines $\mathcal{B}$, which contains $\mathcal{A} \setminus \mathcal{A}_d$, whose socle is $\mathcal{X} \setminus \mathcal{X}_d$, hence $\widehat{\mathcal{C}}$ refines $\mathcal{X} \setminus \mathcal{X}_d$. Thus ∂X is also the union of sets in $\widehat{\mathcal{C}}$, hence of $\widehat{\mathcal{D}}$.

Now let $D, E \in \mathcal{D}$ be such that E meets $\overline{D}$, let $X = \widehat{D}$ and $Y = \widehat{E}$. By construction $\partial^0_Y D$ and $\partial^1_Y D$ are cells (if non-empty) and cover $\overline{D} \cap (Z \times K)$. So there is $i \in \{0,1\}$ such that $\partial^i_Y D$ is a cell which meets E. We have to prove that $E \lhd^n \partial^i_X D$. Note that Y meets the socle of $\overline{D}$, which is contained in $\overline{X}$, hence $Y = X$ or $Y \subseteq \partial X$ because $\widehat{\mathcal{D}}$ is a complex. So, if $\dim X < d$ then also $\dim Y < d$ hence $D, E \in \mathcal{C}$. In that case $E \lhd^n \partial^j_X D$ because $\mathcal{C}$ is a $\lhd^n$–complex. Thus we can assume that $\dim X = d$, that is $D \in \mathcal{A}_d$. We know that $Y = X$ or $Y \subseteq \partial X$. In the first case $Y = X$ hence $\partial^i_X D \in \mathcal{A}_d \subseteq \mathcal{D}$ by construction, so $E = \partial^i_X D$ because $\mathcal{D}$ is a partition. In the second case $Y \in \widehat{\mathcal{C}}$ hence $E \in \mathcal{C}$. Now Y is contained in some $Z \in \mathcal{X} \setminus \mathcal{X}_d$ because $\widehat{\mathcal{C}}$ refines $\mathcal{X} \setminus \mathcal{X}_d$, and E meets $\partial^i_Z D$. By construction $\partial^i_Z D$ belongs to $\mathcal{B}$. Since $\mathcal{C} \lhd^n \mathcal{B}$ it follows that $E \lhd^n \partial^i_Z D$ hence *a fortiori* $E \lhd^n \partial^i_Y D$ because $E \subseteq Y \times K$ and $\partial^i_Y D = \partial^i_Z D \cap (Y \times K)$.
■

Before entering in more complicated constructions, let us mention here two elementary properties of fitting cells which will be of some use later.

Proposition 5.5. *Let $A \subseteq K^{m+1}$ be a cell mod $\mathbf{G}$ of type 1. Then:*

- *μ_A is a fitting bound if and only if $\mu_A = \infty$ or $v\mu_A(\widehat{A}) \subseteq vG_A$.*
- *ν_A is a fitting bound if and only if $\nu_A = 0$ or $v\nu_A(\widehat{A}) \subseteq vG_A$).*

Proof: The case where $\mu_A = \infty$ being trivial, we can omit it. If $\mu_A \neq \infty$ is a fitting bound then obviously $v\mu_A(\widehat{A}) \subseteq vG_A$ because $v(t - c_A(x)) \in vG_A$ for every $(x,t) \in A$. Conversely assume that $v\mu_A(\widehat{A}) \subseteq vG_A$. Let x be any element of $\widehat{A}$. We have to prove that $|\mu_A(x)| = \max\{|d| : d \in D_x\}$ where $D_x = \{t - c_A(x) : (x,t) \in A\}$. D_x is bounded since $\mu_A \neq \infty$, hence by Corollary 2.6 it contains an element d of maximal norm. By construction $|d| \leq |\mu_A(x)|$. Assume for a contradiction that $|d| < |\mu_A(x)|$, that is $v(d/\mu_A(x)) > 0$. By construction $v(d)$ and $v\mu_A(x)$ belong to $vG_A = v\lambda_A + v\mathbf{G}$ hence $v(d/\mu_A(x)) \in v\mathbf{G} = N_0\mathcal{Z}$. Thus $v(d/\mu_A(x)) \geq N_0$, that is $|d| \leq |\pi^{N_0}\mu_A(x)|$. Pick any $g \in \mathbf{G}$ such that

$v(g) = N_0$ and let $t' = c_A(x) + d/g$. We have $t' - c_A(x) = d/g \in G_A$, $|\nu_A(x)| \leq |d| \leq |d/g|$ and $|d/g| \leq |\mu_A(x)|$, hence $(x, t') \in A$. So $t' - c_A(x) \in D_x$ and $|d| < |t' - c_A(x)|$, a contradiction. The proof for ν_A is similar and left to the reader.
■

Proposition 5.6. *For every fitting cell A mod Q_{N_0,M_0} in K^{m+1}, if $A \subseteq R^{m+1}$ then $v\mu_A \geq -M_0$.*

Since $A \subseteq R^{m+1}$, one may naively expect that $|\mu_A| \leq 1$, that is $v\mu_A \geq 0$. The presented cell $A = (-\pi^{-M_0}, \pi^{-M_0}, \pi^{-M_0}, Q_{N_0,M_0})$ is a counterexample in K: it is contained in R (it is actually equal to R) and $v\mu_A = -M_0 < 0$.

Proof: Assume the contrary, that is $v\mu_A(x) < -M_0$ for some $x \in \widehat{A}$. Since A is a fitting cell there is $t \in K$ such that $(x, t) \in A$ and $v(t - c_A(x)) = v\mu_A(x)$. Since $A \subseteq R^{m+1}$, $t \in R$ hence $v(t - c_A(x)) < 0 = v(t)$ implies that $vc_A(x) = v(t - c_A(x)) = v\mu_A(x)$. So there are $a \in R$ and $g \in Q_{N,M}$ such that $c_A(x) = a\pi^{M_0+1}$ and $t - c_A(x) = \lambda_A g$. In particular $v(\lambda_A g) = v(t - c_A(x)) = v\mu_A(x) < -M_0$ so $\pi^{M_0}\lambda_A g \notin R$. Now let $t' = t + \pi^{M_0}\lambda_A g$, then $t' \notin R$ since $t \in R$ and $\pi^{M_0}\lambda_A g \notin R$. On the other hand $1 + \pi^{M_0} \in Q_{N_0,M_0}$ and

$$t' - c_A(x) = t - c_A(x) + \pi^{M_0}\lambda_A g = \lambda_A g + +\pi^{M_0}\lambda_A g = \lambda_A(1 + \pi^{M_0})g.$$

So $t' - c_A(x) \in \lambda_A Q_{N_0,M_0}$ and $v(t' - c_A(x)) = v(\lambda_A(1 + \pi^{M_0})g) = v(\lambda_A g) = v\mu_A(x)$. Thus $(x, t') \in A$, a contradiction since $t' \notin R$ and $A \subseteq R^{m+1}$.
■

6 Cellular monoplexes

We keep as in Section 5 a semi-algebraic clopen subgroup $\mathbf{G}$ of $K^\times$ with finite index, and $N_0 \geq 1$ an integer such that $v\mathbf{G} = N_0\mathcal{Z}$. Lemma 6.1 below (together with Lemma 7.11) is the technical heart of this paper. This section is entirely devoted to its proof.

Lemma 6.1. *Assume $\mathbf{T}_m$. Let $\mathcal{A}$ be a finite set of bounded, largely continuous, fitting cells mod $\mathbf{G}$ in K^{m+1}. Let $\mathcal{F}_0$ be a finite family of definable functions with domains in $\widehat{\mathcal{A}}$. Let $n, N \geq 1$ be a pair of integers. For some integers e, $M > 2v(e)$ which can be made arbitrarily large (in the sense of footnote 12), there is a tuple $(\mathcal{V}, \varphi, \mathcal{D}, \mathcal{F}_\mathcal{D})$ such that:*

- *$\mathcal{D}$ is a cellular monoplex mod $\mathbf{G}$ refining $\mathcal{A}$ such that $\mathcal{D} \lhd^n \mathcal{A}$.*
- *$\mathcal{F}_\mathcal{D}$ is a $\lhd^n$-system for $(\mathcal{D}, \mathcal{A})$.*
- *$(\mathcal{V}, \varphi)$ is a triangulation of*[19] *$\mathcal{F}_0 \cup \mathcal{F}_\mathcal{D} \cup \mathrm{CB}(\mathcal{D})$ with parameters (n, N, e, M), such that $\widehat{\mathcal{D}} = \varphi(\mathcal{V})$.*

Note that, in order to obtain this result, it does not suffice to find a continuous monoplex $\mathcal{D}$ of well presented cells mod $\mathbf{G}$ refining $\mathcal{A}$ such that $\mathcal{D} \lhd^n \mathcal{A}$, and then to select an arbitrary $\lhd^n$-system $\mathcal{F}_\mathcal{D}$ for $(\mathcal{D}, \mathcal{A})$ and to apply $\mathbf{T}_m$ to

[19]Recall that $\mathrm{CB}(\mathcal{D})$ denotes the family of center and bounds of the cells in $\mathcal{D}$.

$\mathcal{F}_0 \cup \mathcal{F}_{\mathcal{D}} \cup \mathrm{CB}(\mathcal{D})$. Indeed, this will give a triangulation $(\mathcal{V}, \varphi)$ of $\mathcal{F}_0 \cup \mathcal{F}_{\mathcal{D}} \cup \mathrm{CB}(\mathcal{D})$. But $\varphi(\mathcal{V})$ will then be a refinement of $\widehat{\mathcal{D}}$, not $\widehat{\mathcal{D}}$ itself. It is then tempting to vertically refine $\mathcal{D}$, that is to replace $\mathcal{D}$ by the family $\mathcal{E}$ of cells $D \cap (\varphi(V) \times K)$ for $D \in \mathcal{D}$ and $V \in \mathcal{V}$ such that $\varphi(V) \subseteq \widehat{D}$. This ensures that $\widehat{\mathcal{E}} = \varphi(\mathcal{V})$ and $\mathcal{E}$ is a cellular complex such that $\mathcal{E} \lhd^n \mathcal{A}$. But $\mathcal{E}$ is no longer a monoplex.

In order to break this vicious circle we have to build $\mathcal{V}$, $\mathcal{D}$ and $\mathcal{F}_{\mathcal{D}}$ simultaneously. The remainder of this section is devoted to this construction. It is divided in three parts: (6.a) preparation, (6.b) vertical refinement, (6.c) horizontal refinement.

6.a Preparation

Given a family $\mathcal{X}$ of subsets of K^m, we let $\mathcal{F}_{0|\mathcal{X}}$ denote the family of all the restrictions $f_{|X}$ with $f \in \mathcal{F}_0$ and $X \in \mathcal{X}$ contained in the domain of f. By the same argument as in the beginning of the proof of Proposition 5.4, we can assume that $\bigcup \mathcal{A}$ is closed. Finally, replacing if necessary $\mathcal{A}$ by a refinement $\mathcal{D}$ given by Proposition 5.4 and $\mathcal{F}_0$ by $\mathcal{F}_{0|\mathcal{X}}$ with $\mathcal{X} = \widehat{\mathcal{D}}$, we are reduced to the case where $\mathcal{A}$ is a closed $\lhd^n$–complex of bounded cells mod $\mathbf{G}$. Enlarging $\mathcal{F}_0$ if necessary, we can, and will, assume that it contains $\mathrm{CB}(\mathcal{A})$ and an inner $\lhd^n$-system for $\mathcal{A}$. For some integers e, $M > 2v(e)$ which can be made arbitrarily large, $\mathbf{T}_m$ gives a triangulation $(\mathcal{S}, \varphi)$ of $\mathcal{F}_0$ with parameters (n, N, e, M). For every $A \in \mathcal{A}$ we let $S_A = \varphi^{-1}(\widehat{A})$.

Since $\bigcup \mathcal{A}$ is bounded and closed in K^{m+1}, its image $\bigcup \widehat{\mathcal{A}}$ by the coordinate projection is closed in K^m by Theorem 2.5. Now φ is a homeomorphism from $\biguplus \mathcal{S}$ to $\bigcup \widehat{\mathcal{A}}$, hence $\mathcal{S}$ is closed by Remark 2.19.

Let $\mathcal{A}'$ be the family of cells $A \cap (\varphi(S) \times K)$ for $A \in \mathcal{A}$ and $S \in \mathcal{S}$ such that $\varphi(S) \subseteq \widehat{A}$, and let $\mathcal{F}'_0 = \mathcal{F}_{0|\varphi(\mathcal{S})}$. Since every cell in $\mathcal{A}'$ has the same center and bounds as the unique cell in $\mathcal{A}$ which contains it, clearly $\mathcal{A}'$ is still a closed $\lhd^n$–complex, $\mathcal{F}'_0$ contains $\mathrm{CB}(\mathcal{A}')$ and an inner $\lhd^n$-system for $\mathcal{A}'$, and $(\mathcal{S}, \varphi)$ is still a triangulation of $\mathcal{F}'_0$. Thus, replacing $(\mathcal{A}, \mathcal{F}_0)$ by $(\mathcal{A}', \mathcal{F}'_0)$ if necessary, we can assume that $\varphi(\mathcal{S}) = \widehat{\mathcal{A}}$, that is $S_A \in \mathcal{S}$ for every $A \in \mathcal{A}$.

A **preparation** for $(\mathcal{S}, \varphi, \mathcal{A}, \mathcal{F}_0)$ is a tuple $(\mathcal{T}, \mathcal{B}, \mathcal{F}_{\mathcal{B}})$ such that:

(P1) $\mathcal{T}$ is a simplicial subcomplex of $\mathcal{S}$. We let $\mathcal{S}_{|\mathcal{T}} = \{S \in \mathcal{S} : S \subseteq \biguplus \mathcal{T}\}$, and $\mathcal{A}_{|\mathcal{T}}$ be the family of cells $A \in \mathcal{A}$ such that $S_A \in \mathcal{S}_{|\mathcal{T}}$. Note that:

- $\mathcal{T}$ is closed because $\biguplus \mathcal{T}$ is closed in $\biguplus \mathcal{S}$.
- By Remark 2.19 it follows that the image by φ of $\bigcup \mathcal{T}$, that is the socle of $\mathcal{A}_{|\mathcal{T}}$, is closed too.
- Hence $\mathcal{A}_{|\mathcal{T}}$ is closed because $\bigcup \mathcal{A}_{|\mathcal{T}}$ is the inverse image of its socle by the (continuous) coordinate projection of $\bigcup \mathcal{A}$ onto $\bigcup \widehat{\mathcal{A}}$.

(P2) $\mathcal{B}$ is a cellular monoplex mod $\mathbf{G}$ refining $\mathcal{A}_{|\mathcal{T}}$ such that $\varphi(\mathcal{T}) = \widehat{\mathcal{B}}$. For every $B \in \mathcal{B}$ we let $T_B = \varphi^{-1}(\widehat{B})$. Note that $\mathcal{B}$ is closed because $\bigcup \mathcal{B} = \bigcup \mathcal{A}_{|\mathcal{T}}$.

(P3) $\mathcal{B} \lhd^n \mathcal{A}_{|\mathcal{T}}$ and $\mathcal{F}_{\mathcal{B}}$ is a $\lhd^n$-system for $(\mathcal{B}, \mathcal{A}_{|\mathcal{T}})$.

