Model completion of varieties of co-Heyting algebras

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Abstract

It is known that exactly eight varieties of Heyting algebras have a model-completion. However no concrete axiomatization of these model-completions were known by now except for the trivial variety (reduced to the one-point algebra) and the variety of Boolean algebras. For each of the six remaining varieties we introduce two axioms and show that 1) these axioms are satisfied by all the algebras in the model-completion, and 2) all the algebras in this variety satisfying these two axioms satisfy a certain remarkable embedding theorem. For four of these six varieties (those which are locally finite) these two results provide a new proof of the existence of a model-completion with, in addition, an explicit and finite axiomatization.

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1 Introduction

It is known from a result of Maksimova [Mak77] that there are exactly eight varieties of Heyting algebras that have the amalgamation property (numbered $\mathcal{H}_1$ to $\mathcal{H}_8$, see section 2). Only these varieties (more exactly their theories) can have a model completion[1] and it is known since the 1990's that this is indeed the case[2]. On the other hand no model-theoretic proof of these facts were known until now and these model-completions still remain very mysterious except for $\mathcal{H}_7$ and $\mathcal{H}_8$: the latter is the trivial variety reduced to the one point Heyting algebra, and the former is the variety of Boolean algebras whose model completion is well known.

In this paper we partly fill this lacuna by giving new proofs for some of these results using algebraic and model-theoretic methods, guided by some geometric intuition. We first give in section 3 a complete classification of all the minimal

[1]Basic model theoretic notions are recalled in section 2 but we may already point out a remarkable application of the existence of a model-completion for these varieties, namely that for each of the corresponding super-intuitionistic logics the second order propositional calculus IPC^2 is interpretable in the first order propositional calculus IPC^1 (in the sense of [Pit92]).

[2]See [GZ97]. For $\mathcal{H}_3$ to $\mathcal{H}_8$, which are locally finite, the existence of a model-completion follows from the amalgamation property and [Whe76], corollary 5. For the variety H_1 of all Heyting algebras it is a translation in model-theoretic terms of a theorem of Pitts [Pit92] combined with [Mak77], as is explained in [GZ97]. It is also claimed in [GZ97] that the same holds for $\mathcal{H}_2$ up to minor adaptations of [Pit92].

finite extensions of a Heyting algebra L. This is done by proving that these extension are in one-to-one correspondence with certain special triples of elements of L. Each of the remaining sections 4 to 9 is devoted to one of the varieties $\mathcal{H}_i$. We introduce for each of them two axioms that we call "density" and "splitting" and prove our main results:

Theorem 1.1 *Every existentially closed model of $\mathcal{H}_i$ satisfies the density and splitting axioms of $\mathcal{H}_i$.*

Theorem 1.2 *Given a Heyting algebra L in $\mathcal{H}_i$ and a finite substructure L_0 of L, if L satisfies the density and splitting axioms of $\mathcal{H}_i$ then every finite extension L_1 of L_0 admits an embedding into L which fixes L_0 pointwise.*

By standard model-theoretic arguments (see fact 2.1) it follows immediately that if $\mathcal{H}_i$ is locally finite then $\mathcal{H}_i$ has a model-completion which is axiomatized by the axioms of Heyting algebras augmented by the density and the splitting axioms of $\mathcal{H}_i$. So this gives a new proof of the previously known model-completion results for $\mathcal{H}_3$ to $\mathcal{H}_6$, which provides in addition a simple axiomatization of these model completions.

Unfortunately we do not fully recover the existence of a model-completion for $\mathcal{H}_1$ and $\mathcal{H}_2$. However our axioms shed some new light on the algebraic structure of the existentially closed Heyting algebras in these varieties. Indeed it is noticed in [GZ97] that such algebras satisfy the density axiom of $\mathcal{H}_1$, but neither the splitting property nor any condition sufficient for theorem 1.2 to hold, seem to have been suspected until now. Moreover all the algebraic properties of existentially closed Heyting algebras in $\mathcal{H}_1$ which are listed in [GZ97] can be derived from our two axioms, as we shall see in the appendix.

Let us also point out an easy consequences of theorem 1.2.

Corollary 1.3 *If L is an algebra in $\mathcal{H}_i$ which satisfies the density and splitting axioms of $\mathcal{H}_i$ then every finite algebra in $\mathcal{H}_i$ embeds into L, and every algebra in $\mathcal{H}_i$ embeds into an elementary extension of L.*

Remark 1.4 In this paper we do not actually deal with Heyting algebras but with their duals, obtained by reversing the order. They are often called **co-Heyting algebras** in the literature. Readers familiar with Heyting algebras will certainly find annoying this reversing of the order. We apologise for this discomfort but there are good reasons for doing so. Indeed, the present work has been entirely build on a geometric intuition coming from the fundamental example[3] of the lattice of all subvarieties of an algebraic variety, and their counterparts in real algebraic geometry. Such lattices are co-Heyting algebras, not Heyting algebras. To see how this intuition is used in finding the proofs (and then hidden while writing the proofs) look at figure 4 in lemma 4.2.

