Topological cell decomposition and dimension theory in P-minimal fields

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Abstract

This paper addresses some questions about dimension theory for P-minimal structures. We show that, for any definable set A, the dimension of $\overline{A}\backslash A$ is strictly smaller than the dimension of A itself, and that A has a decomposition into definable, pure-dimensional components. This is then used to show that the intersection of finitely many definable dense subsets of A is still dense in A. As an application, we obtain that any definable function $f : D \subseteq K^m \to K^n$ is continuous on a dense, relatively open subset of its domain D, thereby answering a question that was originally posed by Haskell and Macpherson.

In order to obtain these results, we show that P-minimal structures admit a type of cell decomposition, using a topological notion of cells inspired by real algebraic geometry.

1 Introduction

Inspired by the successes of o-minimality [vdD98] in real algebraic geometry, Haskell and Macpherson [HM97] set out to create a p-adic counterpart, a project which resulted in the notion of P-minimality. One of their achievements was to build a theory of dimension for definable sets which is in many ways similar to the o-minimal case. Still, some questions remained open.

The theorem below is one of the main results of this paper. It gives a positive answer to one of the questions raised at the end of their paper (Problem 7.5). We will assume K to be a P-minimal expansion of a p-adically closed field with value group $|K|$. When we say definable, we mean definable (with parameters) in a P-minimal structure.

Theorem (Quasi-Continuity) *Every definable function f with domain $X \subseteq K^m$ and values in K^n (resp. $|K|^n$) is continuous on a definable set U which is dense and open in X, and $\dim(X \setminus U) < \dim U$.*

Haskell and Macpherson already included a slightly weaker version of the above result in Remark 5.5 of their paper [HM97], under the additional assumption that K has definable Skolem functions. However, they only gave a

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sketch of the proof, leaving out some details which turned out to be more subtle than expected. The authors agreed with us that some statements in the original proof required further clarification.

One of the motivations for writing this paper was to remedy this, and also to show that the assumption of Skolem functions could be removed. This seemed worthwhile given that the result had already proven to be a useful tool for deducing other properties about the dimension of definable sets in P-minimal structures. For example, in [KL14] one of the authors showed how the Quasi-Continuity Theorem would imply the next result.

Theorem (Small Boundaries) *Let A be a non-empty definable subset of K^m. Then it holds that $\dim(\overline{A} \setminus A) < \dim A$.*

That both theorems are very much related is further illustrated by the approach in this paper: we will first prove the Small Boundaries Property, and use it to derive the Quasi-Continuity Property. The tool used to prove these results is a 'topological cell decomposition', which we consider to be the second main result of this paper.

Theorem (Topological Cell Decomposition) *For every definable function f from $X \subseteq K^m$ to K^n (resp. $|K|^n$) there exists a good t-cell decomposition $\mathcal{A}$ of X, such that for every $A \in \mathcal{A}$, the restriction $f_{|A}$ of f to A is continuous.*

The notions of 't-cell' and 'good t-cell decomposition' were originally introduced by Mathews, whose paper [Mat95] has been a major source of inspiration for us. They are analogous to a classical notion of cells coming from real algebraic geometry (see for example the definition of cells in [BCR87]). Exact definitions will be given in the next section.

By now, there exist many cell decomposition results for P-minimal structures, which can be quite different in flavour, depending on their aims and intended level of generality. Historically, the most influential result is probably Denef's cell decompositon for semi-algebraic sets [Den86] (which in turn was inspired by Cohen's work [Coh69]). This has inspired adaptations to the sub-analytic context by Cluckers [Clu04], and to multi-sorted semi-algebraic structures by Pas [Pas90]. Results like [CKL, Mou09, DH15] give generalizations of Denef-style cell decomposition. Note that full generality is hard to achieve: whereas [CKL] works for all p-minimal structures without restriction, it is somewhat weaker than these more specialized results. On the other hand, [Mou09, DH15] are closer to the results cited above, but require some restrictions on the class of P-minimal fields under consideration. A somewhat different result is the Cluckers-Loeser cell decomposition [CL07] for b-minimal structures.

Each of these decompositions has its own strengths and weaknesses. The topological cell decomposition proposed here seems to be the best for our purposes, since it is powerful enough to fill the remaining lacunas in the dimension theory of definable sets over P-minimal fields, without restriction.

