Model-completions of (co-)Heyting algebras

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(with Markus Junker)

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1 – Model-completion

Theorem (Pitts 1992)

IPC_2 is interpretable in IPC_1.

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For every propositionnal formula $\varphi(\bar{p}, v)$ there are propositionnal formulae $\varphi_R(\bar{p})$ and $\varphi_L(\bar{p})$ such that, for any formula $\psi(\bar{p}, \bar{q})$ not containing v,

$$\varphi \vdash_\psi \iff \varphi_R \vdash \varphi \qquad \psi \vdash \varphi \iff \psi \vdash \phi_L.$$

Theorem (Ghilardi - Zawadowski 1997)

The theory of Heyting algebras has a model-completion.

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Theorem (Ghilardi - Zawadowski 1997)

The theory of Heyting algebras has a model-completion.

Question: Which (theory of) varieties $\mathcal{H}$ of Heyting algebras have a model-completion?

Remark: A necessary condition is is that $\mathcal{H}$ has the amalgamation property.

Theorem (Maksimova 1977)

Exactly 8 varieties $\mathcal{H}_1, \ldots, \mathcal{H}_8$ of Heyting algebras have the amalgation property.

Theorem (Ghilardi - Zawadowski 1997)

Each of the 8 eight varieties of Heyting algeras which has the amalgamation property, has a model-completion.

Proof based on Pitts + Maksimova + some model-theoretic non-sense.

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~~End of the story!~~

Question: What are these model-completions? Can we give a (meaningfull) axiomatisation of them? *Is there a model-theoretic proof?*

From now on and for $i = 1, 2, \ldots, 8$ let $\mathcal{H}_i^*$ be the variety of coHA dual (opposite? reverse?) to $\mathcal{H}_i$:

$$L \in \mathcal{H}_i^* \iff L^* \in \mathcal{H}_i.$$

$\mathcal{L}_{\mathsf{lat}} = \{\mathbf{0}, \mathbf{1}, \vee, \wedge\}$.
$\mathcal{L}_{\mathsf{HA}} = \mathcal{L}_{\mathsf{lat}} \cup \{\rightarrow\}$ and $\mathcal{L}_{\mathsf{HA}^*} = \mathcal{L}_{\mathsf{lat}} \cup \{-\}$.

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From HA to coHA (and way back) without pain

In a coHA, $a - b := \min\{c \mid a \leqslant b \vee c\}$.

Let E be a poset and a an element of E.

- E^* := E, with the opposite order.
- a^* := a, but seen as an element of E^*.

$$b \leqslant a \iff a^* \leqslant b^*$$

If E is a lattice:

$$a \wedge b = (a^* \vee b^*)^* \qquad a \vee b = (a^* \wedge b^*)^*$$

If E is a coHA:

$$a - b = (b^* \to a^*)^*$$

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What means "much smaller"?

Given two subsets S, T of a topological space X,

$$T \ll S \iff T \subseteq S \text{ and } \overline{S \setminus T} = \overline{S}.$$

Given two elements a, b of a (distributive and bounded) lattice L,

$$b \ll a \iff P(b) \ll P(a)$$

whith $P(a) := \{\mathfrak{p} \in \mathrm{Spec}^{\uparrow}(X) \mid a \in \mathfrak{p}\}$.

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Remark: This is a strict order on $L \setminus \{\mathbf{0}\}$ (not on L: $\mathbf{0} \ll \mathbf{0}$!).

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Remark: If a is $\vee$-irreducible then $b \ll a$ iff $b < a$.

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Remark: If L is a coHA then $P(a - b) = \overline{P(a) \setminus P(b)}$ hence

$$b \ll a \iff b \leqslant a \text{ and } a - b = a$$

is quantifier-free definable in L.

$\mathcal{H}_1^*$ = variety of all co-Heyting algebras.