[3] This geometric intuition also played a role in the very beginning of the study of Heyting algebras. Indeed, co-Heyting algebras were born Brouwerian lattices in the paper of Mckinsey and Tarski [MT46] which originated much of the later interest in Heyting algebras. Also the introduction of "slices" in [Hos67] seems inspired by the same geometric intuition that we use in this paper. Indeed, the dual of the co-Heyting algebra of all subvarieties of an algebraic variety V belongs to the $d+1$-th slice if and only if V has dimension $\leq d$ (see [DJ11] for more on this topic).

2 Other notation, definitions and prerequisites

We denote by $\mathcal{L}_{\text{lat}} = \{\mathbf{0}, \mathbf{1}, \vee, \wedge\}$ the language of distributive bounded lattice, these four symbols referring respectively to the least element, the greatest element, the join and meet operations. $\mathcal{L}_{\text{HA}} = \mathcal{L}_{\text{lat}} \cup \{\rightarrow\}$ and $\mathcal{L}_{\text{HA}^*} = \mathcal{L}_{\text{lat}} \cup \{-\}$ are the language of Heyting algebras and co-Heyting algebras respectively. Finite joins and meets will be denoted $\bigvee$ and $\bigwedge$, with the natural convention that the join (resp. meet) of an empty family of elements is $\mathbf{0}$ (resp. $\mathbf{1}$).

The logical connectives 'and', 'or', and their iterated forms will be denoted $\bigwedge$, $\bigvee$, $\bigwedge\!\!\bigwedge$ and $\bigvee\!\!\bigvee$ respectively.

We denote by $\mathcal{I}^{\vee}(L)$ the set of **join irreducible** elements of a lattice L, that is the elements a of L which can not be written as the join of any finite subset of L not containing a. Of course $\mathbf{0}$ is never join irreducible since it is the join of the empty subset of L. The set $\mathcal{I}^{\vee}(L)$ inherits the order induced by L.

Dualizing rules. In order to help the reader more familiar with Heyting algebras than co-Heyting algebras, we recommend the use of the following conversion rules. For any ordered set L we denote by L^* the **dual** of L, that is the same set with the reverse order. If a is an element of L we denote by a^* the same element *seen as en element of* L^*, so that we can write for any $a, b \in L$:

$$a \leq b \iff b^* \leq a^*$$

Indeed the star indicates that the second symbol $\leq$ refers to the order of L^*, and the first one to the order of L. Similarly if L is a co-Heyting algebra we can write:

$$\mathbf{0}^* = \mathbf{1} \quad \text{and} \quad \mathbf{1}^* = \mathbf{0}$$
$$(a \vee b)^* = a^* \wedge b^* \quad \text{and} \quad (a \wedge b)^* = a^* \vee b^*$$

The minus operation of $\mathcal{L}_{\text{HA}^*}$ stands of course for the dual of the arrow operation of $\mathcal{L}_{\text{HA}}$, but beware of the order of the operands:

$$a - b = \min\{c : a \leq b \vee c\} = (b^* \rightarrow a^*)^*$$

The topological symmetric difference is defined as:

$$a \bigtriangleup b = (a - b) \vee (b - a) = (a^* \leftrightarrow b^*)^*$$

We will make extensive use of the following relation:

$$b \ll a \iff a - b = a \text{ and } b \leq a$$

Note that $b \ll a$ and $b \not< a$ if and only if $a = b = \mathbf{0}$, hence $\ll$ is a strict order on $L \setminus \{\mathbf{0}\}$. Note also that if a is join irreducible in L (hence non zero) then $b \ll a$ if and only if $b < a$.

The varieties of Maksimova. We can now describe the varieties $\mathcal{H}_1$ to $\mathcal{H}_8$ introduced by Maksimova, more exactly the corresponding varieties $\mathcal{V}_1$ to $\mathcal{V}_8$ of co-Heyting algebras. Note that the intuitionistic negation $\neg\varphi$ being defined as $\varphi \rightarrow \bot$, the corresponding operation in co-Heyting algebras is $\mathbf{1} - a$:

$$\mathbf{1} - a = (a^* \rightarrow \mathbf{1}^*)^* = (a^* \rightarrow \mathbf{0})^* = \big(\neg(a^*)\big)^*$$

- $\mathcal{V}_1$ is the variety of all co-Heyting algebras.
- $\mathcal{V}_2 = \mathcal{V}_1 + \big[(\mathbf{1} - x) \wedge (\mathbf{1} - (\mathbf{1} - x)) = \mathbf{0}\big]$ is the dual of the variety of the logic of the weak excluded middle ($\neg x \vee \neg\neg x = \mathbf{1}$).
- $\mathcal{V}_3 = \mathcal{V}_1 + \big[\big(((\mathbf{1} - x) \wedge x) - y\big) \wedge y = \mathbf{0}\big]$ is the dual of the second slice of Hosoi. With the terminology of [DJ11], $\mathcal{V}_3$ is the variety of co-Heyting algebras of dimension ≤ 1. So a co-Heyting algebra L belongs to $\mathcal{V}_3$ if and only if any prime filter of L which is not maximal is minimal (with respect to inclusion, among the prime filters of L).
- $\mathcal{V}_4 = \mathcal{V}_3 + \big[(x - y) \wedge (y - x) \wedge (x \mathbin{\triangle} (\mathbf{1} - y)) = \mathbf{0}\big]$ is the variety generated by the co-Heyting algebra $\mathbf{L}_5$ (see figure 1).
- $\mathcal{V}_5 = \mathcal{V}_2 + \big[\big(((\mathbf{1} - x) \wedge x) - y\big) \wedge y = \mathbf{0}\big]$ is the variety generated by $\mathbf{L}_3$ (see figure 1).
- $\mathcal{V}_6 = \mathcal{V}_1 + \big[(x - y) \wedge (y - x) = \mathbf{0}\big]$ is the variety generated by the chains.
- $\mathcal{V}_7 = \mathcal{V}_1 + \big[(1 - x) \wedge x = \mathbf{0}\big]$ is the variety of boolean algebras (which are exactly the co-Heyting algebras of dimension ≤ 0).
- $\mathcal{V}_8$ is the trivial variety $\mathcal{V}_1 + \big[\mathbf{1} = \mathbf{0}\big]$ reduced to $\mathbf{L}_1$ (see figure 1).