The rest of this paper will be organized as follows. In section 2, we recall some definitions and known results, and we set the notation for the remaining sections. In section 3, we will prove the t-cell decomposition theorem

(Theorem 3.2) and deduce the Small Boundaries Property (Theorem 3.5) as a corollary. Finally, in section 4, we prove the Quasi-continuity Property (Theorem 4.6). The key ingredient of this proof is the following result (see Theorem 4.5), which is also interesting in its own right.

Theorem *Let $A_1, \dots, A_r \subseteq A$ be a family of definable subsets of K^m. If the union of the A_k's has non empty interior in A then at least one of them has non empty interior in A.*

Note that the above statement shows that, if $B_1, \dots, B_r$ are definable subsets which are dense in A, then their intersection $B_1 \cap \cdots \cap B_r$ will also be dense in A. Indeed, a definable subset is dense in A if and only if its complement in A has empty interior inside A.

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2 Notation and prerequisites

Let K be a p-adically closed field, i.e., elementarily equivalent to a p-adic field, and $K^* = K \setminus \{0\}$. We use multiplicative notation for the p-valuation, which we then denote by $|\,.\,|$ so $|ab| = |a||b|$, $|a+b| \leqslant \max |a|, |b|$, and so on[1]. For every set $X \subseteq K$ we will use the notation $|X|$ for the image of X by the valuation. A natural way to extend the valuation to K^m is by putting

$$\|(x_1, \dots, x_m)\| := \max_{i \leqslant m} \{|x_i|\}.$$

This induces a topology, with balls

$$B(x, \rho) := \{y \in K^m : \|x - y\| < \rho\}$$

as basic open sets, where $x \in K^m$ and $\rho \in |K^*|$. For every $X \subseteq K^m$, write $\overline{X}$ for the closure of X and $\operatorname{Int} X$ for the interior of X (inside K^m). The relative interior of a subset A of X inside X, that is $X \setminus \overline{X \setminus A}$, is denoted $\operatorname{Int}_X A$.

Let us now recall the definition of P-minimality:

Definition 2.1 *Let $\mathcal{L}$ be a language extending the ring language $\mathcal{L}_{ring}$. A structure $(K, \mathcal{L})$ is said to be P-minimal if, for every structure $(K', \mathcal{L})$ elementarily equivalent to $(K, \mathcal{L})$, the $\mathcal{L}$-definable subsets of K' are semi-algebraic.*

In this paper, we always work in a P-minimal structure $(K, \mathcal{L})$. Abusing notation, we simply denote it as K. The word definable means definable using parameters in K. A set $S \subseteq K^m \times |K|^n$ is said to be definable if the related set $\{(x, y) \in K^m \times K^n : (x, |y_1|, \dots |y_n|) \in S\}$ is definable.

[1] Compared with additive notation this reverses the order: $|a| \leqslant |b| \Leftrightarrow v(b) \leqslant v(a)$.

A function f from $X \subseteq K^m$ to K^n (or to $|K|^n$) is definable if its graph is a definable set. For every such function, let $\mathcal{C}(f)$ denote the set

$$\mathcal{C}(f) := \{a \in X : f \text{ is continuous on a neighbourhood of } a \text{ in } X\}.$$

It is easy to see that this is a definable set.

We will use the following notation for the fibers of a set. For any set $S \subseteq K^m$, the subsets $I = \{i_1, \ldots, i_r\}$ of $\{1, \ldots, m\}$ induce projections $\pi_I : K^m \to K^r$ (onto the coordinates listed in I) . Given an element $y \in K^r$, the fiber $X_{y,I}$ denotes the set $\pi_I^{-1}(y) \cap X$. In most cases, we will drop the sub-index I and simply write X_y instead of $X_{y,I}$ when the projection π_I is clear from the context. In particular, when $S \subseteq K^{m+n}$ and $x \in K^m$, we write S_x for the fiber with respect to the projection onto the first m coordinates.

One can define a strict order on the set of non-empty definable subsets of K^m, by putting

$$B \ll A \quad \Leftrightarrow \quad B \subseteq A \text{ and } B \text{ lacks interior in } A.$$

The rank of A for this order is denoted $D(A)$. It is defined by induction: $D(A) \geqslant 0$ for every non-empty set A, and $D(A) \geqslant d+1$ if there is a non-empty definable set $B \ll A$ such that $D(B) \geqslant d$. Then $D(A) = d$ if $D(A) \geqslant d$ and $D(A) \ngeqslant d+1$. By convention $D(\emptyset) = -\infty$.