Density D1 For every a, c such that $c \ll a \neq \mathbf{0}$ there exists a non zero element b such that:

$$c \ll b \ll a$$

Splitting S1 For every a, b_1, b_2 such that $b_1 \vee b_2 \ll a \neq \mathbf{0}$ there exists non zero elements $a_1 \geqslant b_1$ and $a_2 \geqslant b_2$ such that:

$$a - a_2 = a_1$$
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$$a_1 \wedge a_2 = b_1 \wedge b_2$$

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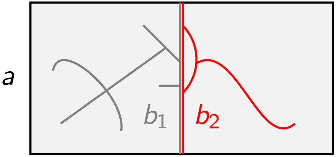

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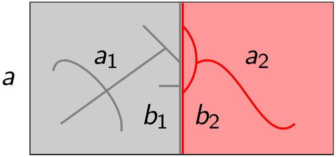

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- $\mathcal{H}_2^* = \mathcal{H}_1^* + \left[(\mathbf{1} - x) \wedge (\mathbf{1} - (\mathbf{1} - x)) = \mathbf{0}\right]$.

This is the dual (opposite? reverse?) of the variety of the logic of the weak excluded middle ($\neg x \vee \neg\neg x = \mathbf{1}$).

Density D2 Same as D1.

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- $\mathcal{H}_3^* = \mathcal{H}_1^* + \left[(((\mathbf{1} - x) \wedge x) - y) \wedge y = \mathbf{0}\right]$

This is the dual of the second slice of Hosoi: a coHA $L \in \mathcal{H}_3^*$ iff every $\mathfrak{p} \in \mathrm{Spec}^{\uparrow} L$ is minimal or maximal.

Density D3 For every a such that $a = \mathbf{1} - (\mathbf{1} - a) \neq \mathbf{0}$ there exists a non zero element b such that $b \ll a$.

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- And so on for $\mathcal{H}_4^*, \ldots, \mathcal{H}_8^*$.

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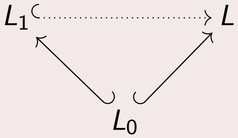
Theorem (Darnière - Junker 2011-18)

For $i = 1, 2, \dots, 8$*:*

1. *Every coHA existentially closed in* $\mathcal{H}_i^*$ *satisfies* $D_i + S_i$.
2. *For every* $L_0, L_1, L \in \mathcal{H}_i^*$ *such that* $L_0 \subseteq L_1$ *and* $L_0 \subseteq L$ *, if* L_1 *is finite and if* L *satisfies* $D_i + S_i$*, there exist an* $\mathcal{L}_{\mathrm{HA}^*}$*-embedding of* L_1 *into* L *over* L_0.



Fact: $\mathcal{H}_1^*$ and $\mathcal{H}_2^*$ are not locally finite, but every other $\mathcal{H}_i^*$ is.

Corollary

For $i = 3, 4, \dots, 8$, $\mathcal{H}_i^*$ *has a model-completion, which is axiomatized by* $D_i + S_i$ *and the axioms of* $\mathcal{H}_i^*$.

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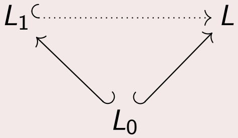
2. *For every $L_0, L_1, L \in \mathcal{H}_i^*$ such that $L_0 \subseteq L_1$ and $L_0 \subseteq L$, if L_1 is finite and if L satisfies $D_i + S_i$, there exist an $\mathcal{L}_{\mathrm{HA}^*}$-embedding of L_1 into L over L_0.*



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2 – Dimension theory

Let $<$ be strict order on a set E, and $x \in E$. The **foundation rank** of x in E for $<$ is defined as follows:

$$\mathrm{rk}(x, <) \geqslant n \iff \exists x_0 < x_1 < \cdots < x_n = x.$$

Then $\mathrm{rk}(x, <) = n \iff \mathrm{rk}(x, <) \geqslant n$ and $\mathrm{rk}(x, <) \not\geqslant n + 1$.

The **cofoundation rank** $\mathrm{cork}(x, <) = \mathrm{rk}(x, >)$.

Examples:

- $\mathrm{rk}(x, <) = 0$ iff x in minimal in E.
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For every a in a distributive bounded lattice L,

$$\dim_L a := \max\{\operatorname{cork}(\mathfrak{p}, \subset) \mid \mathfrak{p} \in P(a)\}.$$

(Reminder: $P(a) = \{\mathfrak{p} \in \operatorname{Spec} L \mid a \in \mathfrak{p}\}$.)
By convention $\dim \mathbf{0} = -\infty$.