(P4) $\mathcal{T}$ together with the restriction of φ to $\biguplus \mathcal{T}$, which we will denote $\varphi_{|\mathcal{T}}$, is a triangulation of $\mathcal{F}_{\mathcal{B}} \cup \mathrm{CB}(\mathcal{B})$ with parameters (n, N, e, M). Note that, since $\mathcal{T}$ refines $\mathcal{S}_{|\mathcal{T}}$ and $(\mathcal{S},\varphi)$ is a triangulation of $\mathcal{F}_0$, $(\mathcal{T},\varphi_{|\mathcal{T}})$ is also a triangulation of $\mathcal{F}_{0|\mathcal{X}}$ with $\mathcal{X}=\varphi(\mathcal{T})$.

Remark 6.2. Obviously $(\emptyset,\emptyset,\emptyset)$ is preparation for $(\mathcal{S},\varphi,\mathcal{A},\mathcal{F}_0)$. Given an arbitrary preparation $(\mathcal{T},\mathcal{B},\mathcal{F}_\mathcal{B})$ for $(\mathcal{S},\varphi,\mathcal{A},\mathcal{F}_0)$ such that $\bigcup\mathcal{T}\neq\bigcup\mathcal{S}$, and S a minimal element in $\mathcal{S}\setminus\mathcal{S}_{|\mathcal{T}}$, it suffices to build from it a preparation $(\mathcal{U},\mathcal{C},\mathcal{F}_\mathcal{C})$ such that $\biguplus\mathcal{U}=\biguplus\mathcal{T}\cup S$. Indeed, $\mathcal{S}_{|\mathcal{U}}$ contains one more element of $\mathcal{S}$ than $\mathcal{S}_{|\mathcal{T}}$ thus, starting from $(\emptyset,\emptyset,\emptyset)$ and repeating the process inductively we will finally get a preparation $(\mathcal{V},\mathcal{D},\mathcal{F}_\mathcal{D})$ such that $\biguplus\mathcal{V}=\biguplus\mathcal{S}$, hence $\mathcal{A}_{|\mathcal{V}}=\mathcal{A}$. (P4) then implies that $(\mathcal{V},\varphi)$ is a triangulation of $\mathcal{F}_0\cup\mathcal{F}_\mathcal{B}\cup\mathrm{CB}(\mathcal{B})$ with parameters (n,N,e,M). So the tuple $(\mathcal{V},\varphi,\mathcal{D},\mathcal{F}_\mathcal{D})$ satisfies the conclusion of Lemma 6.1, which finishes the proof.

So from now on, let $(\mathcal{T},\mathcal{B},\mathcal{F}_\mathcal{B})$ be a given preparation for $(\mathcal{S},\varphi,\mathcal{A},\mathcal{F}_0)$ such that $\biguplus\mathcal{T}\neq\biguplus\mathcal{S}$. Let S be a minimal element in $\mathcal{S}\setminus\mathcal{S}_{|\mathcal{T}}$ and $\mathcal{A}_S=\{A\in\mathcal{A}:\widehat{A}=\varphi(S)\}$. The minimality of S ensures that every proper face of S belongs to $\mathcal{S}_{|\mathcal{T}}$, hence $\biguplus\mathcal{S}\cup S$ and $\bigcup(\mathcal{A}_{|\mathcal{T}}\cup\mathcal{A}_S)$ are closed.

Claim 6.3. *Let A be a cell of type 1 in $\mathcal{A}_S$, T a simplex in $\mathcal{T}$ contained in $\overline{S}$, and $Y=\varphi(T)$. If $\bar\nu_A=0$ on Y then $\mathrm{Gr}\,\bar c_{A|Y}=\partial^0_Y A$ belongs to $\mathcal{B}$. If moreover $\bar\mu_A\neq 0$ on Y then $\partial^1_Y A$ is covered by the cells in $\mathcal{B}$ that it meets, and among them there is a unique cell B^1_T whose closure meets $\partial^0_Y A$. More precisely:*

$$B^1_T=\big(\bar c_{A|Y},0,\mu_{B^1_T},G_A\big)$$

and either $|\mu_{B^1_T}|=|\bar\mu_A|$ on Y, or $|\mu_{B^1_T}|\leq|\pi^{N_0}\bar\mu_A|$ on Y. In particular the closure of B^1_T contains $\partial^0_Y A$.

Proof: Note first that for every $i\in\{0,1\}$, $\partial^i_Y A$ is contained in $\overline{A}$ hence in $\bigcup\mathcal{A}$ since it is closed by assumption. Every cell in $\mathcal{A}$ which meets $\partial^i_Y A$ is contained in it since $\mathcal{A}$ is a $\lhd^n$-complex, and belongs to $\mathcal{A}_{|\mathcal{T}}$ (otherwise its socle would not meet Y since $Y\subseteq\bigcup\varphi(\mathcal{T})$). Since $\mathcal{B}$ refines $\mathcal{A}_{|\mathcal{T}}$ it follows that $\partial^i_Y A$ is the union of the cells B in $\mathcal{B}$ which it contains.

In particular, if $\bar\nu_A=0$ on A then $\partial^0_Y A\neq\emptyset$ hence it contains a cell $B\in\mathcal{B}$. Necessarily B is of type 0 since so is $\partial^0_Y A$, and thus $B=\partial^0_Y A$ since they have the same socle Y. This proves the first point.

For the second point, since $\bar\nu_A=0\neq\bar\mu_A$ on Y both $\partial^0_Y A$ and $\partial^1_Y A$ are non-empty. Now $\partial^0_Y A$ is contained in the closure of $\partial^1_Y A$, which is the union of the closure of the cells in $\mathcal{B}$ contained in $\partial^1_Y A$. Hence necessarily the closure of at least one of them, say B, meets $\partial^0_Y A$.

$\widehat{B}$ meets Y and both of them belong to $\widehat{\mathcal{B}}$ so $\widehat{B}=Y$. Since $\overline{B}\cap(Y\times K)$ meets $\partial^0_Y A$ and $B\subseteq\partial^1_Y A$ is disjoint from $\partial^0_Y A$, B must be of type 1 with $\nu_B=0$ because otherwise B would be closed in $Y\times K$. It follows that $\overline{B}\cap(Y\times B)$ is the union of B and $\partial^0_Y B$, and the latter meets $\partial^0_Y A$. By the first point $\partial^0_Y A\in\mathcal{B}$. By Proposition 5.3 applied to $\mathcal{B}$, $\partial^0_Y B\in\mathcal{B}$. Thus $\partial^0_Y B=\partial^0_Y A$, in particular they have the same center so $c_B=\bar c_{A|Y}$. Pick any $(x,t)\in B$, so that $t-c_B(x)\in G_B$. B is contained in $\partial^1_Y A$ hence $t-\bar c_A(x)\in G_A$. Since $c_B(x)=\bar c_A(x)$ it follows that $G_B\cap G_A\neq\emptyset$ hence $G_A=G_B$.

This proves that $B=(\bar c_{A|Y},0,\mu_B,G_A)$. The uniqueness of B follows. Indeed if B' is any cell in $\mathcal{B}$ contained in $\partial^1_Y A$ whose closure meets $\partial^0_Y A$, the same argument shows that $B'=(\bar c_{A|Y},0,\mu_{B'},G_A)$. This implies that for any $t'\in K$

such that $t' - \bar{c}_A(x)$ is small enough and belongs to G_A, the point (x, t') will belong both to B and B', so $B = B'$.

If $|\mu_B| = |\bar{\mu}_{A|Y}|$ we are done, so let us assume the contrary. Then $|\mu_B(x)| \neq |\bar{\mu}_A(x)|$ for some $x \in Y$. B is a fitting cell so let $t \in K$ be such that $(x, t) \in B$ and $|t - c_B(x)| = |\mu_B(x)|$. We have $|t - \bar{c}_A(x)| \leq |\bar{\mu}_A(x)|$ because $(x, t) \in \partial^1_Y A$, so $|\mu_B(x)| < |\bar{\mu}_A(x)|$. We are going to show that $|\mu_B| < |\bar{\mu}_A|$ on Y. Since $\partial^1_Y A$ is a fitting cell it follows that $\partial^1_Y A$ is not contained in B, so there is at least one other cell C in $\mathcal{B}$ contained in $\partial^1_Y A$. Now $\widehat{C}$ is contained in Y and both of them belong to $\widehat{\mathcal{B}}$ so $\widehat{B} = Y$. For each y in Y fix t_y in K such that $(y, t_y) \in C$. Since C is contained in $\partial^1_Y A$ we have:

$$0 \leq |t_y - \bar{c}_A(y)| \leq |\bar{\mu}_A(y)| \text{ and } t_y - \bar{c}_A(y) \in G_A$$

Necessarily $|\mu_B(y)| < |t_y - \bar{c}_A(y)|$ because otherwise (y, t_y) would belong both to C and B, a contradiction. Hence *a fortiori* $|\mu_B(y)| < |\bar{\mu}_A(y)|$. By Proposition 5.5 this implies that $|\mu_B(y)| \leq |\pi^{N_0}\bar{\mu}_A(y)|$ because B and A are fitting cells mod $\mathbf{G}$, $v\mathbf{G} = N_0\mathcal{Z}$ and $G_B = G_A$. So $|\mu_B| \leq |\pi^{N_0}\bar{\mu}_{A|Y}|$ in that case, which proves our claim.

■

We can now begin our construction of a preparation $(\mathcal{U}, \mathcal{C}, \mathcal{F}_\mathcal{C})$ for $(\mathcal{S}, \varphi, \mathcal{A}, \mathcal{F}_0)$ such that $\biguplus \mathcal{U} = \biguplus \mathcal{T} \cup S$. We are going to refine $\mathcal{A}_S$ twice. First "vertically", according to the image by φ of a certain partition of S which, together with $\mathcal{T}$, forms a simplicial subcomplex $\mathcal{U}$ of $\mathcal{S}$ refining $\mathcal{S}_{|\mathcal{T}} \cup \{S\}$ (Claim 6.5). Then "horizontally" by enlarging the cells in $\mathcal{B}$ contained in the closure of $\bigcup \mathcal{A}_S$ in such a way that the family of these new cells, together with $\mathcal{B}$, forms a cellular monoplex $\mathcal{C}$ mod $\mathbf{G}$ refining $\mathcal{A}_{|\mathcal{T}} \cup \mathcal{A}_S = \mathcal{A}_{|\mathcal{U}}$ such that $\mathcal{C} \lhd^n \mathcal{A}_{|\mathcal{U}}$. The point of the construction is to ensure that $\mathcal{C}$ comes with a $\lhd^n$-system $\mathcal{F}_\mathcal{C}$ for $(\mathcal{C}, \mathcal{A}_{|\mathcal{U}})$ such that $(\mathcal{U}, \mathcal{C}, \mathcal{F}_\mathcal{C})$ is a preparation for $(\mathcal{S}, \varphi, \mathcal{A}, \mathcal{F}_0)$.

6.b Vertical refinement

Let $\mathcal{T}_S$ be the list of proper faces of S. We first deal with the case where S is not closed, that is $\mathcal{T}_S \neq \emptyset$. For every A in $\mathcal{A}_S$ let:

$$(c^\circ_A, \nu^\circ_A, \mu^\circ_A) = (c_A \circ \varphi, \nu_A \circ \varphi, \mu_A \circ \varphi)_{|S}$$

For every $T \in \mathcal{T}_S$ and every $A \in \mathcal{A}_S$ let $\Phi_{T,A}(t, \varepsilon)$ be the formula saying that $(t, \varepsilon) \in T \times R^*$ and that one of the following conditions hold, with $n_1 = \max(n, 1 + 2vN)$:

(A1)$_{t,\varepsilon}$**:** $\bar{\nu}^\circ_A(t) \neq 0$ and for every $s \in S$ such that $\|s - t\| \leq |\varepsilon|$:

$$|c^\circ_A(s) - \bar{c}^\circ_A(t)| \leq |\pi^{n_1}\bar{\nu}^\circ_A(t)|$$

and $|\nu^\circ_A(s)| = |\bar{\nu}^\circ_A(t)|$ and $|\mu^\circ_A(s)| = |\bar{\mu}^\circ_A(t)|$

(A2)$_{t,\varepsilon}$**:** $\bar{\nu}^\circ_A(t) = 0$, $\bar{\mu}^\circ_A(t) \neq 0$ and for every $s \in S$ such that $\|s - t\| \leq |\varepsilon|$:

$$|c^\circ_A(s) - \bar{c}^\circ_A(t)| \leq |\pi^{n_1 - N_0}\mu^\circ_B(t)|$$

and $|\nu^\circ_A(s)| \leq |\mu^\circ_B(t)| \leq |\bar{\mu}^\circ_A(t)| = |\mu^\circ_A(s)|$ where B is the cell B^1_T given by Claim 6.3.

(A3)$_t$: $\bar{\nu}_A^\circ(t) = \bar{\mu}_A^\circ(t) = 0$.

Let $\Phi(t,\varepsilon)$ be the conjunction of the (finitely many) $\Phi_{T,A}(t,\varepsilon)$'s as T ranges over $\mathcal{T}_S$ and A over $\mathcal{A}_S$. Finally let $\Psi(t,\varepsilon)$ be the formula saying that that $|\varepsilon|$ is maximal among the elements ε' in R^* such that $K \models \Phi(t,\varepsilon')$. Obviously $\Psi(t,\varepsilon)$ implies $\Phi(t,\varepsilon)$.

By continuity of the center and bounds of A, for every $t \in T$ there exists $\varepsilon_{t,T,A} \in R^*$ such that $K \models \Phi_{T,A}(t,\varepsilon_{t,T,A})$. Hence for every $t \in \partial S$ there is $\varepsilon \in R^*$ such that $K \models \Phi(t,\varepsilon)$ (every $\varepsilon \in R^*$ such that $|\varepsilon| \leq |\varepsilon_{t,T,A}|$ for every $(T,A) \in \mathcal{T}_S \times \mathcal{A}_S$ is a solution). For every $t \in \partial S$, the set E_t of elements ε of R^* such that $K \models \Phi(t,\varepsilon)$ is semi-algebraic, bounded and non-empty. So by Corollary 2.6 there is $\varepsilon_t \in E_t$ such that $|\varepsilon_t|$ is maximal in $|E_t|$, that is $K \models \Psi(t,\varepsilon_t)$. Theorem 2.7 then gives a semi-algebraic function $\varepsilon : \partial S \to R^*$ such that $K \models \Psi(t,\varepsilon(t))$ for every $t \in \partial S$, hence *a fortiori*:

$$\forall t \in \partial S,\ K \models \Phi(t,\varepsilon(t)). \tag{15}$$

Claim 6.4. *Let $\varepsilon : \partial S \to R^*$ be the semi-algebraic function defined above. Then the restriction of $|\varepsilon|$ to every proper face T of S is continuous.*

Proof: Note first that if $K \models \Phi_{T,A}(t,\varepsilon')$ for some $t \in T$ and $\varepsilon' \in R^*$ then $K \models \Phi_{T,A}(t',\varepsilon')$ for every $t' \in T \cap B$ where $B = B(t,\varepsilon')$ is the ball with center t and radius ε'.