The notion of dimension used by Haskell and Macpherson in [HM97] (which they denoted as topdim A) is defined as follows:

Definition 2.2 *The* ***dimension*** *of a set $A \subset K^m$ (denoted as $\dim A$) is the maximal integer r for which there exists a subset I of $\{1, \ldots, m\}$ such that $\pi_I^m(A)$ has non-empty interior in K^r, where $\pi_I^m : K^m \to K^r$ is defined by*

$$\pi_I^m : (x_1, \ldots, x_m) \mapsto (x_{i_1}, \ldots, x_{i_r})$$

with $i_1 < \cdots < i_r$ an enumeration of I.

We will omit the super-index m in π_I^m when it is clear from the context, and put $\dim \emptyset = -\infty$. Given a set $S \subseteq K^{m+1}$, $\pi_{\{1,\ldots,m\}}^{m+1}(S)$ is simply denoted $\widehat{S}$.

Note that by P-minimality, if $A \subseteq K^m$ is a definable set and $\dim A = 0$, then A is a finite set. Also, $\dim A = m$ if and only if A has non-empty interior.

Let us now recall some of the properties of this dimension that were already proven by Haskell and Macpherson in [HM97]:

(HM$_1$) *Given definable sets $A_1, \ldots, A_r \subseteq K^m$, it holds that* $\dim A_1 \cup \cdots \cup A_r = \max(\dim A_1, \ldots, \dim A_r)$. *(Theorem 3.2)*

(HM$_2$) *For every definable function $f : X \subseteq K^m \to |K|$, $\dim X \setminus \mathcal{C}(f) < m$. (Theorem 3.3 and Remark 3.4 (rephrased))*

(HM$_3$) *For every definable function $f : X \subseteq K^m \to K$, $\dim X \setminus \mathcal{C}(f) < m$. (Theorem 5.4)*

Recall that a complete theory T satisfies the Exchange Principle if the model-theoretic algebraic closure for T does so. In every model of a theory satisfying the Exchange Principle, there is a well-behaved notion of dimension for definable sets, which is called model theoretic rank. Haskell and Macpherson showed that

($\mathbf{HM_4}$) *The model-theoretic algebraic closure for* $\mathrm{Th}(K)$ *satisfies the Exchange Principle. (Corollary 6.2)*

($\mathbf{HM_5}$) *For every definable* $X \subseteq K^m$, $\dim X$ *coincides with the model-theoretic* $\mathrm{rk}\, X$*. (Theorem 6.3)*

The following Additivity Property (Lemma 2.3 below) is known to hold for the model theoretic rank rk, in theories satisfying the exchange principle. For a proof, see Lemma 9.4 in [Mat95]. Hence, theorems (HM_4) and (HM_5) imply that dim also satisfies the Additivity Property. This fact was not explicitly stated by Haskell and Macpherson in [HM97], and seems to have been somewhat overlooked until now. It plays a crucial role in our proof of Theorem 3.2, hence in all our paper.

Lemma 2.3 (Additivity Property) *Let $S \subseteq K^{m+n}$ be a definable set. For $d \in \{-\infty, 0, 1, \ldots, n\}$, write $S(d)$ for the set*

$$S(d) := \{a \in K^m : \dim S_a = d\}.$$

Then $S(d)$ is definable and

$$\dim \bigcup_{a \in S(d)} S_a = \dim(S(d)) + d.$$

Combining this with the first point (HM_1), it follows easily that dim is a dimension function in the sense of van den Dries [vdD89].

Haskell and Macpherson also proved that P-minimal structures are **model-theoretically bounded** (also known as "algebraically bounded" or also that "they eliminate the $\exists^\infty$ quantifier"), *i.e.*, for every definable set $S \subseteq K^{m+1}$ such that all the fibers of the projection of S onto K^m are finite, there exists an integer $N \geqslant 1$ such that all of them have cardinality $\leqslant N$.

While it is not known whether general P-minimal structures admit definable Skolem functions, we do have the following weaker version for coordinate projections with finite fibers.