Note that the product of an empty family of co-Heyting algebras is just $\mathbf{L}_1$.

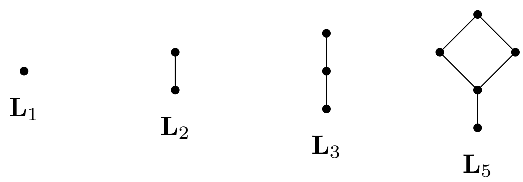


Figure 1: Four basic co-Heyting algebras

Model-completion and super-intuitionistic logics. For an introduction to the basic notions of first-order model-theory (language, formulas, elementary classes of structures, models and existentially closed models of a theory) we refer the reader to any introductory book, such as [Hod97] or [CK90].

Every model of a universal theory T embeds in an existentially closed model. If the class of all existentially closed models of T is elementary, then the corresponding theory $\overline{T}$ is called the **model companion** of T. The model companion eliminates quantifiers if and only if T has the amalgamation property, in which case $\overline{T}$ is called the **model completion** of T. By abuse of language we will speak of the model completion of a variety in place of the model completion of the theory of this variety.

It is an elementary fact that formulas in the first order intuitionistic propositional calculus (IPC^1) can be considered as terms (in the usual model-theoretical sense) in the language of Heyting algebras, and formulas in the second order intuitionistic propositional calculus (IPC^2) as first order formulas in the language of Heyting algebras. In particular, if a variety of Heyting algebras has a model completion then it appears, following [GZ97] that the corresponding super-intuitionistic logic has the property that IPC^2 is interpretable in IPC^1, in the sense of Pitts [Pit92].

Finally let us recall the criterion for model completion which makes the link with theorems 1.1 and 1.2.

Fact 2.1 *A theory $\overline{T}$ is the model completion of a universal theory $T \subseteq \overline{T}$ if and only if it satisfies the two following conditions.*

1. *Every existentially closed model of T is a model of $\overline{T}$.*
2. *Given a model L of $\overline{T}$, a finitely generated substructure L_0 of L and a finitely generated model L_1 of T containing L_0, there is an embedding of L_1 into an elementary extension of L which fixes L_0 pointwise.*

The finite model property. A variety $\mathcal{V}$ of co-Heyting algebras has the finite model property if any equation valid on every finite algebra in $\mathcal{V}$ is valid on every algebra of $\mathcal{V}$.

Proposition 2.2 *For a variety $\mathcal{V}$ of co-Heyting algebras the following properties are equivalent.*

1. *$\mathcal{V}$ has the finite model property.*
2. *For every quantifier-free $\mathcal{L}_{\mathrm{HA}^*}$-formula $\varphi(x)$ and every algebra L in $\mathcal{V}$ such that $L \models \exists x\ \varphi(x)$, there exists a finite algebra L' in $\mathcal{V}$ such that $L' \models \exists x\ \varphi(x)$.*

Proof: For every equation $\theta(x)$, if there is an algebra L in $\mathcal{V}$ on which $\theta(x)$ is not valid then condition 2 applied to $\varphi(x) \equiv \neg\theta(x)$ gives a finite algebra in $\mathcal{V}$ on which $\theta(x)$ is not valid. This proves that (2)⇒(1). For the converse, see [DJ11] proposition 8.1. $\dashv$

The finite model property holds obviously for every locally finite variety, but also[4] for $\mathcal{V}_1$ and $\mathcal{V}_2$. We combine it with a bit of model-theoretic non-sense in the following lemmas.