Indeed, assume for example that $\bar{\nu}_A^\circ(t) \neq 0$, hence $(\mathrm{A1})_{t,\varepsilon'}$ holds. It claims that for every $s \in S \cap B$

$$|c_A^\circ(s) - \bar{c}_A^\circ(t)| \leq |\pi^{n_1}\bar{\nu}_A(t)| \tag{16}$$

and $|\nu_A(s)| = |\bar{\nu}_A^\circ(t)|$ and $|\mu_A(s)| = |\bar{\mu}_A^\circ(t)|$. Now $T \cap B \subseteq \overline{S \cap B}$ hence, as s tends in $S \cap B$ to any given $t' \in T \cap B$ we get

$$|c_A^\circ(t') - \bar{c}_A^\circ(t)| \leq |\pi^{n_1}\bar{\nu}_A(t)| \tag{17}$$

and $|\nu_A(t')| = |\bar{\nu}_A^\circ(t)|$ and $|\mu_A(t')| = |\bar{\mu}_A^\circ(t)|$. By combining (16) and (17) with the triangle inequality we obtain that for every $s \in S \cap B$

$$|c_A^\circ(s) - \bar{c}_A^\circ(t')| \leq |\pi^{n_1}\bar{\nu}_A(t)| = |\pi^{n_1}\bar{\nu}_A(t')|$$

and $|\mu_A(s)| = |\bar{\mu}_A^\circ(t')|$, that is $(\mathrm{A1})_{t',\varepsilon'}$.

Assume now that $\bar{\mu}_A^\circ(t) = 0$, hence $\bar{\nu}_A^\circ(t) = 0$, that is $(\mathrm{A3})_t$ holds. Then $\varphi(T) \in \widehat{A}$ and $\bar{\mu}_A(\varphi(t)) = 0$ imply that $\bar{\mu}_A = 0$ on $\varphi(T)$ because $\mathcal{A}$ is a closed $\lhd^n$–complex (see footnote 18). So $\bar{\mu}_A^\circ = \bar{\nu}_A^\circ = 0$ on T, and $(\mathrm{A3})_{t'}$ follows.

The intermediate case $(\mathrm{A2})_{t,\varepsilon}$ where $\bar{\nu}_A^\circ(t) = 0$ and $\bar{\mu}_A^\circ(t) \neq 0$ is similar, and left to the reader.

Now it follows that if $K \models \Psi(t,\varepsilon')$ and $\|t' - t\| \leq |\varepsilon'|$, then $K \models \Psi(t,\varepsilon'')$ if and only if $|\varepsilon'| = |\varepsilon''|$. So $|\varepsilon(t)| = |\varepsilon(t')|$ for every $t, t' \in T$ such that $\|t - t'\| \leq |\varepsilon(t)|$. Thus $|\varepsilon|$ is locally constant, hence continuous on T.
■

Theorem 2.17 applies to S, $\mathcal{T}_S$ and the function ε. It gives a partition $\mathcal{U}_S$ of S such that $\mathcal{U}_S \cup \mathcal{T}_S$ is a simplicial complex, for each $T \in \mathcal{T}_S$ there is a unique $U \in \mathcal{U}_S$ with facet T, and for every $u \in U$:

$$\|u - \pi_U(u)\| \leq |\varepsilon(\pi_U(u))| \tag{18}$$

where π_U is the coordinate projection of U onto T (see Remark 2.13). On $\varphi(U)$ let $\sigma_U = \varphi \circ \pi_U \circ \varphi^{-1}$. This is a continuous retraction of $\varphi(U)$ onto $\varphi(T)$.

For every $U \in \mathcal{U}_S$ and every $A \in \mathcal{A}_S$ let

$$A_U = A \cap \big(\varphi(U) \times K\big). \tag{19}$$

Let $T_U = \emptyset$ if U is closed, and $T_U \in \mathcal{T}_S$ be the facet of U otherwise. Finally let $\mathcal{U} = \mathcal{U}_S \cup \mathcal{T}_S$.

Claim 6.5. *With the notation above, $\mathcal{U}$ is a simplicial subcomplex of $\mathcal{S}$ refining $\mathcal{S}_{|\mathcal{T}} \cup \{S\}$ and containing $\mathcal{T}$. For every $U \subseteq S$ in $\mathcal{U}$ and every $A \in \mathcal{A}_S$, A_U is a largely continuous fitting cell mod* **G**. *Moreover if U is not closed then:*

1. *If $|\bar{\nu}_A| \neq 0$ on $\varphi(T_U)$, then for every $x \in \widehat{A}_U$:*

$$\big|c_A(x) - \bar{c}_A(\sigma_U(x))\big| \leq \big|\pi^{n_1}\bar{\nu}_A(\sigma_U(x))\big| \tag{20}$$

$$|\nu_A(x)| = |\bar{\nu}_A(\sigma_U(x))| \quad \text{and} \quad |\mu_A(x)| = |\bar{\mu}_A(\sigma_U(x))| \tag{21}$$

2. *If $|\bar{\nu}_A| = 0 < |\bar{\mu}_A|$ on $\varphi(T_U)$, then for every $x \in \widehat{A}_U$:*

$$\big|c_A(x) - \bar{c}_A(\sigma_U(x))\big| \leq \big|\pi^{n_1 - N_0}\mu_B(\sigma_U(x))\big| \tag{22}$$

$$|\nu_A(x)| \leq |\mu_B(\sigma_U(x))| \leq |\bar{\mu}_A(\sigma_U(x))| = |\mu_A(x)| \tag{23}$$

where B is the cell B^1_T given by Claim 6.3.

Proof: By construction $\mathcal{U}$ is clearly a simplicial complex refining $\mathcal{T} \cup \{S\}$, hence refining $\mathcal{S}_{|\mathcal{T}} \cup \{S\}$ since $\mathcal{T}$ refines $\mathcal{S}_{|\mathcal{T}}$. For every $U \subseteq S$ in $\mathcal{U}$ and every $A \in \mathcal{A}_S$, A_U is a largely continuous fitting cell mod **G** by (19), because so is A. If moreover U is not closed let $T = T_U \in \mathcal{T}_S$ be its facet, let x be any element of $\widehat{A}_U = \varphi(U)$, $s = \varphi^{-1}(x) \in U$ and $t = \pi_U(s) \in T$, where π_U is the coordinate projection of U onto T (see Remark 2.13). Note that $\sigma_U(x) = \varphi \circ \pi_U(s) = \varphi(t)$ hence $\bar{c}_A(\sigma_U(x)) = \bar{c}^\circ_A(t)$, and similarly for $\bar{\nu}_A(\sigma_U(x))$ and $\bar{\mu}_A(\sigma_U(x))$. By (15) we have $K \models \Phi(t, \varepsilon(t))$.

If $|\bar{\nu}_A| \neq 0$ on $\varphi(T)$ then $\bar{\nu}^\circ_A(t) = \bar{\nu}_A(\sigma_U(x)) \neq 0$ hence $\Phi(t, \varepsilon(t))$ says that $(\mathrm{A1})_{t,\varepsilon(t)}$ holds for t. By (18), $\|s - t\| \leq |\varepsilon(t)|$ so (20) and (21) follow from $(\mathrm{A1})_{t,\varepsilon(t)}$. Similarly, if $|\bar{\nu}_A| = 0 < |\bar{\mu}_A|$ on $\varphi(T)$ then (22) and (23) follow from $(\mathrm{A2})_{t,\varepsilon(t)}$.
■

This finishes the construction of the vertical refinement of $\mathcal{A}_S$ if S is not closed. When S is closed we simply take $\mathcal{U} = \mathcal{S}_{|\mathcal{T}} \cup \{S\}$. Claim 6.5 holds in this case too, for the trivial reason that there is no non-closed $U \subseteq S$ in $\mathcal{U}$.

Remark 6.6. For every $U \in \mathcal{U}_S$, if $\nu_{A_U} = 0$ then $\mathrm{Gr}\, c_{A_U} = B_U$ for some $B \in \mathcal{A}_S$. Indeed $\nu_{A|\varphi(U)} = \nu_{A_U} = 0$ implies that $\nu_A = 0$ (thanks to our definition of presented cells) hence $\mathrm{Gr}\, c_A = \partial^0_{\varphi(S)} A$ belongs to $\mathcal{A}$: it is contained in $\overline{A}$, hence in $\bigcup \mathcal{A}$ since the latter is closed, in particular it meets at least one cell B in $\mathcal{A}$, and the last point of Proposition 5.3 then gives that $B = \mathrm{Gr}\, c_A$. Thus $B = \mathrm{Gr}\, c_A \in \mathcal{A}_S$, and clearly $\mathrm{Gr}\, c_{A_U} = B_U$.

6.c Horizontal refinement

For every $A \in \mathcal{A}_S$ we are going to construct for each $U \in \mathcal{U}_S$ a partition $\mathcal{E}_{A,U}$ of A_U, and for each E in $\mathcal{E}_{A,U}$ a semi-algebraic function $h_{E,A_U} : \varphi(U) \to K$ such that:

(Pres) $\widehat{E} = \varphi(U) = \widehat{A_U}$ and E is a largely continuous fitting cell mod $\mathbf{G}$.

(Fron) One of the following holds:

- **(∂1)** $\partial E = \emptyset$.
- **(∂2)** $\partial E = \overline{\operatorname{Gr} c_E}$ and $\operatorname{Gr} c_E \in \mathcal{E}_{C,U}$ for some $C \in \mathcal{A}_S$.
- **(∂3)** $\partial E = \overline{B}$ for some $B \in \mathcal{B}$, in which case U is not closed, $\widehat{B} = \varphi(T_U)$ and:
$$(c_B, \nu_B, \mu_B) = (\bar{c}_E, \bar{\nu}_E, \bar{\mu}_E)_{|\varphi(T_U)}.$$

(Out) $E \lhd^n A_U$ and h_{E,A_U} is a $\lhd^n$-transition for (E, A_U).

(Mon) $c_E \circ \varphi_{|U}$, $\mu_E \circ \varphi_{|U}$, $\nu_E \circ \varphi_{|U}$ and $h_{E,A_U} \circ \varphi_{|U}$ are N-monomial mod $U_{e,n}$.

This last construction will finish the proof of Lemma 6.1. Indeed, assuming that it is done, let $\mathcal{C}$ be the union of $\mathcal{B}$ and all the cells E in $\mathcal{E}_{A,U}$ for $A \in \mathcal{A}_S$ and $U \in \mathcal{U}_S$. Let $\mathcal{F}_\mathcal{C}$ be the union of the family of the corresponding functions h_{E,A_U} and of $\mathcal{F}_\mathcal{B}$. By Claim 6.5, $\mathcal{U}$ is a simplicial subcomplex of $\mathcal{S}$ such that $\biguplus \mathcal{U} = \biguplus \mathcal{T} \cup S$. The assumption (P2) for $\mathcal{B}$, together with (Pres) and (Fron), give that $\mathcal{C}$ is a cellular monoplex mod $\mathbf{G}$ refining $\mathcal{A}_{|\mathcal{T}} \cup \mathcal{A}_S$ and that $\varphi(\mathcal{U}) = \widehat{\mathcal{C}}$. The assumption (P3) for $\mathcal{B}$ and $\mathcal{F}_B$, together with (Out) above, give that $\mathcal{C} \lhd^n \mathcal{A}_{|\mathcal{U}}$ and $\mathcal{F}_\mathcal{C}$ is a $\lhd^n$-system for $(\mathcal{C}, \mathcal{A}_{|\mathcal{U}})$. Finally the assumption (P4) for $(\mathcal{T}, \varphi_{|\mathcal{T}})$ together with (Mon) ensure that $(\mathcal{U}, \varphi_{|\mathcal{U}})$ is a triangulation of $\mathcal{F}_\mathcal{C} \cup \operatorname{CB}(\mathcal{C})$ with parameters (n, N, e, M). So $(\mathcal{U}, \mathcal{C}, \mathcal{F}_\mathcal{C})$ is a preparation of $(\mathcal{S}, \varphi, \mathcal{A}, \mathcal{F}_0)$, and since $\biguplus \mathcal{U} = \biguplus \mathcal{T} \cup S$ we conclude by Remark 6.2.

So let $A \in \mathcal{A}_S$ and $U \in \mathcal{U}_S$ be fixed once and for all in the remainder.

Remark 6.7. Recall that $(\mathcal{S}, \varphi)$ is a triangulation of $\mathcal{F}_0$, and $\mathcal{F}_0$ contains $\operatorname{CB}(\mathcal{A})$. In particular $c_A \circ \varphi_{|S}$ is N-monomial mod $U_{e,n}$ hence *a fortiori* so is $c_A \circ \varphi_{|U}$. By (19) $c_{A_U} = c_{A|\varphi(U)}$ hence $c_{A_U} \circ \varphi_{|U} = c_A \circ \varphi_{|U}$. Thus $c_{A_U} \circ \varphi_{|U}$ is N-monomial mod $U_{e,n}$, and so are $\mu_{A_U} \circ \varphi_{|U}$ and $\nu_{A_U} \circ \varphi_{|U}$ by the same argument.

Let us first assume that U is closed. We distinguish two elementary cases.

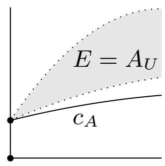
Case 1.1: $\mu_{A_U} = 0$ or $\nu_{A_U} \neq 0$.
Then A_U is closed. We let $\mathcal{E}_{A,U} = \{A_U\}$ and $h_{A_U,A_U} = 1$. (Pres), (∂1), (Out) and (Mon) are obvious (using Remark 6.7 for the latter).

Case 1.2: $0 = |\nu_{A_U}| < |\mu_{A_U}|$.
We let $\mathcal{E}_{A,U} = \{A_U\}$ and $h_{A_U,A_U} = 1$. Again (Pres), (Out) and (Mon) are obvious (same as Case 1.1). Moreover $\partial A_U = \operatorname{Gr} c_{A_U} = \operatorname{Gr} c_A$, which belongs to $\mathcal{A}_S$ by Remark 6.6. If we let $B = \operatorname{Gr} c_A$, we have $B_U = \operatorname{Gr} c_{A_U}$ and $\mu_{B_U} = 0$ hence $\mathcal{E}_{B,U} = \{B_U\}$ by the previous case. So $\partial A_U = \operatorname{Gr} c_A$ and $\operatorname{Gr} c_A \in \mathcal{E}_{B,U}$,

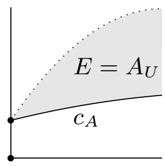
which finishes the proof of the statement that ($\partial 2$) holds.

These cases being solved, we assume in the remainder that U is not closed. Recall that T_U is then the facet of U and belongs to $\mathcal{T}$. By construction $\partial\varphi(U) = \varphi(\partial U) = \varphi(\overline{T_U}) = \overline{\varphi(T_U)}$. For the convenience of the reader, each of the following cases is illustrated by a geometric representation of its conditions (almost like if we were dealing with a cell A over a real closed field, except that the vertical intervals representing the fibres of A over $\widehat{A}$ can be clopen). In these figures A_U is represented by a gray area in K^2, its bounds by dotted lines, its socle $\varphi(U)$ by the horizontal axe, $\partial\varphi(U) = \varphi(T_U)$ by a dot on the left bound of $\varphi(U)$, and $\overline{A} \cap (\varphi(T_U) \times K)$ by a thick line or dot on the vertical axe above $\varphi(T_U)$.

Case 2.1: $|\bar{\mu}_{A_U}| = 0$ on $\varphi(T_U)$.



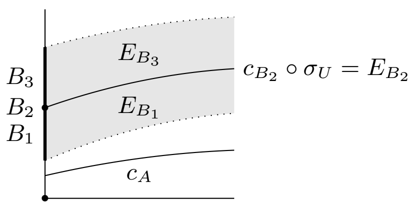
or



We let $\mathcal{E}_{A,U} = \{A_U\}$ and $h_{A_U,A_U} = 1$. (Pres), (Out) and (Mon) are obvious as in the previous cases.