Lemma 2.4 *Let $S \subseteq K^{m+1}$ be a definable set. Assume that all fibers S_x with respect to the projection onto the first m coordinates are finite. Then there exists a definable function $\sigma : \widehat{S} \to K^{m+1}$ such that $\sigma(x) \in S$ for every $x \in \widehat{S}$.*

Proof: In Lemma 7.1 of [Den84], Denef shows that this is true on the condition that the fibers are not only finite, but uniformly bounded. (The original lemma was stated for semi-algebraic sets, but the same proof holds for general P-minimal structures.) Since uniformity is guaranteed by model-theoretic boundedness, the lemma follows.

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From this it follows by an easy induction that

Corollary 2.5 (Definable Finite Choice) *Let $f : X \subseteq K^m \to K^n$ be a new definable function. Assume that for every $y \in f(X)$, $f^{-1}(y)$ is finite. Then there exists a definable function $\sigma : f(X) \to X$, such that*

$$\big(\sigma \circ f(x), f(x)\big) \in \mathit{Graph}(f)$$

for all $x \in X$.

Using the coordinate projections from Definition 2.2, we will now give a definition of t-cells and t-cell decomposition:

Definition 2.6 *A set $C \subseteq K^m$ is a* **topological cell** *(or* **t-cell** *for short) if there exists some (non unique) $I \subseteq \{1, \dots, n\}$ such that π_I^m induces a homeomorphism from C to a non-empty open set.*

In particular, every non-empty open subset of K^m is a t-cell, and the only finite t-cells in K^m are the points. For any definable set $X \subseteq K^m$, a **t-cell decomposition** is a partition $\mathcal{A}$ of X in finitely many t-cells. We say that the t-cell decomposition is **good**, if moreover each t-cell in $\mathcal{A}$ is either open in X or lacks interior in X.

3 Topological cell decomposition

Recall that K is a P-minimal expansion of a p-adically closed field. We will first show that every set definable in such a structure admits a decomposition in t-cells:

Lemma 3.1 *Every definable set $X \subseteq K^m$ has a good t-cell decomposition.*

Proof: Put $d = \dim X$ and let $e = e(X)$ be the number of subsets I of $\{1, \dots, m\}$ for which $\pi_I(X)$ has non-empty interior in K^d. The proof goes by induction on pairs (d, e) (in lexicographic order). The result is obvious for $d \leqslant 0$ so let us assume that $1 \leqslant d$, and that the result is proved for smaller pairs.

Let $I \subseteq \{1, \dots, m\}$ be such that $\pi_I(X)$ has non-empty interior in K^d. For every y in $\operatorname{Int} \pi_I(X)$, we write X_y for the fiber with respect to the projection π_I. For every integer $i \geqslant 1$ let W_i be the set

$$W_i := \{y \in \operatorname{Int} \pi_I(X) : \operatorname{Card} X_y = i\}.$$

By model-theoretic boundedness, there is an integer $N \geqslant 1$ such that W_i is empty for every $i > N$. We let $\mathcal{I}$ denote the set of indices i for which W_i has non-empty interior in K^d.

For each $i \in \mathcal{I}$, Definable Finite Choice (Corollary 2.5) induces a definable function

$$\sigma_i := (\sigma_{i,1}, \dots, \sigma_{i,i}) : \operatorname{Int} W_i \to K^{mi},$$

such that $X_y = \{\sigma_{i,j}(y)\}_j$ for every $y \in \operatorname{Int} W_i$. Put $V_i := \mathcal{C}(\sigma_i)$, and $C_{i,j} := \sigma_{i,j}(V_i)$. Notice that $C_{i,j}$ is a t-cell for every $i \in \mathcal{I}$ and $j \leqslant i$. Indeed, the restrictions of π_I and $\sigma_{i,j}$ are reciprocal homeomorphisms between $C_{i,j}$ and the open set V_i. We show that each $C_{i,j}$ is open in X.