Proposition

For every $a, b \in L$, $\dim_L(a \vee b) = \max(\dim_L a, \dim_L b)$.

Proof: $P(a \vee b) = P(a) \cup P(b)$.

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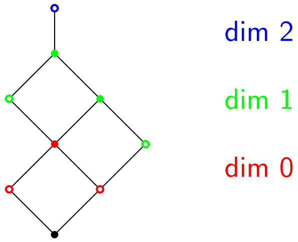
Proof: $P(a \vee b) = P(a) \cup P(b)$.

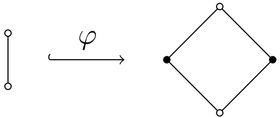
The finite case

If L is finite:

- Every $a \in L$ is the join of finitely many $\vee$-irreducible elements.
- For every $c \in \mathcal{I}^{\vee}(L)$, $\dim c$ is the foundation rank of c in $\mathcal{I}^{\vee}(L)$.



Remark: $\dim_L a$ strongly depends on L.



$\dim_{L_2} \mathbf{1} = 1$ $\quad$ $\dim_{B_2} \varphi(\mathbf{1}) = 0$

Proposition

If $\varphi : L_0 \to L_1$ is an $\mathcal{L}_{\mathrm{HA}^*}$-embedding then $\dim_{L_0} a \leqslant \dim_{L_1} \varphi(a)$.

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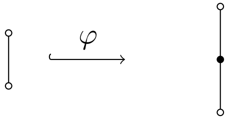

$\dim_{L_2} \mathbf{1} = 1 \qquad \dim_{L_3} \varphi(\mathbf{1}) = 2$

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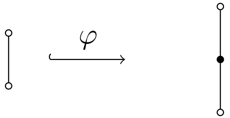

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The geometric case

k = algebraically closed field.
S = an algebraic variety (= Zariski-closed subset of k^n).

$$\dim S = \max\{\text{cork}(\mathfrak{p}, \subset) \mid \mathfrak{p} \in P(\text{Ann}(S))\}$$

where $\text{Ann}(S) = \{f \in k[x_1, \ldots, x_n] \mid f = 0 \text{ on } S\}$, and
$P(\text{Ann}(S)) = \{\mathfrak{p} \in \text{Spec}\, k[X_1, \ldots, X_n] \mid \text{Ann}(S) \subseteq \mathfrak{p}\}$.

Theorem ($\simeq$ Hilbert's Nullstellensatz)

$$\text{Spec}\, k[X_1, \ldots, X_n] \underset{homeo.}{\simeq} \text{Spec}\, L(k^n)$$

where $L(k^n) = \{$Zariski-closed subsets of $k^n\}$.

As a consequence, $\dim S = \dim_{L(k^n)} S$.

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Fact: For non-empty $S, T \in L(k^n)$, $T \ll S \Rightarrow \dim T < \dim S$.

Proposition

For every non-zero elements a,b of a distributive bounded lattice L,

$$b \ll a \Rightarrow \dim b < \dim a.$$

Hence $\exists a_0 \ll \cdots \ll a_n = a$ in $L \setminus \{\mathbf{0}\} \Rightarrow \dim_L a \geqslant n$. That is

$$\dim a \geqslant \mathrm{rk}(a, \ll).$$

Proposition

If L is a coHA then $\dim_L a = \mathrm{rk}(a, \ll)$ *for every* $a \in L \setminus \{\mathbf{0}\}$.
As a consequence "$\dim a = n$*" is first-order definable in* $\mathcal{L}_{\mathrm{HA}^*}$.

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If L is a coHA then $\dim_L a = \mathrm{rk}(a, \ll)$ *for every* $a \in L \setminus \{\mathbf{0}\}$.
As a consequence "$\dim a = n$*" is first-order definable in* $\mathcal{L}_{\mathrm{HA}^*}$.