- *Sub-case 2.1.a:* $\nu_{A_U} \neq 0$ or $\mu_{A_U} = \nu_{A_U} = 0$. Then ∂A_U is the closure of $\operatorname{Gr} \bar{c}_{A_U|\varphi(T_U)} = \operatorname{Gr} \bar{c}_{A|\varphi(T_U)}$. The latter belongs to $\mathcal{B}$ by Claim 6.3, which proves ($\partial 3$).
- *Sub-case 2.1.b:* $\nu_{A_U} = 0 \neq \mu_{A_U}$. Then ∂A_U is the closure of $\operatorname{Gr} c_{A_U}$. By Remark 6.6, there is a cell $C \in \mathcal{A}_S$ such that $C_U = \operatorname{Gr} c_{A_U}$. Then $\mu_{C_U} = 0$ (because A_U is a fitting cell) hence $\mathcal{E}_{C,U} = \{C_U\}$ by the previous sub-case. So $\partial A_U = \overline{\operatorname{Gr} c_{A_U}}$ and $\operatorname{Gr} c_{A_U} \in \mathcal{E}_{C,U}$, which proves that ($\partial 2$) holds.

Case 2.2: $0 < |\bar{\nu}_A|$ on $\varphi(T_U)$.



In this case, by Claim 6.3, $\partial^1_{\varphi(T_U)} A = \bar{A} \cap (\varphi(T_U) \times K)$ is the union of the cells $B \in \mathcal{B}$ which it contains. For every such B, $\widehat{B} = \varphi(T_U)$ (because $\widehat{\mathcal{B}} = \varphi(\mathcal{T})$ and $\widehat{B} \subseteq \varphi(T_U)$) and we let:

$$E_B = (c_B \circ \sigma_U, \nu_B \circ \sigma_U, \mu_B \circ \sigma_U, G_B)$$

These E_B's form a family $\mathcal{E}_{A,U}$ of two by two disjoint largely continuous cells because the various cells B involved are so and:

$$(x,t) \in E_B \iff x \in \widehat{A_U} \text{ and } (\sigma_U(x), t) \in B. \qquad (24)$$

Each E_B has socle $\varphi(U) = \widehat{A_U}$ and for every $x \in \varphi(U)$, $\sigma_U(x)$ belongs to $\varphi(T_U) = \widehat{B}$. If B is of type 0, then so is E_B and $\mu_B(\sigma_U(x)) = 0$ (because B is a fitting cell of type 0) hence E_B is a fitting cell. If B is of type 1, then $\mu_B(\sigma_U(x)) \in vG_B$ by Proposition 5.5 (because B is a fitting cell of type 1). That is $\mu_{E_B}(x) \in G_{E_B}$ hence μ_{E_B} is a fitting bound by Proposition 5.5. Similarly ν_{E_B} is a fitting bound, so E_B is a fitting cell. This proves (Pres), and one can easily derive from (24) that $\partial E_B = \overline{B}$ so that (∂3) holds. Note also that $c_{E_B} \circ \varphi_{|U}$ is N-monomial mod $U_{e,n}$ because so is $c_B \circ \varphi_{|T_U}$ and $c_{E_B} \circ \varphi_{|U} = c_B \circ \sigma_U \circ \varphi_{|U} = c_B \circ \varphi \circ \pi_U = c_B \circ \varphi_{|T_U}$. The same reasoning applies to ν_{E_B} and μ_{E_B}. So the next claim finishes to prove that $\mathcal{E}_{A,U}$ is a partition of A_U and that (out), (Mon) hold.

Claim 6.8. *$E_B \lhd^n A_U$ and there is a semi-algebraic $\lhd^n$-transition h_{E_B,A_U} for (E_B, A_U) such that $h_{E_B,A_U} \circ \varphi_{|U}$ is N-monomial mod $U_{e,n}$.*

Proof: For every (x,t) in E_B, let us prove that (x,t) belongs to A_U. Since $x \in \widehat{A_U}$ it suffices to prove that $(x,t) \in A$. By construction $(\sigma_U(x), t)$ belongs to B hence to $\partial^1_{\varphi(T_U)}A$ so:

$$\big|\bar{\nu}_A(\sigma_U(x))\big| \leq \big|t - \bar{c}_A(\sigma_U(x))\big| \leq \big|\bar{\mu}_A(\sigma_U(x))\big| \text{ and } t - \bar{c}_A(\sigma_U(x)) \in G_A \quad (25)$$

By (21) $|\nu_A(x)| = |\bar{\nu}_A(\sigma_U(x))|$ and $|\mu_A(x)| = |\bar{\mu}_A(\sigma_U(x))|$. Moreover by (20):

$$\begin{aligned} \big|\big(t - c_A(x)\big) - \big(t - \bar{c}_A(\sigma_U(x))\big)\big| &= \big|c_A(x) - \bar{c}_A(\sigma_U(x)\big| \\ &\leq \big|\pi^{n_1}\bar{\nu}_A(\sigma_U(x))\big| \\ &< \big|t - \bar{c}_A(\sigma_U(x))\big| \end{aligned} \quad (26)$$

Thus $|t - c_A(x)| = |t - \bar{c}_A(\sigma_U(x))|$ and by (25):

$$|\nu_A(x))| \leq |t - c_A(x)| \leq |\mu_A(x)|$$

Moreover by (26):

$$\left|\frac{t - c_A(x)}{t - \bar{c}_A(\sigma_U(x))} - 1\right| \leq \left|\pi^{n_1}\frac{\bar{\nu}_A(\sigma_U(x))}{t - \bar{c}_A(\sigma_U(x)}\right| \leq \big|\pi^{n_1}\big| \quad (27)$$

Recall that $n_1 = \max(n, 1+2vN)$, in particular $n_1 > 2vN$ hence $1+\pi^{n_1}R \subseteq P_N$ by Hensel's lemma. Since $t - \bar{c}_A(\sigma_U(x)) \in G_A$ by (25) and $G_A \in K^\times/P_N^\times$, it follows that $t - c_A(x) \in G_A$. So $(x,t) \in A$ which proves that $E_B \subseteq A$.

It remains to check that $E_B \lhd^n A_U$, and to find a $\lhd^n$-transition for (E_B, A_U). For every $(x,t) \in E_B$ let:

$$\omega_B(x,t) = \pi^{-n}\bigg(\frac{t - c_A(x)}{t - \bar{c}_A(\sigma_U(x))} - 1\bigg)$$

By (27) ω_B takes values in $\pi^{n_1-n}R$ hence in R since $n_1 \geq n$, thus for every $(x,t) \in E_B$:

$$t - c_A(x) = \mathcal{U}_n(x,t)\big(t - \bar{c}_A(\sigma_U(x))\big) \quad (28)$$

with $\mathcal{U}_n = 1 + \pi^n\omega_B$ in this case. We have $B \subseteq \partial^1_{\varphi(T_U)}A$ and by (P3) $\mathcal{B} \lhd^n \mathcal{A}_{|\mathcal{T}}$. Since $\mathcal{A}$ is a closed $\lhd^n$-complex this implies that for some $A' \in \mathcal{A}$ we have $B \lhd^n A' \lhd^n \partial^1_{\varphi(T_U)}A$. Let $h_0 \in \mathcal{F}_0$ be a $\lhd^n$-transition function for $(A', \partial^1_{\varphi(T_U)}A)$, and $h_1 \in \mathcal{F}_B$ a $\lhd^n$-transition function for (B, A'). Then for some $\alpha_0, \alpha_1 \in \{0, 1\}$ and every (x', t') in B we have

$$t' - \bar{c}_A(x') = \mathcal{U}_n(x', t') h_0^{\alpha_0}(x') \big(t' - c_{A'}(x')\big)^{1-\alpha_0}$$

and

$$t' - c_{A'}(x') = \mathcal{U}_n(x', t') h_1^{\alpha_1}(x') \big(t' - c_B(x')\big)^{1-\alpha_1}$$

hence $t' - \bar{c}_A(x') = \mathcal{U}_n(x', t') h(x')^{\alpha}(t' - c_B(x'))^{1-\alpha}$ with $h = h_0^{1-\alpha_0} h_1^{(1-\alpha_0)\alpha_1}$ and $\alpha = \alpha_0 + \alpha_1 - \alpha_0\alpha_1$. So h is a $\lhd^n$-transition function for $(B, \partial^1_{\varphi(T_U)} A)$. Moreover $h_0 \circ \varphi_{|T_U}$ and $h_1 \circ \varphi_{|T_U}$ are N-monomial mod $U_{e,n}$ by (P4), hence so is $h \circ \varphi_{|T_U}$. For every (x, t) in E_B, $(\sigma_U(x), t) \in B$ so

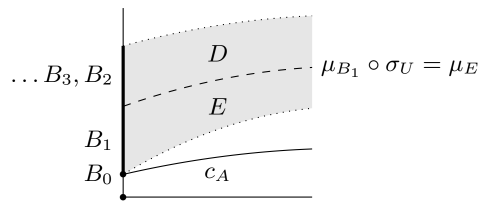
$$t - \bar{c}_A\big(\sigma_U(x)\big) = \mathcal{U}_n(x, t) h\big(\sigma_U(x)\big)^{\alpha} \big[t - c_B\big(\sigma_U(x)\big)\big]^{1-\alpha}.$$

Combining this with (28) and the definition of $c_{E_B} = c_B \circ \sigma_U$ we get

$$t - c_A(x) = \mathcal{U}_n(x, t) h\big(\sigma_U(x)\big)^{\alpha} \big[t - c_{E_B}(x)\big]^{1-\alpha}.$$

So $E_B \lhd^n A_U$ and $h \circ \sigma_U$ is a $\lhd^n$-transition for (E_B, A_U). Moreover $h \circ \sigma_U \circ \varphi_{|U} = h \circ \varphi \circ \pi_U$ by definition of σ_U. The coordinate projection π_U of U onto T_U is obviously 1-monomial, and $h \circ \varphi_{|T_U}$ is N-monomial mod $U_{e,n}$ by construction. So $h \circ \sigma_U \circ \varphi$ is also N-monomial mod $U_{e,n}$ and we can take $h_{E_B, A_U} = h \circ \sigma_U$. ■

Case 2.3: $0 = |\bar{\nu}_A| < |\bar{\mu}_A|$ on $\varphi(T_U)$ and $\nu_A \neq 0$.



Let $B_0 = B^0_{T_U}$ and $B_1 = B^1_{T_U}$ the two cells in $\mathcal{B}$ given by claim 6.3. Let:

$$E = (c_A, \nu_A, \mu_{B_1} \circ \sigma_U, G_A)_{|\varphi(U)}$$

If $|\mu_{B_1}| = |\bar{\mu}_A|$ on $\varphi(T_U)$ then $|\mu_{B_1} \circ \sigma_U| = |\mu_A|$ on $\varphi(U)$ by (23). Thus E and A_U have the same underlying set. In this case we let $\mathcal{E}_{A,U} = \{E\}$ and properties (Pres), (Mon), (∂3) are trivially true. So is (out), using Remark 6.7 for $c_A \circ \varphi_{|U}$, $\nu_A \circ \varphi_{|U}$, and (P4) for $\mu_{B_1} \circ \sigma_U \circ \varphi_{|U} = \mu_{B_1} \circ \varphi_{|T_U}$.

Otherwise $|\mu_{B_1}| < |\bar{\mu}_A|$ on $\varphi(T_U)$ by Claim 6.3 and we let:

$$D = (c_A, \pi^{-N_0} \mu_{B_1} \circ \sigma_U, \mu_A, G_A)_{|\varphi(U)}$$

$|\mu_{B_1}| \leq |\pi^{N_0} \bar{\mu}_A|$ on $\varphi(T_U)$ by Claim 6.3, $|\bar{\mu}_A \circ \sigma_U| = |\mu_A|$ on $\varphi(U)$ by (23), so $|\nu_D| = |\pi^{-N_0} \mu_{B_1} \circ \sigma_U| \leq |\bar{\mu}_A \circ \sigma_U| \leq |\mu_A| = |\mu_D|$ on $\varphi(U)$. Moreover A is a fitting cell hence for every $x \in \varphi(U)$ there is $t \in K$ such that $(x, t) \in A$ and $|t - c_A(x)| = |\mu_A(x)|$, so $(x, t) \in D$. Thus D is indeed a cell, with socle $\varphi(U)$. It is actually a largely continuous cell, and $\mu_D = \mu_A$ is a fitting bound. Let us check that $\nu_D = \pi^{-N_0}\mu_{B_1}\circ\sigma_U$ is a fitting bound too. B_1 is a fitting cell of type 1 with socle $\varphi(T_U)$ hence $\mu_{B_1}(\varphi(T_U)) \subseteq vG_{B_1}$ by Proposition 5.5. But $G_{B_1} = G_A$ by Claim 6.3, $G_A = G_D$ and $\varphi(T_U) = \sigma_U(\varphi(U))$ by construction, and $N_0 \in v\mathbf{G}$ so $\nu_D(\varphi(U)) \subseteq vG_D$. Thus ν_D is indeed a fitting bound by Proposition 5.5.

Clearly A_U is the disjoint union of E and D. Moreover the cells in $\mathcal{B}$ contained in $\bar{D} \cap (\varphi(T_U) \times K)$ are exactly those contained in $\bar{A} \cap (\varphi(T_U) \times K)$ except B_0 and B_1. Thus the construction that we have done for A_U in case 2.2 applies to D because $\bar{\nu}_D = \pi^{-N_0}\mu_{B_1} \neq 0$ on $\varphi(T_U)$ and because the analogues of conditions (20) and (21) that we used for A_U in case 2.2 hold for D in the present case. Indeed by (22) we have

$$\big|c_{A_U}(x) - \bar{c}_{A_U}(\sigma_U(x))\big| \leq \big|\pi^{n_1-N_0}\bar{\mu}_{B_1}(\sigma_U(x))\big|.$$

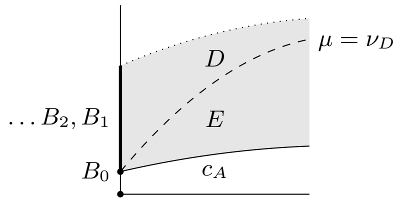
This is just condition (20) for D since $c_D = c_{A_U}$ and $\nu_D = \pi^{-N_0}\mu_{B_1}$. Moreover condition (21) for D is:

$$|\nu_D| = |\bar{\nu}_D \circ \sigma_U| \text{ and } |\mu_D| = |\bar{\mu}_D \circ \sigma_U|$$

The first equality is true by definition of ν_D as $\pi^{-N_0}\mu_{B_1} \circ \sigma_U$. The second one is true because $\mu_D = \mu_A$ and because of (23).

So the construction of Case 2.2 gives a partition $\mathcal{E}'$ of D and for each $E' \in \mathcal{E}'$ a semi-algebraic function[20] $h_{E',A_U} : \varphi(U) \to K$ satisfying conditions (Pres), (∂3), (out) and (Mon). Since E also has these properties (with $h_{E,A_U} = 1$ since $c_E = c_A$ on $\varphi(U)$) we can take $\mathcal{E}_{A,U} = \{E\} \cup \mathcal{E}'$.

Case 2.4: $\bar{\mu}_A \neq 0$ on $\varphi(T_U)$ and $\nu_A = 0$.