Fix $i \in \mathcal{I}$ and $j \leqslant i$. Let x_0 be an element of $C_{i,j}$ and $y_0 = \pi_I(x_0)$, so that $x_0 = \sigma_{i,j}(y_0)$. By construction, $\bigcup_k C_{i,k} = \pi_I^{-1}(V_i) \cap X$ is open in X (because

$V_i = \mathcal{C}(\sigma_i)$ is open in $\operatorname{Int} W_i$, hence open in K^d). So there is $\rho \in |K^\times|$ such that $B(x_0, \rho) \cap X$ is contained in $\bigcup_k C_{i,k}$. Let ε be defined as

$$\varepsilon := \min_{k \neq j} \|\sigma_{i,k}(y_0) - \sigma_{i,j}(y_0)\| = \min_{k \neq j} \|\sigma_{i,k}(y_0) - x_0\|$$

Because σ_i is continuous on the open set V_i, there exists δ such that

$$B(y_0, \delta) \subseteq \sigma_i^{-1}(B(\sigma_i(y_0), \rho)) \subseteq V_i,$$

and such that for all $y \in B(y_0, \delta)$, we have that

$$\|\sigma_i(y) - \sigma_i(y_0)\| < \varepsilon.$$

Making δ smaller if necessary, we may assume that $\delta < \min\{\varepsilon, \rho\}$. We will show that $B(x_0, \delta) \cap X \subseteq C_{i,j}$. Let x be in $B(x_0, \delta) \cap X$, and put $y := \pi_I(x)$. Since $\delta < \rho$, we know that there must exist k such that $x = \sigma_{i,k}(y)$. Assume that $k \neq j$. Since $\delta < \varepsilon$, we now have that

$$\begin{aligned}
\|\sigma_{i,k}(y) - \sigma_{i,k}(y_0)\| &\leqslant \|\sigma_i(y) - \sigma_i(y_0)\| \\
&< \varepsilon \\
&\leqslant \|\sigma_{i,j}(y_0) - \sigma_{i,k}(y_0)\| \\
&= \|\sigma_{i,k}(y) - \sigma_{i,j}(y_0)\| \\
&= \|\sigma_{i,k}(y) - x_0\|,
\end{aligned}$$

but this means that $\sigma_{i,k}(y) \notin B(x_0, \delta)$, and hence we can conclude that $x = \sigma_{i,j}(y) \in C_{i,j}$.

Given that each $C_{i,j}$ is a t-cell which is open in X, it remains to show the result for $Z := X \setminus (\bigcup_{i \in \mathcal{I}, j \leqslant i} C_{i,j})$. We will check that $\pi_I(Z)$ has empty interior (in K^d), or equivalently that $\dim \pi_I(Z) < d$.

Note that $\pi_I(Z)$ is a disjoint union $A_1 \sqcup A_2 \sqcup A_3$, where $A_1 := \pi_I(X) \setminus \operatorname{Int} \pi_I(X)$, $A_2 := \operatorname{Int} \pi_I(X) \setminus \bigcup_{i \leqslant N} W_i$, and A_3 is the set

$$A_3 := \Big(\bigcup_{i \in \mathcal{I}} (W_i \setminus \operatorname{Int} W_i) \cup (\operatorname{Int} W_i \setminus V_i) \Big) \cup \bigcup_{i \notin \mathcal{I}} W_i.$$

By (HM_1) it suffices to check that each of these parts has dimension $< d$. Clearly A_1 has empty interior, hence dimension $< d$. For every y in A_2, the fiber X_y is infinite, hence A_2 must have dimension $< d$ by the Additivity Property.

Next, we need to check that A_3 also has dimension smaller than d. By (HM_1), it is sufficient to do this for each part separately. The set $W_i \setminus \operatorname{Int} W_i$ has empty interior for every $i \in \mathcal{I}$, and hence dimension $< d$. For $i \in \mathcal{I}$, $\operatorname{Int} W_i \setminus V_i$ has dimension $< d$ by (HM_3). And finally, W_i has empty interior for every $i \notin \mathcal{I}$ by definition of $\mathcal{I}$, hence dimension $< d$. So $\dim \pi_I(Z) < d$ by (HM_1), hence $\pi_I(Z)$ has empty interior.

A fortiori, the same holds for $\pi_I(Z_1)$ and $\pi_I(Z_2)$ where $Z_1 := \operatorname{Int}_X Z$ and $Z_2 := Z \setminus \operatorname{Int}_X Z$. This implies that, for each $k \in \{1, 2\}$, either $\dim Z_k < d$, or $\dim Z_k = d$ and $e(Z_k) < e$. Hence, the induction hypothesis applies to each Z_k separately and gives a good partition $(D_{k,l})_{l \leqslant l_k}$ of Z_k. Since Z_1 is open in X and Z_2 has empty interior in X, the sets $D_{k,l}$ will also be either open in X, or have empty interior in X. It follows that the family of consisting of the t-cells $C_{i,j}$ and $D_{k,l}$ forms a good t-cell decomposition of X.