Lemma 2.3 *Let $\mathcal{V}$ be a variety of co-Heyting algebras having the finite model property. Let $\theta(x)$ and $\phi(x, y)$ be quantifier-free $\mathcal{L}_{\mathrm{HA}^*}$-formulas. Assume that for every finite co-Heyting algebra L_0 and every tuple a of elements of L_0 such that $L_0 \models \theta(a)$, there exists an extension L_1 of L_0 which satisfies $\exists y\ \phi(a, y)$. Then every algebra existentially closed in $\mathcal{V}$ satisfies the following axiom:*

$$\forall x\ (\theta(x) \longrightarrow \exists y\ \phi(x, y))$$

Proof: Let L be an existentially closed co-Heyting algebra which satisfies $\theta(a)$ for some tuple a. Let Σ be its quantifier-free diagram, that is the set of all atomic and negatomic $\mathcal{L}_{\mathrm{HA}^*}$-formulas (with parameters) satisfied in L. Let Σ_0 be an arbitrary finite subset of Σ. The conjunction of $\theta(a)$ and the elements of Σ_0 is a quantifier-free formula (with parameters) $\Delta(a, b)$. Since $L \models \exists x, y\ \Delta(x, y)$ and $\mathcal{V}$ has the finite model property, by proposition 2.2 there exists a finite co-Heyting algebra L_0 and a tuple (a_0, b_0) of elements of

[4]For example corollary 2.2.1 of [McK68] applies to $\mathcal{V}_2$, as well as to $\mathcal{V}_1$.

L_0 such that $L_0 \models \Delta(a_0, b_0)$. In particular $L_0 \models \theta(a_0)$ hence by assumption L_0 admits an extension L_1 which satisfies $\exists y\ \phi(a_0, y)$. So L_1 is a model of this formula and of Σ_0 (because Σ_0 is quantifier free and already satisfied in L_0). We have proved that the union of Σ and $\exists y\ \phi(a, y)$ is finitely satisfiable hence by the model-theoretic compactness theorem, it admits a model L' in which L embeds naturally. Since L is existentially closed it follows that L itself satisfies $\exists y\ \phi(a, y)$. $\dashv$

Lemma 2.4 *Let $\mathcal{V}$ be a variety of co-Heyting algebras having the finite model property. Let $\theta'(x)$ and $\phi'(x, y)$ be $\mathcal{L}_{\mathrm{HA}^*}$-formulas that are conjunctions of equations. Assume that:*

1. *There is a subclass $\mathcal{C}$ of $\mathcal{V}$ such that a finite co-Heyting algebra belongs to $\mathcal{V}$ if and only if it embeds into the direct product of a finite (possibly empty) family of algebras in $\mathcal{C}$.*

2. *For every algebra L in $\mathcal{C}$ and every $a = (a_1, \ldots, a_m) \in L^m$ such that $L \models \theta'(a)$ there is an extension L' of L in $\mathcal{V}$ and some $b = (b_1, \ldots, b_n) \in L'^n$ such that $L' \models \phi'(a, b)$. If moreover $a_1 \neq \mathbf{0}$ then one can require all the b_i's to be non zero.*

Then every algebra existentially closed in $\mathcal{V}$ satisfies:

$$\forall x \quad \left[\left(\theta'(x) \bigwedge x_1 \neq \mathbf{0} \right) \to \exists y \quad \left(\phi'(x, y) \bigwedge \bigwedge_{i \leq n} y_i \neq \mathbf{0} \right) \right]$$

Although somewhat tedious, this lemma will prove to be helpful for the varieties $\mathcal{H}_2$, $\mathcal{H}_4$, $\mathcal{H}_5$ and $\mathcal{H}_6$.

Proof: Let L be a finite algebra in $\mathcal{V}$ and $a = (a_1, \ldots, a_m) \in L^m$. Assume that $L \models \theta'(a) \bigwedge a_1 \neq \mathbf{0}$.

By assumption there are $L_1, \ldots, L_r$ in $\mathcal{C}$ such that L embeds into the direct product of the L_j's. So each a_i can be identified with $(a_i^1, \ldots, a_i^r) \in L_1 \times \cdots \times L_r$.

For every $j \leq r$ let $a^j = (a_1^j, \ldots, a_m^j) \in L_j^m$. Since $\theta'(x)$ is a conjunction of equations and $L \models \theta'(a)$, we have $L_j \models \theta'(a^j)$. Thus by assumption there is an extension L'_j of L_j and a tuple $b^j = (b_1^j, \ldots, b_n^j) \in L_j'^n$ such that $L'_j \models \phi'(a^j, b^j)$. Moreover if $a_1^j \neq \mathbf{0}$ then we do require $b_i^j \neq \mathbf{0}$ for every $i \leq n$.

Let L' be the direct product of the L'_j's. For every $i \leq n$ let $b_i = (b_i^1, \ldots, b_i^r)$ and $b = (b_1, \ldots, b_n) \in L'^n$. The algebra L' is an extension of L in $\mathcal{V}$, and since $\phi'(x, y)$ is a conjunction of equation by construction $L' \models \phi'(a, b)$.

Moreover $a_1 = (a_1^1, \ldots, a_1^r)$ is non zero, so there is an index $j \leq r$ such that $a_1^j \neq 0$. Then by construction for every $i \leq n$, $b_i^j \neq \mathbf{0}$ hence b_i is non zero.