Let again $B_1 = B^1_{T_U}$ be the cell given by claim 6.3. We are going to split A_U in two cells E and D to which previous cases apply. In order to do so, choose any $i \in \operatorname{Supp} U \setminus \operatorname{Supp} T_U$. For every $u \in U$ let $\xi_i(u) = u_i$, the i-th coordinate of u. Clearly ξ_i is largely continuous and $\bar{\xi_i} = 0$ on $\partial U = \bar{T}_U$. So the function:

$$\mu = (\xi_i \circ \varphi^{-1})^N.(\mu_{B_1} \circ \sigma_U)$$

is largely continuous on $\widehat{A_U} = \varphi(U)$ and $\bar{\mu} = 0$ on $\varphi(T_U)$, hence also on $\overline{\varphi(T_U)} = \partial\varphi(U)$. Note that $\mu \circ \varphi_{|U}$ is N-monomial mod $U_{e,n}$. Let:

$$E = (c_A, 0, \pi^{N_0}\mu, G_A)_{|\varphi(U)}$$

$$D = (c_A, \mu, \mu_A, G_A)_{|\varphi(U)}$$

[20] Case 2.2 applied to D actually gives for each $E' \in \mathcal{E}'$ a $\lhd^n$-transition $h_{E',D}$ for (E', D). But $D \subseteq A_U$ and $c_D = c_{A_u}$ so $h_{E',D}$ is also a $\lhd^n$-transition for (E', A_U) and we can set $h_{E',A_U} = h_{E',D}$.

E and D are largely continuous fitting cells mod $\mathbf{G}$ which define a partition of A_U. (Here we use that A is a fitting cell: for every $x \in \varphi(U)$ there is $t \in K$ such that $(x,t) \in A$ and $|t - c_A(x)| = |\mu_A(x)|$ so $(x,t) \in D$, which proves that D is really a cell. That D, E are fitting cells and $A_U = E \cup D$ then follows from Proposition 5.5.) In particular E satisfies condition (Pres). Since $\nu_E = 0$ and $\bar\mu_E = \pi^{N_0}\bar\mu = 0$ on $\partial\varphi(U)$, we have $\partial E = \overline{\operatorname{Gr} c_E}$. By Remark 6.6, $\operatorname{Gr} c_{A_U} = C_U$ for some $C \in \mathcal{A}_S$, and by Sub-case 2.1.1 applied to C_U, $\operatorname{Gr} c_{A_U} \in \mathcal{E}_{C,U}$. This proves $(\partial 2)$ for E since $c_E = c_{A_U}$. Let $h_{E,A_U} = 1$, this is a $\lhd^n$-transition for (E, A_U) since they have the same center, so E satisfies (out). It also satisfies (Mon), thanks to Remark 6.7 for $c_E = c_{A_U}$ and because $\mu_E \circ \varphi_{|U} = \pi^{N_0}\mu \circ \varphi_{|U}$ is N-monomial mod $U_{e,n}$.

Case 2.3 applies to D because $\nu_D = \mu \neq 0$, $|\bar\nu_D| = |\bar\mu| = 0$ on $\varphi(T_U)$ and $|\bar\mu_D| = |\bar\mu_A| \neq 0$ on $\varphi(T_U)$, and because the analogues of conditions (22) and (23) that we used for A_U in case 2.3 hold for D in the present case. Indeed (22) holds for D because it holds for A_U, and because D and A_U have the same center. Condition (23) for D is:

$$|\nu_D| \leq |\mu_{B_1} \circ \sigma_U| \leq |\bar\mu_D \circ \sigma_U| = |\mu_D|$$

The first inequality is true because $|\nu_D| = |\mu| \leq |\mu_{B_1} \circ \sigma_U|$ by construction, the second one is true by claim 6.3 and because $\mu_D = \mu_A$, and the last equality is true because it is true for A_U by (23) and because $\mu_D = \mu_{A_U} = \mu_{A|\varphi(U)}$.

So the construction of case 2.3 gives a partition $\mathcal{E}'$ of D and for each $E' \in \mathcal{E}'$ a semi-algebraic function[21] $h_{E',A_U} : \varphi(U) \to K$ satisfying conditions (Pres), (Fron), (out) and (Mon). Since E also has these properties we can take $\mathcal{E}_{A,U} = \{E\} \cup \mathcal{E}'$.

7 Cartesian morphisms

Let $\mathcal{A}$ be a cellular monoplex mod $\mathbf{G}$ such that $\bigcup \mathcal{A}$ is a closed subset of R^{m+1}. Let $(\mathcal{U}, \psi)$ be a triangulation of $\mathrm{CB}(\mathcal{A})$ with parameters (n, N, e, M) such that for every $A \in \mathcal{A}$, $\psi^{-1}(A) \in \mathcal{U}$ (we will denote it U_A). Note that this is essentially the data given by the conclusion of Lemma 6.1. The aim of this section is to build a triangulation $(\mathcal{S}, \varphi)$ of $\mathcal{A}$ with the same parameters (n, N, e, M), together with a continuous projection $\Phi : \biguplus \mathcal{S} \to \biguplus \mathcal{U}$ such that the following diagram is commutative.

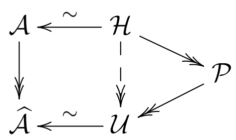
$$\begin{array}{ccc} \bigcup \mathcal{A} & \overset{\varphi}{\dashleftarrow} & \biguplus \mathcal{S} \\ \downarrow & & \downarrow \Phi \\ \bigcup \widehat{\mathcal{A}} & \overset{\psi}{\longleftarrow} & \biguplus \mathcal{U} \end{array}$$

We will make the assumption that $\mathbf{G} = Q_{N,M'}$ with $M' = M + v(N)$ and $M > v(N)$. In addition we temporarily assume that $\mathcal{A}$ is a rooted tree, and $\mathcal{U}$ a simplicial complex in $D^M R^{q_1}$ for some q_1. We keep these data and assumptions until the end of this section, where we finally state our result in a more precise and slightly more general form.

The construction is done below through a series of claims, which are connected in the following way. The idea is to prepare the construction of $\mathcal{S}$, φ, Φ

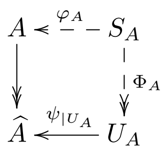
[21] Same remark as in footnote 20.

by building first the tree $\mathcal{H}$ of supports[22] of S for $S \in \mathcal{S}$, together with an epimorphism of trees from $\mathcal{H}$ to $\mathcal{U}$. In order to do so, we construct a pair of trees of finite subsets of $\mathbf{N}^*$ ordered by inclusion, $\mathcal{H} = (H(A))_{A\in\mathcal{A}}$ and $\mathcal{P} = (P(A))_{A\in\mathcal{A}}$, which come naturally with increasing maps[23] making the following diagram commutative (see Claim 7.3 and the comments after).



(29)

For each $A \in \mathcal{A}$, a simplex S_A will then be constructed inside $F_{H(A)}(D^M R^{q_2})$ (for some $q_2 \in \mathbf{N}^*$ large enough), together with a semi-algebraic isomorphism φ_A and a semi-algebraic projection Φ_A defined by means of these maps from $\mathcal{H}$ to $\mathcal{A}$ and from $\mathcal{H}$ to $\mathcal{U}$. This will ensure not only that the following diagram is commutative (Claim 7.7)



but also that $\mathcal{S} = (S_A)_{A\in\mathcal{A}}$ is a simplicial complex (Claim 7.8) and that the resulting maps φ, Φ defined by glueing all the local maps φ_A, Φ_A are continuous on $\bigcup \mathcal{S}$ (Claims 7.5 and 7.9).

Claim 7.1. *The faces of U_A are exactly the sets U_B with $B \le A$ in $\mathcal{A}$.*

Proof: Let $B \le A$ in $\mathcal{A}$, $Y = \widehat{B}$ and $i = \operatorname{tp} B$. Then, with the notation of Section 5, $B = \partial^i_Y A$ because $\mathcal{A}$ is a cellular complex. Since A is bounded, the socle of $\overline{A}$ is closed hence Y must be contained in it. Since $\psi^{-1}(Y) = U_B$, it follows that U_B is a face of U_A. Conversely for every face V of U_A, the set $B = \partial^0_{\psi(V)} A$ (resp. $B = \partial^1_{\psi(V)} A$) is non-empty if $\bar\nu_{A|Y} = 0$ (resp. $\bar\mu_{A|Y} \neq 0$) hence belongs to $\mathcal{A}$. One of these two cases necessarily happens (because $|\bar\nu_A| \le |\bar\mu_A|$ on Y), which gives $B \in \mathcal{A}$ such that $U_B = V$.
■

Claim 7.2. *Given any two cells $B \le A$ in $\mathcal{A}$, $B < A$ if and only if either $U_B < U_A$ or $\operatorname{tp} B < \operatorname{tp} A$. In particular if B is the predecessor of A in $\mathcal{A}$ then either U_B is the facet of U_A, or $U_B = U_A$, in which case $\operatorname{tp} B = 0$ and $\operatorname{tp} A = 1$*

Proof: Recall that $B = \partial^j_Y A$ with $Y = \widehat{B} = \psi(U_B)$ and $j = \operatorname{tp} B$. In particular $A = \partial^i_X A$ with $X = \widehat{A}$ and $i = \operatorname{tp} A$. Thus $B \neq A$ if and only if $U_B \neq U_A$ or $\operatorname{tp} B \neq \operatorname{tp} A$. Since $U_B \le U_A$ by the previous claim, and obviously $\operatorname{tp} B \le \operatorname{tp} A$ (otherwise $\partial^j_Y A = \emptyset$) this proves the equivalence. In particular if $U_B = U_A$ then $\operatorname{tp} B < \operatorname{tp} A$ hence $\operatorname{tp} B = 0$ and $\operatorname{tp} A = 1$.

If B is the predecessor of A in $\mathcal{A}$ and $U_B \neq U_A$, then $U_B < U_A$ by Claim 7.1. Let V be the facet of U_A. Then $U_B \le V < A$ hence $B \le \partial^j_{\psi(V)} A < A$. On the

[22] See Remark 2.16.

[23] $\mathcal{A}$, $\widehat{\mathcal{A}}$ and $\mathcal{U}$ are ordered by specialisation, while $\mathcal{H}$ and $\mathcal{P}$ are ordered by inclusion.

other hand B is the predecessor of A in $\mathcal{A}$, hence $B = \partial^j_{\psi(V)} A$. So $\widehat{B} = \psi(V)$ and finally $U_B = V$.

■

Given a strictly increasing map $\sigma : I \to J$ with $I \subseteq [\![1, r]\!]$ and $J \subseteq [\![1, s]\!]$, we let $[\sigma] : K^s \to K^r$ be defined by $[\sigma](y) = u$ where $u_i = y_{\sigma(i)}$ if $i \in I$, and $u_i = 0$ otherwise. We say that a function $f : S \subseteq K^r \to K^s$ is a **Cartesian map** if for every $I \subseteq [\![1, r]\!]$ the restriction of f to $S \cap F_I(K^r)$ is of that form, that is if there is $J \subseteq [\![1, s]\!]$ and a strictly increasing map $\sigma : I \to J$ such that $f(y) = [\sigma](y)$ for every $y \in S$ with support I. If X is the disjoint union of finitely many sets $X_k \subseteq K^{r_k}$ for various k, then a Cartesian map on X is simply the data of a Cartesian map on each X_k. A **Cartesian morphism** is a continuous Cartesian map.

Claim 7.3. *There exists a pair of functions H, P from $\mathcal{A}$ to $\mathcal{P}(\mathbf{N}^*)$ such that H is strictly increasing and for every $B \leq A$ and every C in $\mathcal{A}$:*

(C0) *If* $\operatorname{tp} A = 0$ *then* $H(A) = P(A)$.

(C1) *If* $\operatorname{tp} A = 1$ *then* $H(A) = P(A) \cup \{r_A\}$ *for some* $r_A > \max P(A)$.

(C2) $\operatorname{Card} P(A) = \operatorname{Card}(\operatorname{Supp} U_A)$.

(C3) $P(B) = H(B) \cap P(A)$ *(in particular P is increasing and $P(B) \subseteq H(B)$).*

(C4) *If* $\sigma_A : \operatorname{Supp} U_A \to P(A)$ *denotes the increasing bijection given by (C2) then* $\sigma_A(\operatorname{Supp} U_B) = P(B)$.

(C5) *If* $P(C) \subseteq P(A)$ *then* $U_C \leq U_A$.

According to this claim, $H : \mathcal{A} \to \mathcal{H}$ is an increasing bijection and $P : \mathcal{A} \to \mathcal{P}$ an increasing surjection. Thus $P \circ H^{-1} : \mathcal{H} \to \mathcal{P}$ is an increasing surjection. H^{-1} and $P \circ H^{-1}$ are respectively the maps $\mathcal{H} \to \mathcal{A}$ and $\mathcal{H} \to \mathcal{P}$ in the diagram (29) at the beginning of this section. The maps $A \mapsto \widehat{A}$ and $U_A \mapsto A$ are $\mathcal{A} \to \widehat{\mathcal{A}}$ and $\mathcal{U} \to \widehat{\mathcal{A}}$ respectively. The last[24] map, from $\mathcal{P}$ to $\mathcal{U}$, is $P(A) \mapsto U_A$. This is a well defined increasing map by (C5), and obviously a surjective one. The commutativity of the diagram follows by construction.

Remark 7.4. Since σ_A and σ_B are strictly increasing, (C4) implies that $\sigma_A(i) = \sigma_B(i)$ for every $i \in \operatorname{Supp} U_B$.

Proof: The construction goes by induction in $\operatorname{Card} \mathcal{A}$. For the root A of $\mathcal{A}$ we let $P(A) = \operatorname{Supp} U_A$, $H(A) = P(A)$ if $\operatorname{tp} A = 0$, and $H(A) = P(A) \cup \{q_1 + 1\}$ if $\operatorname{tp} A = 1$ (recall that $\mathcal{U}$ is a simplicial complex in $D^M R^{q_1}$). If $\mathcal{A} = \{A\}$ we are done. Otherwise let A be a maximal element of $\mathcal{A}$ and apply the induction hypothesis to $\mathcal{A} \setminus \{A\}$. This defines $P(A')$, $H(A')$ for every $A' \in \mathcal{A} \setminus \{A\}$ so that H is strictly increasing on $\mathcal{A} \setminus \{A\}$ and properties (C0) to (C4) hold for every $B' \leq A'$ in $\mathcal{A} \setminus \{A\}$.

Let B be the predecessor of A in $\mathcal{A}$ and $k = \operatorname{Card}(\operatorname{Supp} U_A \setminus \operatorname{Supp} U_B) + 1$. For every $A' \in \mathcal{A} \setminus \{A\}$ let $P_k(A') = \{ki\}_{i \in P(A')}$ and $H_k(A') = \{ki\}_{i \in H(A')}$. Clearly P_k and H_k inherit all the properties of P and H. Thus, replacing if necessary P and H by P_k and H_k we can assume that $H(A') \subseteq k\mathbf{N}^*$ for every $A' \in \mathcal{A} \setminus \{A\}$.

[24]The dashed map from $\mathcal{H}$ to $\mathcal{U}$ is just the compositum of $\mathcal{H} \to \mathcal{P} \to \mathcal{U}$.

Let q' be the maximum of the integers in all these sets $H(A')$. We have to define $P(A)$ and $H(A)$ so that the resulting maps P, H satisfy: (C0) to (C5) for every $B' \leq A'$ and every C' in $\mathcal{A}$; $H(B') \subseteq H(A')$ and $H(B') \neq H(A')$ if $B' \neq A'$. By the induction hypothesis it suffices to check these properties when $A' = A$, $B' = B$ and $C' \in \mathcal{A}'$.