■

We will now show that this decomposition can be chosen in such a way as to ensure that definable functions are piecewise continuous, which is one of the main theorems of this paper.

Theorem 3.2 (Topological Cell Decomposition) *For every definable function f from $X \subseteq K^m$ to K^n (or to $|K|^n$) there exists a good t-cell decomposition $\mathcal{C}$ of X, such that for every $C \in \mathcal{C}$ the restriction $f_{|C}$ of f to C is continuous.*

Proof: We prove the result for functions $f : X \subseteq K^m \to K^n$, by induction on pairs (m, d) where $d = \dim X$. Our claim is obviously true if $m = 0$ or $d \leqslant 0$, so let us assume that $1 \leqslant d \leqslant m$ and that the theorem holds for smaller pairs.

Note that it suffices to prove the result for each coordinate function f_i of $f := (f_1, \ldots, f_n)$ separately. Indeed, suppose the theorem is true for the functions $f_i : X \to K$. This means that, for each $1 \leqslant i \leqslant n$, there exists a good t-cell decomposition $\mathcal{C}_i$ of X adapted to f_i. It is then easy, by means of Lemma 3.1, to build a common, finer good t-cell decomposition of X having the required property simultaneously for each f_i, and hence for f. Thus, we may as well assume that $n = 1$.

Consider the set $X \setminus \operatorname{Int} \mathcal{C}(f)$, which can be partitioned as $A_1 \sqcup A_2$, where $A_1 := X \setminus \mathcal{C}(f)$ and $A_2 := \mathcal{C}(f) \setminus \operatorname{Int} \mathcal{C}(f)$. It follows from ($\mathrm{HM}_3$) that $\dim A_1 < m$. Also, $\dim A_2 < m$ since it has empty interior (inside K^m), and therefore the union, $X \setminus \operatorname{Int} \mathcal{C}(f)$, has dimension $< m$ by (HM_1). Hence, by throwing away $\operatorname{Int} \mathcal{C}(f)$ if necessary (which is a definable open set contained in X, hence a t-cell open in X if non-empty), we may assume that $\dim X < m$.

Using Lemma 3.1, one can obtain a good t-cell decomposition $(X_j)_{j \in J}$ of X. For each $j \in J$, we get a subset I_j of $\{1, \ldots, m\}$, an open set $U_j \subseteq K^{d_j}$ (with $d_j = \dim X_j < m$), and a definable map $\sigma_j : U_j \to X_j$. These maps σ_j can be chosen in such a way that σ_j and the restriction of π_{I_j} to X_j are reciprocal homeomorphisms. Now apply the induction hypothesis to each of the functions $f \circ \sigma_j$ to get a good t-cell decomposition $\mathcal{C}_j$ of X_j. Putting $\mathcal{C} = \bigcup_{j \in J} \mathcal{C}_j$ then gives the conclusion for f.

The proof for functions $f : X \subseteq K^m \to |K|^n$ is similar, the main difference being that one needs to use (HM_2) instead of (HM_3).

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Remark 3.3 With the notation of Theorem 3.2, let U be the union of the cells in $\mathcal{C}$ which are open in X. Clearly $U \subseteq \mathcal{C}(f)$ and $X \setminus U$ is the union of the other cells in $\mathcal{C}$, each of which lacks interior in X. To conclude that $\mathcal{C}(f)$ is dense in X, it remains to check that this union still has empty interior in X. This will be the subject of section 4.

The Topological Cell Decomposition Theorem is a strict analagon of the Cell Decomposition Property (CDP) considered by Mathews in the more general context of t-minimal structures. In his paper, Mathews showed that the CDP holds in general for such structures, if a number of rather restrictive conditions hold (e.g., he assumes that the theory of a structure has quantifier elimination), see Theorem 7.1 in [Mat95]. Because of these restrictions, we could not simply refer to this general setting for a proof of the CDP for P-minimal structures.