So we can apply lemma 2.3 to the variety $\mathcal{V}$ with the quantifier free formulas $\theta(x)$ and $\phi(x, y)$ defined by:

$$\theta(x) \equiv \theta'(x) \bigwedge x_1 \neq \mathbf{0} \quad \text{and} \quad \phi(x, y) \equiv \phi'(x, y) \bigwedge \bigwedge_{i \leq n} y_i \neq \mathbf{0}$$

$\dashv$

Decreasing subsets and finite duality. For any element a and any subset A of an ordered set E we let:

$$a^{\downarrow} = \{b \in X : b \leq a\} \qquad \text{and} \qquad A{\downarrow} = \bigcup_{a \in A} a^{\downarrow}$$

A **decreasing subset** of E is a subset such that $A = A{\downarrow}$. The family $\mathcal{L}^{\downarrow}(E)$ of all decreasing subsets of E are the closed sets of a topology on E, hence a co-Heyting algebra with operations:

$$A \vee B = A \cup B \qquad A \wedge B = A \cap B \qquad A - B = (A \setminus B){\downarrow}$$

Its completely join irreducible elements are precisely the decreasing sets $x^{\downarrow}$ for x ranging over E.

It is folklore that if L is a finite co-Heyting algebra, then the map $\iota_L : a \mapsto a^{\downarrow} \cap \mathcal{I}^{\vee}(L)$ is an $\mathcal{L}_{\mathrm{HA}^*}$-isomorphism from L to the family $\mathcal{L}^{\downarrow}(\mathcal{I}^{\vee}(L))$ of all decreasing subsets of $\mathcal{I}^{\vee}(L)$, whose inverse is the map $A \mapsto \bigvee A$. This actually defines a contravariant duality between the category of finite co-Heyting algebras and the category of finite posets with morphisms the increasing maps $\psi : I' \to I$ satisfying the property:

Up: $\forall x' \in I'$, $\psi(x'^{\uparrow}) \subseteq \psi(x')^{\uparrow}$.

This property can be rephrased as:

$$\forall x' \in I',\ \forall x \in I\ \big[\psi(x') \leq x \Rightarrow \exists y' \in I',\ \big(x' \leq y' \text{ and } x = \psi(y')\big)\big]$$

For the lack of a reference we provide here a brief overview of this duality, which can be seen as a restriction of the classical Stone's duality between distributive lattices and their prime filter spectrum. The point is that condition Up characterizes among increasing maps (which are exactly the spectral maps, since we restrict to finite spectral spaces) those which come from an $\mathcal{L}_{\mathrm{HA}^*}$-morphism.

- Given a map $\psi : I' \to I$ as above we consider $[\psi] : \mathcal{L}^{\downarrow}(I) \to \mathcal{L}^{\downarrow}(I')$ defined by $[\psi](A) = \psi^{-1}(A)$. Note that $\psi^{-1}(A)$ is indeed a decreasing subset of I' because A itself is decreasing and ψ is an increasing map.

- Conversely, given an $\mathcal{L}_{\mathrm{HA}^*}$-morphism $\varphi : L \to L'$ of finite co-Heyting algebras, we consider $[\varphi] : \mathcal{I}^{\vee}(L') \to \mathcal{I}^{\vee}(L)$ defined by $[\varphi](x') = \min(\varphi^{-1}(x'^{\uparrow}))$. Note that $x'^{\uparrow}$ is a prime filter of L' hence $\varphi^{-1}(x'^{\uparrow})$ is a prime filter of L, which ensures that its generator $\min(\varphi^{-1}(x'^{\uparrow}))$ is indeed a join irreducible element of L.

Fact 2.5 *Let $\psi : I' \to I$ be an increasing map between finite posets satisfying condition Up. Then $[\psi] : \mathcal{L}^{\downarrow}(I) \to \mathcal{L}^{\downarrow}(I')$ is an $\mathcal{L}_{\mathrm{HA}^*}$-morphism. Moreover ψ is surjective if and only if $[\psi]$ is injective.*

Fact 2.6 *Conversely, let $\varphi : L \to L'$ an $\mathcal{L}_{\mathrm{HA}^*}$-morphism between finite co-Heyting algebras. Then $[\varphi] : \mathcal{I}^{\vee}(L') \to \mathcal{I}^{\vee}(L)$ is an increasing map which satisfies condition Up. Moreover φ is injective if and only if $[\varphi]$ is surjective.*

The proofs are good exercises that we leave to the reader. On the other hand we provide a self-contained proof of the next proposition which summarises the only parts of this duality that we will use[5]. Since it provides a flexible tool to build an extension of a finite co-Heyting algebra with prescribed conditions, it will play a central role in our constructions.

Proposition 2.7 *Let L be a finite co-Heyting algebra and $\mathcal{I}$ an ordered set. Assume that there is a surjective increasing map π from $\mathcal{I}$ onto $\mathcal{I}^{\vee}(L)$ which satisfies condition Up. Then there exists an $\mathcal{L}_{\mathrm{HA}^*}$-embedding φ of L into $\mathcal{L}^{\downarrow}(\mathcal{I})$ such that*[6] *$\pi(\varphi(a)) = a^{\downarrow} \cap \mathcal{I}^{\vee}(L)$ for every $a \in L$.*