We are going to build σ_A first, and then let $P(A) = \sigma_A(\operatorname{Supp} U_A)$. Let $j_1 < \cdots < j_r$ be an enumeration of $\operatorname{Supp} U_B$. Let $j_0 = 0$ and $j_{r+1} = q' + 1$. For every $i \in \operatorname{Supp} U_A$ there is a unique $l \in [\![0, r]\!]$ such that $j_l \leq i < j_{l+1}$. We then let $\sigma_A(i) = \sigma_B(j_l) + i - j_l$ (if $j_l = j_0 = 0$ we let $\sigma_B(j_l) = 0$ in this definition). Note that $j_l + k \leq j_{l+1}$ and $\sigma_B(j_{l+1}) \in P(B) \subseteq k\mathbf{N}^*$ hence

$$\sigma_A(j_l) \leq \sigma_A(i) < \sigma_B(j_l) + j_{l+1} - j_l \leq \sigma_B(j_l) + k \leq \sigma_B(j_{l+1}).$$

It follows immediately that σ_A is strictly increasing. Let $P(A) = \sigma_A(\operatorname{Supp} U_A)$, by construction (C2) and (C4) hold, $P(A) \cap k\mathbf{N}^* = P(B)$ and $P(B)$ is strictly contained in $P(A)$ except if $\operatorname{Supp} U_A = \operatorname{Supp} U_B$. Note also that in any case $q' + k \notin H(B) \cup P(A)$. Finally (C5) holds because:

- If $P(C') \subseteq P(A)$ then $P(C') \subseteq P(B)$ by construction (because $C' \in \mathcal{A}'$ hence $P(C') \subseteq k\mathbf{N}^*$). So $U_{C'} \leq U_B$ by the induction hypothesis, and since $U_B \leq U_A$ we get $U_{C'} \leq U_A$.
- If $P(A) \subseteq P(C')$ then in particular $P(A) \subseteq k\mathbf{N}^*$, hence by construction $P(A) = P(B) = P(C')$ and $\operatorname{Supp} U_A = \operatorname{Supp} U_B$. This last point implies that $\dim U_A = \dim U_B$, hence $U_A = U_B$ since $U_B \leq U_A$. On the other hand $P(B) = P(C')$ implies that $U_B = U_{C'}$ by the induction hypothesis. So altogether $U_A = U_{C'}$ and *a fortiori* $U_A \leq U_{C'}$.

It remains to define $H(A)$ and to check (C1) and (C3). We distinguish four cases, according to the types of A and B, and apply Claim 7.2 to each of them.

Case 1: $\operatorname{tp} A = 0$, hence $\operatorname{tp} B = 0$ and U_B is the facet of A. In particular $\operatorname{Supp} U_B$ is strictly contained in $\operatorname{Supp} U_A$, hence so is $P(B)$ in $P(A)$. By the induction hypothesis (C0), $H(B) = P(B)$. Let $H(A) = P(A)$, then $H(B) \subseteq H(A)$, $H(B) \neq H(A)$ and (C0), (C3) are obvious.

Case 2: $\operatorname{tp} A = 1$, $\operatorname{tp} B = 0$ and $U_B = U_A$. Then $P(B) = P(A)$ by construction, and $P(B) = H(B)$ by the induction hypothesis (C0). Let $H(A) = H(B) \cup \{q' + k\}$, then $H(B) \subseteq H(A)$, $H(B) \neq H(A)$ and (C1) are obvious because $q' + k \notin H(B)$, and $H(B) \cap P(A) = P(B) \cap P(A) = P(B)$ which proves (C3).

Case 3: $\operatorname{tp} A = 1$, $\operatorname{tp} B = 0$ and U_B is the facet of U_A. We let $H(A) = P(A) \cup \{q' + 1\}$. By the induction hypothesis (C0) $H(B) = P(B)$. By construction $P(B) \subseteq P(A) \subseteq H(A)$. So $H(B) \subseteq H(A)$, $H(B) \neq H(A)$ and (C1) are obvious because $q' + k \notin H(B) \cup P(A)$. As in Case 2, $H(B) \cap P(A) = P(B) \cap P(A) = P(B)$ which proves (C3).

Case 4: $\operatorname{tp} A = \operatorname{tp} B = 1$ and U_B is the facet of U_A. By the induction hypothesis (C1), $P(B)$ is strictly contained in $P(A)$. Let $H(A) = P(A) \cup H(B)$. Then $H(B) \subseteq H(A)$, $H(B) \neq H(A)$ because $H(A) \setminus H(B) = P(A) \setminus k\mathbf{N}^* =$

$P(A) \setminus P(B) \neq \emptyset$, $H(A) \cap P(B) = (P(A) \cap P(B)) \cup (H(B) \cap P(B)) = P(B) \cup P(B) = P(B)$ which proves (C3), and (C1) follows because then $H(A) \setminus P(A) = H(B) \setminus P(A) = H(B) \setminus P(B)$ is a singleton by the induction hypothesis (C1). ■

With the notation of Claim 7.3, let q_2 be the maximal element of $\bigcup_{A\in\mathcal{A}} H(A)$ and $\mathcal{S}^\dagger = \{F_{H(A)}(D^M R^{q_2})\}_{A\in\mathcal{A}}$. For every $A \in \mathcal{A}$ let $\Phi_A = [\sigma_A] : F_{H(A)}(D^M R^{q_2}) \to D^M R^{q_1}$. Finally let $\Phi : \bigcup \mathcal{S}^\dagger \to D^M R^{q_1}$ be the resulting Cartesian map.

Claim 7.5. Φ *is continuous, hence a Cartesian morphism.*

Proof: We have to show that for every $T \leq S$ in $\mathcal{S}^\dagger$ and every $z \in T$, $\Phi(y)$ tends to $\Phi(z)$ as y tends to z in S. By construction there are A, B in $\mathcal{A}$ such that $H(A) = \operatorname{Supp} S$, $H(B) = \operatorname{Supp} T$, $\Phi(y) = [\sigma_A](y)$ and $\Phi(z) = [\sigma_B](z)$. Since $[\sigma_A]$ is obviously continuous, it tends to $[\sigma_A](z)$ so we have to prove that $[\sigma_A](z) = [\sigma_B](z)$. Let $u = [\sigma_A](z)$ and $u' = [\sigma_B](z)$. Recall that $u, u' \in D^M R^{q_1}$ and for every $i \in [\![1, q_1]\!]$, $u_i = z_{\sigma_A(i)}$ if $i \in \operatorname{Supp} U_A$, $u_i = 0$ otherwise, $u'_i = z_{\sigma_B(i)}$ if $i \in \operatorname{Supp} U_B$, and $u'_i = 0$ otherwise.

Since $T \leq S$ we have $\operatorname{Supp} T \leq \operatorname{Supp} S$, that is $H(B) \leq H(A)$, hence $B \leq A$ since H is strictly increasing. In particular $\operatorname{Supp} U_B \subseteq \operatorname{Supp} U_A$ hence for every $i \in [\![1, q_1]\!]$, we have $u_i = u'_i = 0$ if $i \notin \operatorname{Supp} U_A$, and by Remark 7.4 $z_{\sigma_A}(i) = z_{\sigma_B}(i)$ if $i \in \operatorname{Supp} U_B$, that is $u_i = u'_i$ in this case too. The remaining case occurs when $i \in \operatorname{Supp} U_A \setminus \operatorname{Supp} U_B$, so that $u_i = z_{\sigma_A(i)}$ and $u'_i = 0$. We have to prove that $z_{\sigma_A(i)} = 0$, that is $\sigma_A(i) \notin \operatorname{Supp} z$. By (C4) and the assumption on i, $\sigma_A(i) \in P(A) \setminus P(B)$. By (C3), $P(A) \setminus P(B) = P(A) \setminus H(B)$. So $\sigma_A(i) \notin H(B)$, and we are done since $\operatorname{Supp} z = \operatorname{Supp} T = H(B)$. ■

For every $A \in \mathcal{A}$, $\mu_A \circ \psi$ is N-monomial mod $U_{e,n}$ so there are $\zeta \in K$ and some integers $\beta_{i,A}$ for $i \in \operatorname{Supp} U_A$ such that for every $u \in U_A$

$$\mu_A \circ \psi(u) = U_{e,n}(u) \cdot \zeta \cdot \prod_{i \in \operatorname{Supp} U_A} u_i^{N\beta_{i,A}}.$$

If $\mu_A \neq 0$ then $v\mu_A(\widehat{A}) = vG_A = v\lambda_A + N\mathcal{Z}$ by Proposition 5.5, and by the above displayed equation $v(\zeta) \equiv v(\lambda_A)\ [N]$. So there is $\beta_{0,A} \in \mathcal{Z}$ such that $v(\zeta) = v(\lambda_A) + N\beta_{0,A}$. Let $\mu_A^v : vU_A \to \mathcal{Z}$ be defined by[25] $\mu_A^v(a) = M' + \beta_{0,A} + \sum_{i \in \operatorname{Supp} U_A} \beta_{i,A} a_i$. If $\mu_A = 0$ then we let $\mu_A(a) = +\infty$ for every $a \in vU_A$. Define ν_A^v accordingly. By construction, for every $u \in U_A$ we have

$$v\mu_A\big(\psi(u)\big) = v\lambda_A + N\mu_A^v(vu) - NM' \qquad (30)$$

$$v\nu_A\big(\psi(u)\big) = v\lambda_A + N\nu_A^v(vu) - NM' \qquad (31)$$

In particular μ_A^v (resp. ν_A^v) is uniquely determined by μ_A (resp. ν_A), even if the coefficients $\beta_{i,A}$ are not.

Remark 7.6. Since A is a fitting cell mod $Q_{N,M'}$ contained in R, $v\mu_A + M' \geq 0$ by Proposition 5.6. On the other hand $0 \leq v\lambda_A \leq N - 1$ (see Section 2). So,

[25] We remind the reader that A is a cell mod $Q_{N,M'}$ with $M' = M + v(N)$.

for every $u \in U_A$ we have by (30):

$$\begin{aligned}\mu_A^v(vu) &= v\mu_A(\psi(u)) + NM' - v\lambda_A\\ &\geq -M' + NM' - (N-1)\\ &= (N-1)(M'-1) \ \geq \ 0\end{aligned}$$

Let $S_A \subseteq D^M R^{q_2}$ be defined as follows.

- If $\operatorname{tp} A = 0$, S_A is the set of $y \in F_{H(A)}(D^M R^{q_2}) = F_{P(A)}(D^M R^{q_2})$ such that $\Phi(y) \in U_A$.
- If $\operatorname{tp} A = 1$, S_A is the set of $y \in F_{H(A)}(D^M R^{q_2})$ such that $\Phi(y) \in U_A$ and $\mu_A^v(v\Phi(y)) \leq vy_{r_A} \leq \nu_A^v(v\Phi(y))$.

In both cases, for every $y \in S_A$ let

$$\varphi_A(y) = \big(\psi \circ \Phi(y),\ c_A(\psi \circ \Phi(y)) + \pi^{-NM'}\lambda_A y_{r_A}^N\big)$$

where $r_A = \max H(A)$ (if $H(A) = \emptyset$, which happens when A is a point, then r_A is not defined but in that case $\operatorname{tp} A = 0$, hence $\lambda_A = 0$ and we can let $\lambda_A y_{r_A}^N = 0$ by convention).

Claim 7.7. *$\Phi(S_A) = U_A$ and φ_A is a bijection from S_A to A.*

Proof: If $\operatorname{tp} A = 0$ the result is trivial because in that case $H(A) = P(A)$ hence the restriction of Φ to $F_{H(A)}(D^M R^{q_2})$ is a bijection onto $F_{\operatorname{Supp} U_A}(D^M R^{q_1})$. So from now onwards we assume that $\operatorname{tp} A = 1$, hence $H(A) = P(A) \cup \{r_A\}$ and $r_A \notin P(A)$ by (C1).

Let $y, y' \in S_A$ be such that $\varphi_A(y) = \varphi_A(y')$. Then $\psi(\Phi(y)) = \psi(\Phi(y'))$ and

$$c_A(\psi \circ \Phi(y)) + \pi^{-NM'}\lambda_A y_{r_A}^N = c_A(\psi \circ \Phi(y')) + \pi^{-NM'}\lambda_A y_{r_A}'^N$$

These two equations imply that $y_{r_A}^N = y_{r_A}'^N$. Since $y_{r_A}, y'_{r_A} \in D^M R = Q_{1,M} \cap R$ and $M > v(N)$ it follows that $y_{r_A} = y'_{r_A}$ by Lemma 2.8. On the other hand $\psi(\Phi(y)) = \psi(\Phi(y'))$ implies $\Phi(y) = \Phi(y')$ (because ψ is one-to-one), that is $y_i = y'_i$ for every $i \in P(A)$ (because $\Phi(y) = [\sigma_A](y)$ by construction). Thus $y_i = y'_i$ for every $i \in P(A) \cup \{r_A\} = H(A)$, that is $y = y'$ since $\operatorname{Supp} y = \operatorname{Supp} y' = H(A)$. This proves that φ_A is one-to-one.

Let us check now that $A \subseteq \varphi_A(S_A)$. Pick any $(x,t) \in A$, let $u = \psi^{-1}(x)$ and $\delta = t - c_A(x)$. Since $\delta \in \lambda_A Q_{N,M'}$ and $\pi^{NM'} \in Q_{N,M'}$ we have $\pi^{NM'}\delta \in \lambda_A Q_{N,M'}$. Recall that $M' = M + v(N)$, hence by Lemma 2.8 there is a unique $z \in Q_{1,M}$ such that $\pi^{NM'}\delta = \lambda_A z^N$, hence $t = c_A(x) + \pi^{-NM'}\lambda_A z^N$. On the other hand we have $v\mu_A(\psi(u)) = v\mu_A(x) \leq v\delta$ so by (30)

$$vz = \frac{v(\pi^{NM'}\delta/\lambda_A)}{N} \geq \frac{NM' + v\mu_A(\psi(u)) - v\lambda_A}{N} = \mu_A^v(vu).$$

In particular $vz \geq 0$ by Remark 7.6 so $z \in Q_{1,M} \cap R = D^M R$. Similarly $vz \leq \nu_A^v(vu)$ by (31). Let $y \in D^M R^{q_2}$ be such that $y_i = u_{\sigma_A(i)}$ if $i \in P(A)$, $y_i = z$ if $i = r_A$, $y_i = 0$ otherwise. Then $y \in F_{H(A)}(D^M R^{q_2})$, $\Phi(y) = [\sigma_A](y) = u$ and $\mu_A^v(vu) \leq vy_{r_A} \leq \nu_A^v(vu)$ since $y_{r_A} = z$, so y belongs to S_A. By construction $\varphi_A(y) = (x,t) \in A$, which proves that $A \subseteq \varphi_A(S_A)$.

We turn now to $\Phi(S_A)$. For every $u \in U_A$, $\psi(u) \in \widehat{A}$ so there is $(x,t) \in A$ such that $u = \psi^{-1}(x)$. The above construction gives $y \in S_A$ such that $\varphi_A(y) = (x,t)$. In particular $\psi \circ \Phi(y) = x$, so $\Phi(y) = \psi^{-1}(x) = u$, which proves that $\Phi(S_A) \supseteq U_A$. Since $\Phi(S_A) \subseteq U_A$ by definition of S_A we get that $\Phi(S_A) = U_A$.