Further results from Mathews' paper justify why proving Theorem 3.2 is worth the effort. In Theorem 8.8 of [Mat95] he shows that, if the CDP and the Exchange Principle are satisfied for a t-minimal structure with a Hausdorff topology, then several classical notions of ranks and dimensions, including D and dim, coincide for its definable sets. Because of Theorem 3.2 and (HM$_4$), we can now apply the observation from Theorem 8.8 to P-minimal fields, to get that

Corollary 3.4 *For every definable set $A \subseteq K^m$, $\dim A = D(A)$.*

The Small Boundaries Property then follows easily.

Theorem 3.5 (Small Boundaries Property) *For every definable set $A \subseteq K^m$, one has that $\dim(\overline{A} \setminus A) < \dim A$.*

Proof: First note that $D(\overline{A} \setminus A) < D(\overline{A})$, since $\overline{A} \setminus A$ has empty interior in $\overline{A}$. This means that $\dim(\overline{A} \setminus A) < \dim \overline{A}$ by Corollary 3.4. Applying (HM$_1$), we get that $\dim \overline{A} = \dim A$, and therefore $\dim(\overline{A} \setminus A) < \dim A$.
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4 Relative interior and pure components

Given a definable set $A \subseteq K^m$ and $x \in K^m$, let $\dim(A, x)$ denote the smallest $k \in \mathbf{N} \cup \{-\infty\}$ for which there exists a ball $B \subseteq K^m$ centered at a, such that $\dim A \cap B = k$ (see for example [BCR87]). Note that $\dim(A, x) = -\infty$ if and only if $x \notin \overline{A}$. We call this the **local dimension** of A at x. A is said to be **pure dimensional** if it has the same local dimension at every point $x \in A$.

Claim 4.1 *Let $S \subseteq K^m$ be a definable set of pure dimension d.*

1. *Every definable set dense in $\overline{S}$ has pure dimension d.*
2. *For every definable set $Z \subseteq S$, Z has empty interior in S if and only if $\dim Z < \dim S$.*

Proof: Let $X \subseteq S$ be a definable set dense in S. Consider a ball B with center $x \in X$. Then $B \cap S$ is non-empty, and therefore we have that $\dim B \cap S = d$. Moreover, it is easy to see that $B \cap X$ is dense in $B \cap S$, which implies that $\dim B \cap X = d$ as well, by the Small Boundaries Property and (HM$_1$). This proves the first part.

Let us now prove the second point. If Z has empty interior in S, this means that $S \setminus Z$ is dense in S, and hence Z is contained in $\overline{(S \setminus Z)} \setminus (S \setminus Z)$. But then $\dim Z < \dim(S \setminus Z)$ by the Small Boundaries Property, and therefore $\dim Z < \dim S$. Conversely, if Z has non-empty interior inside S, there exists a ball B centered at a point $z \in Z$ such that $B \cap S \subseteq Z$. By the purity of S, $\dim B \cap S = d$, and hence $\dim Z \geqslant d$. Since $Z \subseteq S$, this implies that $\dim Z = d$.
■

For every positive integer k, we put

$$\Delta_k(A) := \{a \in A \mid \dim(A, a) = k\},$$

and we write $C_k(A)$ for the topological closure of $\Delta_k(A)$ inside A. It is easy to see that $\Delta_k(A)$ is pure dimensional, and of dimension k if the set is non-empty. By part 1 of Claim 4.1, the same holds for $C_k(A)$. Moreover, since $C_k(A)$ is closed in A, one can check that it is actually the largest definable subset of A with pure dimension k (if it is non-empty). For this reason, we call the sets $C_k(A)$ the **pure dimensional components** of A.

Remark 4.2 If $\dim(A,x) < k$ for some $x \in A$, then there exists a ball B centered at x for which $\dim B \cap A < k$. Such a ball must be disjoint from $C_k(A)$, because $C_k(A)$ either has pure dimension k or is empty. But then $C_k(A)$ is disjoint from every $\Delta_l(A)$ with $l < k$, which means that it must be contained in the union of the $\Delta_l(A)$ with $l \geqslant k$.