Proof: Let $L^{\downarrow}$ denote $\mathcal{L}^{\downarrow}(\mathcal{I}^{\vee}(L))$ and $L'^{\downarrow}$ denote $\mathcal{L}^{\downarrow}(\mathcal{I})$. The map $\varphi^{\downarrow} : L^{\downarrow} \to L'^{\downarrow}$ defined by $\varphi^{\downarrow}(A) = \pi^{-1}(A)$ is clearly a morphism of bounded lattices ($\varphi^{\downarrow}(A) \in L'^{\downarrow}$ for every $A \in L^{\downarrow}$ because π is increasing) which is injective because π is surjective. Clearly $\pi(\varphi^{\downarrow}(A)) = A$ for every $A \in L^{\downarrow}$ so it remains to check that $\varphi^{\downarrow}$ is an $\mathcal{L}_{\mathrm{HA}^*}$-embedding of $L^{\downarrow}$ into $L'^{\downarrow}$ in order to conclude that $\varphi : a \mapsto \varphi^{\downarrow}(a^{\downarrow} \cap \mathcal{I}^{\vee}(L))$ is the required $\mathcal{L}_{\mathrm{HA}^*}$-embedding of L into $\mathcal{L}^{\downarrow}(\mathcal{I})$.

Let $A, B \in L^{\downarrow}$ and choose any $x' \in \varphi^{\downarrow}(A - B) = \pi^{-1}\big((A \setminus B){\downarrow}\big)$. Then $\pi(x') \le x$ for some $x \in A \setminus B$. Condition Up gives $y \in \mathcal{I}$ such that $\pi(y) = x$ and $x' \le y$ so $x' \in \pi^{-1}(A \setminus B){\downarrow}$.

$$\pi^{-1}(A \setminus B){\downarrow} = \big(\pi^{-1}(A) \setminus \pi^{-1}(B)\big){\downarrow} = \big(\varphi^{\downarrow}(A) \setminus \varphi^{\downarrow}(B)\big){\downarrow} = \varphi^{\downarrow}(A) - \varphi^{\downarrow}(B)$$

We conclude that $\varphi^{\downarrow}(A - B) \subseteq \varphi^{\downarrow}(A) - \varphi^{\downarrow}(B)$. The reverse inclusion is immediate since $\varphi^{\downarrow}(A) \subseteq \varphi^{\downarrow}\big((A - B) \cup B)\big) = \varphi^{\downarrow}(A - B) \cup \varphi^{\downarrow}(B)$ implies that $\varphi^{\downarrow}(A) - \varphi^{\downarrow}(B) \subseteq \varphi^{\downarrow}(A - B)$. So $\varphi^{\downarrow}(A - B) = \varphi^{\downarrow}(A) - \varphi^{\downarrow}(B)$. $\dashv$

3 Minimal finite extensions

This section is devoted to the study of minimal finite proper extensions of a finite co-Heyting algebra L_0. We are going to show (see remark 3.6) that they are in one-to-one correspondence with **signatures** in L_0, that is triples (g, H, r) such that g is a join irreducible element of L_0, $H = \{h_1, h_2\}$ is a set of one or two elements of L_0 and:

- either $r = 1$ and $h_1 = h_2 < g$;
- or $r = 2$ and $h_1 \vee h_2$ is the unique predecessor of g (both possibilities, $h_1 = h_2$ and $h_1 \ne h_2$ may occur in this case, see example 3.2).

Let L be an $\mathcal{L}_{\mathrm{HA}^*}$-extension of L_0 and $x \in L$. We introduce the following notation.

- For every $a \in L_0$, $a^- = \bigvee\!\!\!\bigvee\{b \in L_0 : b < a\}$.
- $L_0\langle x\rangle$ denotes the $\mathcal{L}_{\mathrm{HA}^*}$-substructure of L generated by $L_0 \cup \{x\}$.
- $g(x, L_0) = \bigwedge\!\!\!\bigwedge\{a \in L_0 : x \le a\}$.

[5]We will actually use facts 2.5 and 2.6 in the proof of the implication (3)⇒(1) of corollary 3.4. But from this corollary we will only use the reverse implication (2)⇒(3), which does not require these facts.

[6]Note that the compositum $\pi \circ \varphi$ is not defined. In this proposition $\varphi(a)$ is a decreasing subset of $\mathcal{I}$ and $\pi(\varphi(a)) = \{\pi(\xi) : \xi \in \varphi(a)\}$.

Clearly $a \in \mathcal{I}^\vee(L_0)$ if and only if a^- is the unique predecessor of a in L_0 (otherwise $a^- = a$). We say that a tuple (x_1, x_2) of elements of L is **primitive over** L_0 if they are both[7] not in L_0 and there exists $g \in \mathcal{I}^\vee(L_0)$ such that:

P1: $g^- \wedge x_1$ and $g^- \wedge x_2$ belong to L_0.

P2: One of the following holds:

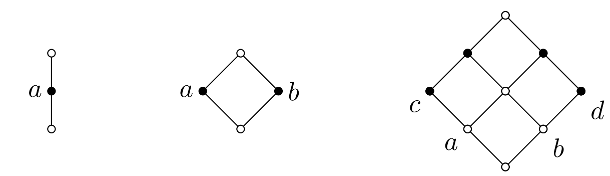
1. $x_1 = x_2$ and $g^- \wedge x_1 \ll x_1 \ll g$.
2. $x_1 \neq x_2$ and $x_1 \wedge x_2 \in L_0$, $g - x_1 = x_2$, $g - x_2 = x_1$.