It only remains to show that $\varphi_A(S_A) \subseteq A$. Pick any $y \in S_A$, let $(x,t) = \varphi_A(y)$ and $\delta = t - c_A(x) = \pi^{-NM'}\lambda_A y_{r_A}^N$. Since $\Phi(y) \in \Phi(S_A) = U_A$, we have $x = \psi(\Phi(y)) \in \psi(U_A) = \widehat{A}$. Since $y_{r_A} \in D^M R = Q_{1,M} \cap R$, by Lemma 2.8 $y_{r_A}^N \in Q_{N,M+v(N)} = Q_{N,M'}$. Hence $\delta = \pi^{-NM'}\lambda_A y_{r_A}^N$ belongs to $\lambda_A Q_{N,M'}$. We have $\mu_A^v(v\Phi(y)) \leq v y_{r_A}$ by definition of S_A, so by (30)

$$v\mu_A\big(\psi(\Phi(y))\big) = v\lambda_A + N\mu_A^v(v\Phi(y)) - NM' \leq v\lambda_A + Nvy_{r_A} - NM'.$$

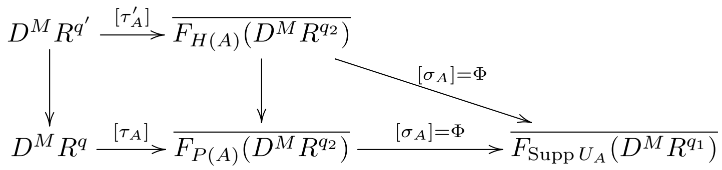
The left hand side is equal to $v\mu_A(x)$. For the right hand side we have

$$v\lambda_A + Nvy_{r_A} - NM' = v\big(\pi^{-NM'}\lambda_A y_{r_A}^N\big) = v\delta.$$

So $v\mu_A(x) \leq v\delta$, that is $|\delta| \leq |\mu_A(x)|$. Similarly $|\nu_A(x)| \leq |\delta|$ hence $(x,t) \in A$. ■

Claim 7.8. *S_A is a simplex in $D^M R^{q_2}$, whose faces are exactly the sets S_B with $B \leq A$ in $\mathcal{A}$.*

Proof: Let $q = \operatorname{Card} P(A)$ and $q' = \operatorname{Card} H(A)$. Let τ_A (resp. τ'_A) be the strictly increasing map from $P(A)$ to $[\![1,q]\!]$ (resp. from $H(A)$ to $[\![1,q']\!]$). By construction and by Claim 7.5 the following diagram is commutative (vertical arrows are the natural coordinate projections).



The horizontal arrows in this diagram are isomorphisms. All of them are obtained simply by an order-preserving renumbering of the coordinates, hence they preserve the faces and the property of being a simplex. It will then be convenient here to identify isomorphic spaces, hence to consider $U_A \subseteq D^M R^q$ and $S_A \subseteq D^M R^{q'}$. Since $\Phi(S_A) = U_A$ by Claim 7.7, after this identification U_A is just the image of S_A by the coordinate projection of $D^M R^{q'}$ to $D^M R^q$. Since $H(A) = \operatorname{Supp} S_A$ we identify also $H(A)$ with $[\![1,q']\!]$, and $P(A)$ with $[\![1,q]\!]$.

If $\operatorname{tp} A = 0$ then $H(A) = P(A)$, $q' = q$ and the vertical arrows are identity maps. Thus S_A identifies with U_A. In particular S_A is a simplex. Every $B \leq A$ is also of type 0 and S_B identifies to U_B. The conclusion follows by Claim 7.1.

From now on, let us assume that $\operatorname{tp} A = 1$. Then $q' = q + 1$ hence U_A is just the socle of S_A. Similarly, U_B is the socle of S_B for every $B \leq A$ (if $\operatorname{tp} B = 0$ we have $S_B = U_B \times \{0\}$). By construction S_A is the inverse image of vS_A by the valuation (restricted to $D^M R^{q+1}$) and

$$vS_A = \big\{a \in \mathcal{Z}^{q+1} : \widehat{a} \in vU_A \text{ and } \mu_A^v(\widehat{a}) \leq a_{q+1} \leq \nu_A^v(\widehat{a})\big\}.$$

Since $\mu_A \circ \psi$ and $\nu_A \circ \psi$ are largely continuous on U_A, (30) and (31) imply that μ_A^v is largely continuous on vU_A. They are affine maps by definition. Since

$0 \leq \mu_A^v$ by Remark 7.6, and $\mu_A^v \leq \nu_A^v$ because $|\nu_A| \leq |\mu_A|$, it follows that vS_A is a polytope in Γ^{q+1}. We are going to check that its faces are exactly the sets vS_B for $B \leq A$ in $\mathcal{A}$. This will finish the proof since S_A will then have the expected faces, which implies that S_A is a simplex because these faces form a chain by specialisation (because $\mathcal{A}$ is a tree).

Step 1. Let $B \leq A$ in $\mathcal{A}$, then $B = \partial_Y^i A$ with $Y = \widehat{B}$ and $i = \operatorname{tp} B$. Let $J = H(B) \subseteq H(A) = [\![1, q+1]\!]$ and $\widehat{J} = P(A) = J \setminus \{q+1\}$. Since $(\mu_B, \nu_B) = (\bar{\mu}_A, \bar{\nu}_A)_{|Y}$, if $\operatorname{tp} B = 1$ we have by construction

$$vS_B = \big\{a \in F_J(\Gamma^{q+1}) : \widehat{a} \in vU_B \text{ and } \bar{\mu}_A^v(\widehat{a}) \leq a_{q+1} \leq \bar{\nu}_A^v(\widehat{a})\big\}. \qquad (32)$$

This remains true also if $\operatorname{tp} B = 0$ because in that case $q+1 \notin J$ and $\bar{\nu}_A^v = +\infty$ on vU_B (because $\bar{\nu}_A = \nu_B = 0$ on Y) so the right hand side is just $vU_B \times \{+\infty\}$, that is vS_B (because $S_B = U_B \times \{0\}$ when $\operatorname{tp} B = 0$). In both cases we also have $vU_B = F_{\widehat{J}}(U_A)$, because vU_B is a face of U_A by Claim 7.1 and $\operatorname{Supp} vU_B = P(B) = \widehat{J}$. So the description of vS_B given by (32) coincides with the description of $F_J(vS_A)$ given by Proposition 2.11, which proves that $vS_B = F_J(vS_A)$.

Step 2. Conversely let $F_J(vS_A) \neq \emptyset$ be a face of vS_A, for some $J \subseteq [\![1, q+1]\!]$, and let $\widehat{J} = J \setminus \{q+1\}$. By Proposition 2.11 the socle of $F_J(vS_A)$ is $F_{\widehat{J}}(vU_A)$ (because vU_A is the socle of vS_A) and two cases can happen: $q+1 \in J$ and $\bar{\mu}_A^v < +\infty$ on $F_{\widehat{J}}(vU_A)$, or $q+1 \notin J$ and $\bar{\nu}_A^v = +\infty$ on $F_{\widehat{J}}(vU_A)$. In both cases

$$F_J(vS_A) = \big\{a \in F_J(\Gamma^{q+1}) : \widehat{a} \in F_{\widehat{J}}(vU_A) \text{ and } \bar{\mu}_A^v(\widehat{a}) \leq a_{q+1} \leq \bar{\nu}_A^v(\widehat{a})\big\}. \quad (33)$$

Since $F_{\widehat{J}}(vU_A)$ is a face of vU_A, by Claim 7.1 there is $C \leq A$ in $\mathcal{A}$ such that $F_{\widehat{J}}(vU_A) = vU_C$. Let $Y = \widehat{C} = \psi(U_C)$.

If $q+1 \notin J$ then by Proposition 2.11, $\bar{\nu}_A^v = +\infty$ on $F_{\widehat{J}}(vU_A) = vU_C$. That is $\bar{\nu}_A = 0$ on $Y = \psi(U_C)$, hence $\partial_Y^0 A \in \mathcal{A}$. Let $B = \partial_Y^0 A$ and apply Step 1 to B. Since $J = \widehat{J}$ is the support of $vU_C = vU_B$ and of S_B (because $\operatorname{tp} B = 0$), we deduce from (32) and (33) that $vS_B = F_J(S_A)$.

If $q+1 \in J$ then by Proposition 2.11, $\bar{\mu}_A^v \neq +\infty$ on $F_{\widehat{J}}(vU_A) = vU_C$. That is $\bar{\mu}_A \neq 0$ on $Y = \psi(U_C)$, hence $\partial_Y^1 A \in \mathcal{A}$. Let $B = \partial_Y^1 A$ and apply Step 1 to B. Since $\widehat{J}$ is the support of $vU_C = vU_B$ and $J = \widehat{J} \cup \{q+1\}$ is the support of S_B (because $\operatorname{tp} B = 1$), we deduce from (32) and (33) that $vS_B = F_J(S_A)$.
■

Finally let $\mathcal{S} = \{S_A : A \in \mathcal{A}\}$ and $\varphi : \bigcup \mathcal{S} \to \bigcup \mathcal{A}$ be defined by $\varphi_{|S_A} = \varphi_A$ for each $A \in \mathcal{A}$.

Claim 7.9. *φ is a homeomorphism from $\bigcup \mathcal{S}$ to $\bigcup \mathcal{A}$.*

Proof: We already know by Claim 7.7 that φ is a bijection from $\bigcup \mathcal{S}$ to $\bigcup \mathcal{A}$. It follows from Claim 7.8 that $\bigcup \mathcal{S}$ is closed, and it is obviously bounded. Thus by Theorem 2.5 it suffices to show that φ is continuous. Since each φ_A is obviously continuous on S_A, we only have to prove that for every $z \in \partial S_A$ and $y \in S_A$, $\varphi_A(y)$ tends to $\varphi(z)$ as y tends to S_A. By Claim 7.8 there is $B \leq A$ in $\mathcal{A}$ such that $z \in S_B$, hence $\varphi(z) = \varphi_B(z)$. By Claim 7.5, $\psi \circ \Phi(y)$ tends to $\psi \circ \Phi(z)$. By Claim 7.7, $\psi \circ \Phi(y) \in \widehat{A}$ and $\psi \circ \Phi(z) \in \widehat{B}$ hence $c_A(\psi \circ \Phi(y))$ tends to

$\bar{c}_A(\psi \circ \Phi(z))$, which is equal to $c_B(\psi \circ \Phi(z))$ since $\bar{c}_{A|\widehat{B}} = c_B$. Thus it only remains to check that $\lambda_A y_{r_A}^N$ tends to $\lambda_B z_{r_B}^N$.

If $\operatorname{tp} A = 0$ then also $\operatorname{tp} B = 0$ hence and we are done, since $\lambda_A y_{r_A}^N = 0 = \lambda_B z_{r_B}^N$. If $\operatorname{tp} B = 1$ then $\lambda_B = \lambda_A$ (because $\mathcal{A}$ is a cellular complex) and $r_A = r_B$ (because by (C1) and (C3), $H(B) \neq P(B) = H(B) \cap P(A)$ implies that $H(B)$ is not contained in $P(A)$, hence $r_A \in H(B)$ since $H(B) \subseteq H(A) = P(A) \cup \{r_A\}$). Hence obviously $\lambda_A y_{r_A}^N$ tends to $\lambda_A z_{r_A}^N = \lambda_B z_{r_B}^N$ in that case. Finally if $\operatorname{tp} A = 1$ and $\operatorname{tp} B = 0$ then $r_A \notin H(B)$ (because by (C0) and (C3), $H(B) = P(B) \subseteq P(A)$), hence $z_{r_A} = 0$ since $\operatorname{Supp} z = \operatorname{Supp} S_B = H(B)$. Thus $\lambda_A y_{r_A}^N$ tends to $\lambda_A z_{R_A}^N = 0$, which proves the result because $\lambda_B z_{r_B}^N = 0$ since $\operatorname{tp} B = 0$.
■

Remark 7.10. By construction $\operatorname{Supp} S_A = H(A)$ and $\operatorname{Supp} S_{A'} = H(A')$ for every $A, A' \in \mathcal{A}$, hence $\operatorname{Supp} S_{A'} \subseteq \operatorname{Supp} S_A$ if and only if $A' \leq A$ (because H is strictly increasing), which implies that $S_{A'} \leq S_A$ by Claim 7.9. So for every $S, S' \in \mathcal{S}$ we have

$$S' \leq S \iff \operatorname{Supp} S' \subseteq \operatorname{Supp} S.$$

We can recap now our construction and state the result which was the aim of this section.

Lemma 7.11. *Let $\mathcal{A}$ be a cellular monoplex mod $Q_{N,M+v(M)}^{\times}$ such that $\bigcup \mathcal{A}$ is a closed subset of R^{m+1}. Let $(\mathcal{U}, \psi)$ be a triangulation of* $\mathrm{CB}(\mathcal{A})$ *with parameters (n, N, e, M) such that $M > v(N)$ and for every $A \in \mathcal{A}$, $\psi^{-1}(A) \in \mathcal{U}$ (let us denote it U_A). Then there exists a simplicial complex $\mathcal{S}$ of index M, a Cartesian morphism $\Phi : \biguplus \mathcal{S} \to \biguplus \mathcal{U}$ and a semi-algebraic homeomorphism $\varphi : \biguplus \mathcal{S} \to \bigcup \mathcal{A}$ such that for every $A \in \mathcal{A}$, $\varphi^{-1}(A) \in \mathcal{S}$ (let us denote it S_A) and for every $y \in S_A$*

$$\varphi(y) = \big(\psi \circ \Phi(y), c_A(\psi \circ \Phi(y)) + \pi^{-NM'} \lambda_A y_{r_A}^N\big)$$

where[26] $r_A = \max(\operatorname{Supp} S_A)$.

Proof: Let $(A_k)_{1 \leq k \leq r}$ be the list of minimal elements in $\mathcal{A}$, and for each k let $\mathcal{A}_k$ be the family of elements in $\mathcal{A}$ greater than A_k. This is a rooted, cellular monoplex mod $Q_{N,M+v(N)}^{\times}$. For every $A \in \mathcal{A}_k$, $\overline{A}$ is the union of the cells $B \leq A$ in $\mathcal{A}$ since $\mathcal{A}$ is a cellular complex and $\bigcup \mathcal{A}$ is closed. All these cells B belong to $\mathcal{A}_k$ hence $\bigcup \mathcal{A}_k$ is closed. Since $\bigcup \mathcal{A} \setminus \bigcup \mathcal{A}_k$ is the union of the finitely many other $\mathcal{A}_l$ it is closed, hence $\bigcup \mathcal{A}_k$ is clopen in $\bigcup \mathcal{A}$. Let $\mathcal{U}_k = \{\psi^{-1}(\widehat{A}) : A \in \mathcal{A}_k\}$, this is a lower subset of $\mathcal{U}$ with smallest element $\psi^{-1}(\widehat{A}_k)$ hence a rooted simplicial complex in $D^{M_1} R^{q_{1,k}}$ for some $q_{1,k}$. Finally let ψ_k be the restriction of ψ to $\bigcup \mathcal{U}_k$.