Lemma 4.3 *For every definable set $A \subseteq K^m$ and every k, one has that*

$$\dim \left(C_k(A) \cap \bigcup_{l \neq k} C_l(A) \right) < k.$$

Proof: By (HM_1), it suffices to check that $\dim C_k(A) \cap C_l(A) < k$ for every $l \neq k$. This is obvious when $l < k$, since in these cases $C_l(A)$ already has dimension l or is empty. Hence we may assume that $l > k$. Using Remark 4.2, one gets that

$$C_k(A) \cap C_l(A) \subseteq C_k(A) \cap \bigcup_{i>k} \Delta_i(A) = \bigcup_{i>k} C_k(A) \cap \Delta_i(A).$$

Using (HM_1) again, it now remains to check that $\dim C_k(A) \cap \Delta_i(A) < k$ whenever $i > k$. But since $\Delta_i(A)$ is disjoint from $\Delta_k(A)$, we find that $C_k(A) \cap \Delta_i(A) \subseteq C_k(A) \setminus \Delta_k(A)$. This concludes the proof because of the Small Boundaries Property.
$\blacksquare$

Lemma 4.4 *Let $Z \subseteq A \subseteq K^m$ be definable sets. Then Z has empty interior inside A if and only if $\dim Z \cap C_k(A) < k$ for every k.*

Proof: For every k, we will consider the set

$$D_k(A) = A \setminus \bigcup_{l \neq k} C_l(A).$$

Clearly, this set is open in A and contained in $\Delta_k(A)$. We claim that $D_k(A)$ is also dense in $C_k(A)$. Indeed, $C_k(A)$ is either empty or has pure dimension k. The first case is obvious, so assume that $C_k(A)$ has pure dimension k. By part 2 of Claim 4.1, it suffices to check that $C_k(A) \setminus D_k(A)$ has dimension $< k$. But this follows from Lemma 4.3, so our claim holds.

If Z has non-empty interior in A, there exists $z \in Z$ and $r \in |K^\times|$, such that $B(z,r) \cap A \subseteq Z$. If we put $k := \dim(A,z)$, then $z \in \Delta_k(A)$. Since $D_k(A)$ is dense in $\Delta_k(A)$, the set $D_k(A) \cap B(z,r)$ is non-empty. Pick a point z' in this intersection. Because $D_k(A)$ is open in A, there exists $r' \in |K^\times|$ such that $B(z',r') \cap A \subseteq D_k(A)$ and $r' \leqslant r$. But then

$$B(z',r') \cap D_k(A) \subseteq B(z',r) \cap A = B(z,r) \cap A \subseteq Z,$$

and $B(z',r') \cap D_k(A)$ is non-empty since it contains z'. This shows that $Z \cap D_k(A)$ has non-empty interior inside $D_k(A)$. Since $D_k(A)$ is open in A (and hence in $C_k(A)$), $Z \cap D_k(A)$ has non-empty interior inside $C_k(A)$ as well. Because $C_k(A)$ is pure dimensional, part 2 of Claim 4.1 implies that $\dim(Z \cap C_k(A)) = k$.

Conversely, assume that $\dim(Z \cap C_k(A)) = k$ for some k. By the Small Boundaries Property, one has that $\dim(C_k(A) \setminus D_k(A)) < k$. From this, we can deduce that $\dim(Z \cap D_k(A)) = k$, using (HM_1). The purity of $D_k(A)$ and part 2 of Claim 4.1 then imply that $Z \cap D_k(A)$ has non-empty interior in $D_k(A)$, and hence in A (since $D_k(A)$ is open in A). A fortiori, Z itself has non-empty interior in A.
■

We can now prove the results which were the aim of this section.

Theorem 4.5 *Let $A_1, \ldots, A_r \subseteq A$ be a finite family of definable subsets of K^m. If their union has non empty interior in A then at least one of them has non empty interior in A. In particular, a piece A_i has non-empty interior in A if $\dim A_i \cap C_k(A) = k$ for some k.*

Proof: If $Z := A_1 \cup \cdots \cup A_r$ has non-empty interior in A, then $\dim(Z \cap C_k(A)) = k$ for some k by Lemma 4.4. Then by (HM_1), $\dim(A_i \cap C_k(A)) = k$ for some i and some k, and thus A_i has non-empty interior in A by Lemma 4.4.
■

Theorem 4.6 *Every definable function f from $X \subseteq K^m$ to K^n (resp. $|K|^n$) is continuous on a definable set U which is dense and open in X, and $\dim(X \setminus U) < \dim X$.*

Proof: The existence of U, dense and open in X on which f is continuous, follows from Theorems 3.2 and 4.5 by Remark 3.3. That $\dim(X \setminus U) < \dim X$ then follows from the Small Boundaries Property.
■

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