60 %

Fact: For non-empty $S, T \in L(k^n)$, $T \ll S \Rightarrow \dim T < \dim S$.

Proposition

For every non-zero elements a,b of a distributive bounded lattice L,

$$b \ll a \Rightarrow \dim b < \dim a.$$

Hence $\exists a_0 \ll \cdots \ll a_n = a$ in $L \setminus \{\mathbf{0}\} \Rightarrow \dim_L a \geqslant n$. That is

$$\dim a \geqslant \operatorname{rk}(a, \ll).$$

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Codimension

For every non-zero element of a distributive bounded lattice L,

$$\mathrm{codim}_L\, a := \min\{\mathrm{rk}(\mathfrak{p}, \subset) \mid \mathfrak{p} \in P(a)\}.$$

By convention $\mathrm{codim}\, \mathbf{0} = +\infty$.

In a nutshell:

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In a nutshell:

- Similar properties as dim.
- Much better if L is a coHA.

For every a, b in a coHA L let

$$\delta(a,b) := 2^{-\operatorname{codim} a \Delta b}$$

where $a\Delta b = (a - b) \vee (b_a) = (a^* \leftrightarrow b^*)^*$.

Proposition

1. *δ is a pseudometric on L. It is an ultrametric iff every non-zero element has finite codimension in L.*
2. *Every $\mathcal{L}_{\mathrm{HA}^*}$-morphism is 1-lipshitzian.*
3. *$\mathcal{L}_{\mathrm{HA}^*}$-operations are uniformly continuous, hence extend uniquely to the Cauchy-completion $\widehat{L}$ of L (so $\widehat{L}$ is still a coHA).*

Theorem (Darnière - Junker 2010)

For every positive integer d,

$$dL := \{a \in L \mid \operatorname{codim}_L a \geqslant d\}$$

is a principal ideal of L.
The family $(L/dL)_{d<\omega}$ forms a projective system, whose projective limits coincides with the Cauchy-completion $\widehat{L}$ of L.

Remark: If L/dL is finite for every d, this implies that $\widehat{L}$ is also the profinite completion of L.

A pseudometric space is **precompact** if its Cauchy-completion is compact.

Theorem (Darniere - Junker 2010)

For every variety $\mathcal{H}^*$ of coHA, the following are equivalent.

1. *$\mathcal{H}^*$ has the finite model property.*
2. *Every L free in $\mathcal{H}^*$ is Hausdorff.*
3. *Every L finitely presented in $\mathcal{H}^*$ is precompact Hausdorff.*

More on this in *Codimension and pseudometric in co-Heyting algebras*, Algebra Universalis 64 (2010), no. 3-4.

3 – Model-completion of coHA of dimension $\leqslant d$

$\dim L := \dim_L \mathbf{1}$.
$\mathcal{D}(d) := \{ \text{ coHA } L \mid \dim L \leqslant d \}$.

Remark: This is the dual (opposite? reverse?) of the $(d+1)$-slice of Hosoi (1967).

Proposition (Hosoi 1967 + Ono 1971)

$\mathcal{D}(d)$ is a variety of coHA's.

Axiomatisation: $\mathcal{D}(d) = \mathcal{H}_1^* + [\Delta_d = \mathbf{0}]$ where $\Delta_n = \Delta_n(x_0, \ldots, x_d)$ is defined inductively by $\Delta_{-1} = \mathbf{1}$ and for $d \geqslant 0$

$$\Delta_d = (\Delta_{d-1} - x_d) \wedge x_d.$$

The point is that $(a - b) \wedge b \ll a$ in every coHA.

Examples:

- $\mathcal{D}(-1) = \mathcal{H}_1^* + [\mathbf{1} = \mathbf{0}]$ is the trivial variety $\mathcal{H}_8^*$.
- $\mathcal{D}(0) = \mathcal{H}_1^* + [(\mathbf{1} - x) \wedge x = \mathbf{0}]$ is the variety $\mathcal{H}_7^*$ of Boolean algebras.
- $\mathcal{D}(1)$ is the variety $\mathcal{H}_3^*$.
- For $n \geqslant 2$, $\mathcal{D}(n)$ doesn't have the amalgamation property, hence doesn't have a model-completion… in $\mathcal{L}_{\mathsf{HA}^*}$!