Remark 3.1 If (x_1, x_2) is a primitive tuple over L_0, and $g \in \mathcal{I}^\vee(L_0)$ satisfies P1 and P2, then $g = g(x_1, L_0) = g(x_2, L_0)$. Indeed, $g(x_i, L_0) \leq g$ because $x_i \leq g$. On the other hand $g(x_i, L_0) \not< g$ since otherwise $g(x_i, L_0) \leq g^-$ hence *a fortiori* $x_i \leq g^-$ and finally $x_i = x_i \wedge g^- \in L_0$, a contradiction.

If an extension L of L_0 is generated over L_0 by a primitive tuple (x_1, x_2) we call it a **primitive extension**. By the above remark, $(g, \{g^- \wedge x_1, g^- \wedge x_2\}, \mathrm{Card}\{x_1, x_2\})$ is then a signature in L_0 which is determined by (x_1, x_2) (actually by any of x_1, x_2). We call it the **signature of the tuple** (x_1, x_2) (or simply of x_1).

Example 3.2 The following primitive extensions and their elements $a, b, c \dots$ are shown in figure 2.

- $\mathbf{L}_2 \subset \mathbf{L}_3$: (a, a) is primitive over $\mathbf{L}_2$ with signature $(\mathbf{1}, \{\mathbf{0}\}, 1)$.
- $\mathbf{L}_2 \subset \mathbf{L}_2 \times \mathbf{L}_2$: (a, b) is a primitive over $\mathbf{L}_2$, with signature $(\mathbf{1}, \{\mathbf{0}\}, 2)$.
- $\mathbf{L}_5^* \subset \mathbf{L}_9$: (c, d) is primitive over $\mathbf{L}_5^*$ with signature $(\mathbf{1}, \{a, b\}, 2)$.



$\mathbf{L}_2 \subset \mathbf{L}_3$ $\mathbf{L}_2 \subset \mathbf{L}_2 \times \mathbf{L}_2$ $\mathbf{L}_5^* \subset \mathbf{L}_9$

Figure 2: Three examples of primitive extensions $L_0 \subset L$.
Inside L, the white points represent L_0.

Dually, the next figure shows the projections of $\mathcal{I}^\vee(L)$ onto $\mathcal{I}^\vee(L_0)$ corresponding to each of the three embeddings $L_0 \subset L$ in figure 2.

Theorem 3.3 *Let L_0 be a finite co-Heyting algebra, L an extension generated over L_0 by a primitive tuple (x_1, x_2), and let $g = g(x_1, L_0)$.*

Then L is exactly the upper semi-lattice generated over L_0 by x_1, x_2. It is a finite co-Heyting algebra and one of the following holds:

[7] The other conditions imply that x_1 and x_2 do not belong to L_0, provided they are both non zero.

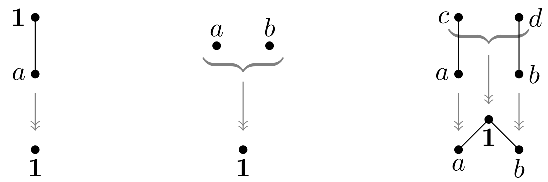


Figure 3: Dual projections.

1. $x_1 = x_2$ *and* $\mathcal{I}^\vee(L) = \mathcal{I}^\vee(L_0) \cup \{x_1\}$.

2. $x_1 \neq x_2$ *and* $\mathcal{I}^\vee(L) = (\mathcal{I}^\vee(L_0) \setminus \{g\}) \cup \{x_1, x_2\}$.

Proof: Let L_1 be the upper semi-lattice generated over L_0 by x_1, x_2. In order to prove that $L_1 = L$ it is sufficient to show that L_1 is an $\mathcal{L}_{\mathrm{HA}^*}$-substructure of L.

We first check that L_1 is a sublattice of L. Any two elements in L_1 can be written $a \vee y$ and $a' \vee y'$ with a, a' in L_0 and y, y' in $\{\mathbf{0}, x_1, x_2\}$, so

$$(a \vee y) \wedge (a' \vee y') = (a \wedge a') \vee (a \wedge y') \vee (a' \wedge y) \vee (y \wedge y').$$

It suffices to show that each of the four elements joined in the right-hand side belong to L_1. That $a \wedge a' \in L_0$ is clear. If $a \geq y'$ then obviously $a \wedge y' = y' \in L_1$. Otherwise, $a \not\geq y'$ implies that $y' = x_i$ for some $i \in \{1, 2\}$ and that $a \not\geq g$, hence $a \wedge g = a \wedge g^-$. Because $g^- \wedge x_i \in L_0$ by definition of a primitive tuple, we have

$$a \wedge x_i = a \wedge (g \wedge x_i) = (a \wedge g^-) \wedge x_i = a \wedge (g^- \wedge x_i) \in L_0.$$

This proves that $a \wedge y' \in L_1$, and by symmetry $a' \wedge y \in L_1$. Finally, if $y \leq y'$ or $y' \leq y$ then obviously $y \wedge y' \in L_1$. Otherwise we have $\{y, y'\} = \{x_1, x_2\}$ and $x_1 \neq x_2$, hence $x_1 \wedge x_2 \in L_0$ by definition of a primitive tuple hence $y \wedge y' \in L_1$ in this case too.