Claims 7.1 to 7.9 apply to $(\mathcal{U}_k, \psi, \mathcal{A}_k)$ and give a simplicial complex $\mathcal{S}_k$ in $D^M R^{q_{2,k}}$ for some $q_{2,k}$, a Cartesian morphism $\Phi_k : \bigcup \mathcal{S}_k \to \bigcup \mathcal{U}_k$ and a semi-algebraic homeomorphism $\varphi_k : \biguplus \mathcal{S}_k \to \bigcup \mathcal{A}_k$ satisfying all the required properties. Since each $\bigcup \mathcal{A}_k$ is clopen in $\bigcup \mathcal{A}$, and each $\bigcup \mathcal{U}_k$ is clopen in $\biguplus \mathcal{U}$, the conclusion follows by taking for $\mathcal{U}$ the family $\{\mathcal{U}_k\}_{1 \leq k \leq r}$ and for Φ (resp. φ) the map obtained by glueing together the various Φ_k (resp. φ_k).
■

[26]If $\operatorname{Supp} S_A = \emptyset$ then r_A is not defined but in that case S_A is a point, hence so is A so $\lambda_A = 0$ and we can let $\lambda_A y_{r_A}^N = 0$ by convention.

8 Triangulation

We have come up to the moment when we can show that $\mathbf{T}_m \Rightarrow \mathbf{T}_{m+1}$. As $\mathbf{T}_0$ is rather obvious, this will finish the proof of $\mathbf{T}_m$ for every m.

Theorem 8.1. *Assume $\mathbf{T}_m$. Let $(\theta_i : A_i \subseteq K^{m+1} \to K)_{i\in I}$ be a finite family of semi-algebraic functions, and $n, N \geq 1$ be any integers. Then for some integers $e, M \geq 1$ which can be chosen arbitrarily large (in the sense of footnote 12), there exists a simplicial complex $\mathcal{T}$ of index M and a semi-algebraic homeomorphism $\varphi : \biguplus \mathcal{T} \to \bigcup_{i\in I} A_i$ such that for every i in I:*

1. *$\{\varphi(T) : T \in \mathcal{T}$ and $\varphi(T) \subseteq A_i\}$ is a partition of A_i.*
2. *$\forall T \in \mathcal{T}$ such that $\varphi(T) \subseteq A_i$, $\theta_i \circ \varphi_{|T}$ is N-monomial mod $U_{e,n}$.*

Proof: By using the same partition of K^{m+1} as in the proof of Lemma 3.3 we are reduced to the case where each A_i is contained in R^{m+1}. We can also extend each θ_i to R^{m+1} by an arbitrary value, and add to this family the indicator functions of each A_i inside R^{m+1}, hence assume that all these functions have domain R^{m+1}, which is closed and bounded. Let $e_* \geq 1$ and $M_* \geq 1$ be any integers.

Theorem 4.7 applies to $(\theta_i)_{i\in I}$. It gives an integer $e_0 \geq 1$, a tuple $\eta \in R^m$, a linear automorphism $u_\eta(x,t) = (x + t\eta, t)$ of K^{m+1} (note that $u_\eta(R^{m+1}) = R^{m+1}$ since $\eta \in R^{m+1}$), a pair of integers $N_0 \geq 1$ and $M_0 > 2v(e_0)$ such that $e_0 N$ divides N_0, and a finite family $\mathcal{A}$ of largely continuous cells mod $Q^\times_{N_0,M_0}$ partitioning $u_\eta^{-1}(R^{m+1}) = R^{m+1}$ such that for every $i \in I$, every $A \in \mathcal{A}$ and every $(x,t) \in A$

$$\theta_i \circ u_\eta(x,t) = \mathcal{U}_{e_0,n}(x,t) h_{i,A}(x) \big[\lambda_A^{-1}\big(t - c_A(x)\big)\big]^{\frac{\alpha_{i,A}}{e_0}} \tag{34}$$

where $h_{i,A} : \widehat{A} \to K$ is a semi-algebraic function and $\alpha_{i,A} \in \mathbf{Z}$.

Let $n_1 = \max(1 + 2v(e_0), n + v(e_0))$, Lemma 6.1 applied to $\mathcal{A}$ and the family $\mathcal{F}_0 = \{h_{i,A} : i \in I,\ A \in \mathcal{A}\}$ gives a pair of integers $e_1 \geq 1$ and $M_1 > 2v(e_1)$, a cellular monoplex $\mathcal{B}$ mod Q_{N_0,M_0} refining $\mathcal{A}$ such that $\mathcal{B} \lhd^{n_1} \mathcal{A}$, a $\lhd^{n_1}$-system $\mathcal{F}_1$ for $(\mathcal{B}, \mathcal{A})$, and a triangulation $(\mathcal{U}, \psi)$ of $\mathcal{F}_0 \cup \mathcal{F}_1 \cup \mathrm{CB}(\mathcal{B})$ with parameters (n_1, N_0, e_1, M_1). Moreover e_1, M_1 can be chosen arbitrarily large, in the sense of footnote 12, so we can require that e_* divides e_1 and $M_1 \geq M_*$, and that $M_1 \geq M_0 - v(N_0)$ and $M_1 > v(N_0) \geq v(e_0)$.

$Q^\times_{N_0,M_1+v(N_0)}$ is a subgroup of $Q^\times_{N_0,M_0}$ (because $M_1 + v(N_0) \geq M_0$) with finite index. Hence every cell in $\mathcal{B}$ is the disjoint union of finitely many cells C mod $Q^\times_{N_0,M_1+v(N_0)}$ with the same socle and bounds as B. Since $vQ^\times_{N_0,M_1+v(N_0)} = N_0\mathcal{Z} = vQ^\times_{N_0,M_0}$, these cells C are still fitting cells by Proposition 5.5. One easily sees that they form a cellular monoplex $\mathcal{C}$ refining $\mathcal{A}$ such that $\mathcal{C} \lhd^{n_1} \mathcal{A}$ and $\mathcal{F}_1$ is a $\lhd^{n_1}$-system for $(\mathcal{C}, \mathcal{A})$. Moreover $\mathrm{CB}(\mathcal{C}) = \mathrm{CB}(\mathcal{B})$ and $\widehat{\mathcal{C}} = \widehat{\mathcal{B}}$ so $(\mathcal{U}, \psi)$ is a triangulation of $\mathcal{B}(\mathcal{C})$ with parameters (n_1, N_0, e_1, M_1) such that $\psi^{-1}(\widehat{C}) \in \mathcal{U}$ for every $C \in \mathcal{C}$.

Since $M_1 > v(N_0)$, Lemma 7.11 applies to $\mathcal{C}$ and $(\mathcal{U}, \psi)$. It gives a simplicial complex $\mathcal{T}$ of index M_1, a Cartesian morphism $\Phi : \biguplus \mathcal{T} \to \biguplus \mathcal{U}$ and a semi-algebraic homeomorphism $\varphi : \biguplus \mathcal{T} \to \bigcup \mathcal{C}$ such that φ^{-1} maps each C in $\mathcal{C}$ onto some T in $\mathcal{T}$, and for every y in T

$$\varphi(y) = \big(\psi \circ \Phi(y), c_C(\psi \circ \Phi(y)) + \pi^{-NM'} \lambda_C y_{r_C}^{N_0}\big) \tag{35}$$

where[27] $r_C = \max(\operatorname{Supp} T)$. Let $\varphi_\eta = u_\eta \circ \varphi$, this is a semi-algebraic homeomorphism from $\biguplus \mathcal{T}$ to R^{m+1}. We are going to check that $\theta_i \circ \varphi_{\eta|T}$ is N-monomial mod $U_{e_0e_1,n}$ for every $i \in I$ and every $T \in \mathcal{T}$. This will prove the result, with $e = e_0e_1$ and $M = M_1$.

So pick any $T \in \mathcal{T}$, let $C = \varphi(T)$ and r_C be as above. There is a unique $B \in \mathcal{B}$ containing C, a unique $A \in \mathcal{A}$ containing B. For every $(x,t) \in C$ let $\delta_C(x,t) = t - c_C(x)$. Let δ_A and δ_B be defined accordingly. Note that $\delta_C = \delta_B$ on C because C has the same center as B by construction. For every $y \in T$, by (34) and (35) we have

$$\theta_i \circ \varphi_\eta(y) = \mathcal{U}_{e_0,n}\big(\varphi(y)\big) h_{i,A}\big(\psi \circ \Phi(y)\big) \big[\lambda_A^{-1}\delta_A\big(\varphi(y)\big)\big]^{\frac{\alpha_{i,A}}{e_0}}.$$

We have $\mathcal{U}_{e_0,n}(\varphi(y)) \in U_{e_0,n} \subseteq U_{e_0e_1,n}$ so the factor $\mathcal{U}_{e_0,n}(\varphi(y))$ can be replaced by $\mathcal{U}_{e_0e_1,n}(y)$. Recalling that $(\mathcal{V},\psi)$ is a triangulation of $\mathcal{F}_0 \cup \mathcal{F}_1$ with parameters (n_1, N_0, e_1, M_1), that Φ is a Cartesian morphism and $h_{i,A} \in \mathcal{F}_0$, we get that the second factor $h_{i,A}(\psi \circ \Phi(y)) = h_{i,A} \circ \psi(\Phi(y))$ is N_0-monomial mod U_{e_1,n_1} hence *a fortiori* N-monomial mod $U_{e_0e_1,n}$ since N divides N_0 and $n_1 \geq n$. So it only remains to prove that the last factor $[\lambda_A^{-1}\delta_A \circ \varphi_{|T}]^{\alpha_{i,A}/e_0}$ is N-monomial mod $U_{e_0e_1,n}$. It suffices to prove it for $[\lambda_A^{-1}\delta_A \circ \varphi]^{1/e_0}$.

We can assume that $\operatorname{tp} A = 1$ otherwise $\lambda_A^{-1}\delta_A = 1$ and the result is trivial (see Remark 4.8). Recall that $\mathcal{C} \lhd^{n_1} \mathcal{A}$ and $\mathcal{F}_1$ is a $\lhd^n$-system for $(\mathcal{C},\mathcal{A})$. For every $(x,t) \in C$ we then have

$$t - c_A(x) = \mathcal{U}_{n_1}(x,t) h_{C,A}(x)^\beta \big(t - c_C(x)\big)^{1-\beta}$$

with $h_{C,A} \in \mathcal{F}_1$ and $\beta \in \{0,1\}$ (depending on A, C). So by (35) we have

$$\delta_A\big(\varphi(y)\big) = \mathcal{U}_{n_1}\big(\varphi(y)\big) h_{C,A}\big(\psi \circ \Phi(y)\big)^\beta \big(\pi^{-NM}\lambda_C y_{r_C}^{N_0}\big)^{1-\beta}. \tag{36}$$

$(\mathcal{V},\psi)$ is a triangulation of $\mathcal{F}_1$ with parameters (n_1, N_0, e_1, M_1) hence $h_{C,A}(\psi \circ \Phi(y))$ is N_0-monomial mod U_{e_1,n_1}. So (36) implies that $\delta_A \circ \varphi_{|T}$ is N_0-monomial mod U_{e_1,n_1}, hence so is $\lambda_A^{-1}\delta_A \circ \varphi_{|T}$. Let $\chi : T \to U_{e_1}$ and $g : T \to K$ be semi-algebraic functions that for every $y \in T$

$$\lambda_A^{-1}\delta_A \circ \varphi(y) = \chi(y)\mathcal{U}_{n_1}(y)\zeta g(y) \quad \text{and} \quad g(y) = \prod_{1 \leq i \leq q} y_i^{\alpha_i N_0}$$

with $\zeta \in K$, $\alpha_1, \ldots, \alpha_q \in \mathbf{Z}$. Let $k = N_0/(e_0N)$, by construction e_0N divides N_0 hence $k \in \mathbf{N}^*$. Since $T \subseteq D^{M_1}R^{q'}$, each $y_i \in D^{M_1}R \subseteq Q_{1,M_1} \subseteq Q_{1,v(e_0)+1}$ (because $M_1 > v(e_0)$) hence $y_i^{e_0} \in Q_{e_0,2v(e_0)+1}$. *A fortiori* $y^{\alpha_i N_0} = y^{e_0 N k \alpha_i}$ belongs to $Q_{e_0,2v(e_0)+1}$ hence g takes values in $Q_{e_0,2v(e_0)+1}$ and g^{1/e_0} is N-monomial:

$$\big(g(y)\big)^{\frac{1}{e_0}} = \bigg(\prod_{1 \leq i \leq q} y_i^{e_0 N k \alpha_i}\bigg)^{\frac{1}{e_0}} = \prod_{1 \leq i \leq q} y_i^{N k \alpha_i}$$

But $\lambda_A^{-1}\delta_A$ also takes values in $Q_{e_0,2v(e_0)+1}$ because $\delta_A(x,t) \in \lambda_A Q_{N_0,M_0}$ for every $(x,t) \in A$, and $Q_{N_0,M_0} \subseteq Q_{e_0,2v(e_0)+1}$ since e_0 divides N_0 and $M_0 > 2v(e_0)$. Thus $(\lambda_A^{-1}\delta_A \circ \varphi_{|T})/g = \mathcal{U}_{n_1}\zeta\chi$ takes values in $Q_{e_0,2v(e_0)+1}$ as well. So

[27] See footnote 26.

does the factor $\mathcal{U}_{n_1}$ since $n_1 > 2v(e_0)$. Hence finally $\zeta\xi(y) \in Q_{e_0,2v(e_0)+1}$ for every $y \in T$, so $(\zeta\chi)^{1/e_0}$ is well defined. Note that $\zeta^{e_1} = \zeta^{e_1}\chi^{e_1} = [(\zeta\chi)^{1/e_0}]^{e_0e_1}$ hence $\zeta^{e_1} \in P_{e_0e_1}$. Pick any $\eta \in K$ such that $\zeta^{e_1} = \eta^{e_0e_1}$, and for every $y \in T$ let $\chi'(y) = (\zeta\chi(y))^{1/e_0}/\eta$. This is a semi-algebraic function taking values in $U_{e_0e_1}$ because

$$\left[(\zeta\chi)^{1/e_0}\right]^{e_0e_1} = \zeta^{e_1} = \eta^{e_0e_1}.$$

By Remark 2.9, $\mathcal{U}_{n_1}^{1/e_0} = \mathcal{U}_{n_1-v(e_0)}$ because $n_1 > 2v(e_0)$, and by definition $\chi'\mathcal{U}_{n_1-v(e_0)} = \mathcal{U}_{e_0e_1,n_1-v(e_0)}$. Altogether this gives that

$$\begin{aligned}
\left[\lambda_A^{-1}\delta_A \circ \varphi_{|T}\right]^{\frac{1}{e_0}} &= \mathcal{U}_{n_1}^{\frac{1}{e_0}}(\zeta\chi)^{\frac{1}{e_0}} g^{\frac{1}{e_0}} \\
&= \chi'\mathcal{U}_{n_1-v(e_0)}\big((\zeta\chi)^{\frac{1}{e_0}}/\chi' i\big) g^{\frac{1}{e_0}} \\
&= \mathcal{U}_{e_0e_1,n_1-v(e_0)}\eta g^{\frac{1}{e_0}}
\end{aligned}$$

Thus $[\lambda_A^{-1}\delta_A \circ \varphi]^{1/e_0}$ is N-monomial mod $U_{e_0e_1,n_1-v(e_0)}$ (because so is $g^{\frac{1}{e_0}}$). It is *a fortiori* N-monomial mod $U_{e_0e_1,n}$ since $n_1 - v(e_0) \geq n$ by construction.
■

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