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$\mathcal{L}_{\mathsf{SC}_d} = \mathcal{L}_{\mathsf{HA}^*} \cup \{\mathrm{C}^i\}_{0\leqslant i\leqslant d}$ where each C^i is a unary function symbol. For every a in an $\mathcal{L}_{\mathsf{SC}_d}$-structure let

$$\text{sc-dim}\, a = \min\{e \leqslant d \mid a = \bigvee_{0\leqslant i\leqslant e} \mathrm{C}^i(a)\}.$$

Remark: Contrary to dim, sc-dim is automatically preserved by $\mathcal{L}_{\mathsf{SC}_d}$-embedding.

For every $S \in L(k^d)$, let $\mathrm{C}^i(S) =$ the pure i-dimensionnal component of S. The expansion of L by these functions C^i is our guiding example of d-scaled lattice.

A ***d*-subscaled lattice** is an $\mathcal{L}_{\mathsf{SC}_d}$-expansion L of a coHA satisfying the following axioms.

SC_1 $\bigvee\!\!\!\!\bigvee_{0 \leqslant i \leqslant d} \mathrm{C}^i(a) = a$

SC_2 $\forall I \subseteq \{0, \dots, d\}$, $\forall k$:

$$\mathrm{C}^k\Big(\bigvee\!\!\!\!\bigvee_{i \in I} \mathrm{C}^i(a)\Big) = \begin{cases} \mathbf{0} & \text{if } k \notin I \\ \mathrm{C}^k(a) & \text{if } k \in I \end{cases}$$

SC_3 $\forall k \geqslant \max(\text{sc-dim}(a), \text{sc-dim}(b))$,
$\mathrm{C}^k(a \vee b) = \mathrm{C}^k(a) \vee \mathrm{C}^k(b)$

SC_4 $\forall i \neq j, \quad \text{sc-dim}\left(\mathrm{C}^i(a) \wedge \mathrm{C}^j(b)\right) < \min(i, j)$

SC_5 $\forall k \geqslant \text{sc-dim}(b), \quad \mathrm{C}^k(a) - b = \mathrm{C}^k(a) - \mathrm{C}^k(b)$
In particular, by SC_1: $\text{sc-dim}\, b < a \Rightarrow \mathrm{C}^k(a) - b = \mathrm{C}^k(a)$.

SC_6 $b \ll a \neq \mathbf{0} \Rightarrow \text{sc-dim}\, b < \text{sc-dim}\, a$.

Because of SC_6, the class of all d-subscaled is not a variety.

Theorem (Darnière 2010-18)

Every finitely generated d-subscaled lattice is finite.

Key of the proof: $\text{sc-dim}(a - b) \wedge b < \text{sc-dim}\, a$.

SC_6 implies that $\dim_L a \leqslant \text{sc-dim}\, a$ for every a in a d-subscaled lattice L. When equality holds L is called a ***d*-scaled lattice**.

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Theorem

The theory of d-subscaled lattices has a model-completion, axiomatised by the axioms of d-scaled lattices and the following conditions.

Catenarity *For every $r < q < p$ and every non-zero elements $c \ll a$, if $c = \mathrm{C}^r(c)$ and $a = \mathrm{C}^p(a)$, there exist an element $b = \mathrm{C}^q(b)$ such that $c \ll b \ll a$.*

Splitting *For every elements b_1, b_2, a, if $b_1 \vee b_2 \ll a \neq \mathbf{0}$, there exist non-zero elements $a_1 \geqslant b_1$ and $a_2 \geqslant b_2$ such that:*

$$\begin{cases} a_1 = a - a_2 \\ a_2 = a - a_1 \\ a_1 \wedge a_2 = b_1 \wedge b_2 \end{cases}$$

For yet another model-completion result based on a Density and a Splitting axiom, see Carai and Ghilardi: *Existentially Closed Brouwerian Semilattices*, arXiv 1702.08352

Thank you!