So L is a sublattice of L. We turn now to the $-$ operation.

$$\begin{aligned} (a \vee y) - (a' \vee y') &= [a - (a' \vee y')] \vee [y - (a' \vee y')] \\ &= [(a - a') - y'] \vee [(y - y') - a'] \end{aligned}$$

So it suffices to prove that $(a - a') - y'$ and $(y - y') - a'$ belong to L_1. For the first one, we have $a - a' \in L_0$ hence it suffices to check that $b - x_1 \in L_1$ for every $b \in L_0$ (that $b - x_2 \in L_1$ will follow by symmetry). For the second one, note that $y - y'$ is either $\mathbf{0}, x_1$ or x_2. This is obvious if $y = y'$ or $y = \mathbf{0}$ or $y' = \mathbf{0}$, and otherwise we have $\{y, y'\} = \{x_1, x_2\}$ and $x_1 \neq x_2$. But then, by definition of a primitive tuple

$$x_1 - x_2 = (x_1 \vee x_2) - x_2 = g - x_2 = x_1$$

and symmetrically $x_2 - x_1 = x_2$. Thus, in order to prove that $(y - y') - a' \in L_1$ it suffices to check that $x_1 - b \in L_1$ for every $b \in L_0$ (that $x_2 - b \in L_1$ will follow by symmetry).

Let b be any element of L_0. We first check that $b - x_1 \in L_1$.

$$\begin{aligned} b - x_1 &= [(b \wedge g) \vee (b - g)] - x_1 \\ &= [(b \wedge g) - x_1] \vee [(b - g) - x_1] \in L_1 \end{aligned}$$

If $g \leq b$ then $(b \wedge g) - x_1 = g - x_1$ is either g or x_2. If $g \not\leq b$ then $b \wedge g \leq g^-$ hence $b \wedge g \wedge x_1$ belongs to L_0 hence so does $(b \wedge g) - x_1 = (b \wedge g) - (b \wedge g \wedge x_1)$. So in any case $(b \wedge g) - x_1$ belongs to L_1. On the other hand $(b - g) - x_1 = b - g$ belongs to L_0, so finally $b - x_1 \in L_1$.

Now we check that $x_1 - b \in L_1$. This is clear if $x_1 \leq b$. If $x_1 \not\leq b$ then $g \not\leq b$ hence $x_1 \wedge b \leq g^-$. It follows that:

$$x_1 \wedge b \leq g^- \wedge x_1 \ll x_1$$

Indeed if $x_1 = x_2$ then $g^- \wedge x_1 \ll x_1$ by assumption, and if $x_1 \neq x_2$ then $g^- \wedge x_1 < g$ and $g^- \wedge x_1 \in L_0$ hence $g^- \wedge x_1 \ll g$ because g is join irreducible. So $x_1 - b = x_1 - (x_1 \wedge b) = x_1$ belongs to L_1.

This proves that L_1 is an $\mathcal{L}_{\mathrm{HA}^*}$-substructure of L. Since L_1 contains L_0 and $\{x_1, x_2\}$ it follows that $L_1 = L$.

We turn now to the description of $\mathcal{I}^\vee(L)$. Since L_0 is finite and L is generated by $L_0 \cup \{x_1, x_2\}$ as an upper semi-lattice, it follows immediately that L is finite and:

$$\mathcal{I}^\vee(L) \subseteq \mathcal{I}^\vee(L_0) \cup \{x_1, x_2\} \tag{1}$$

If $x_1 \neq x_2$ then of course $g = x_1 \vee x_2 \notin \mathcal{I}^\vee(L)$. Conversely if $x_1 = x_2$ then $x_1 \ll g$ by definition of a primitive tuple, so $g - x_1 = g$. Then for any $a \in L_0$ we have $g - (x_1 \cup a) = g - a$ is equal to $\mathbf{0}$ or g by join irreducibility of g in L_0, which proves that $g \in \mathcal{I}^\vee(L)$. So:

$$g \in \mathcal{I}^\vee(L) \iff x_1 = x_2 \tag{2}$$

Of course we cannot have $\mathcal{I}^\vee(L) = \mathcal{I}^\vee(L_0)$ since L is a proper extension of L_0 generated by $\mathcal{I}^\vee(L)$. So by (1) and (2) it only remains to check that

$$\mathcal{I}^\vee(L_0) \setminus \{g\} \subseteq \mathcal{I}^\vee(L) \tag{3}$$

Assume that $\mathcal{I}^\vee(L_0) \not\subseteq \mathcal{I}^\vee(L)$. Let $b \in \mathcal{I}^\vee(L_0) \setminus \mathcal{I}^\vee(L)$ and let $y_1, \ldots, y_r$ ($r \geq 2$) be its $\vee$-irreducible components in L. By (1), each y_i either belongs to L_0 or to $\{x_1, x_2\}$, and at least one of them does not belong to L_0. We may assume without loss of generality that $y_1 = x_1$. Then $x_1 \leq b$ hence $g \leq b$. If $g < b$ then $g \ll b$ since $b \in \mathcal{I}^\vee(L_0)$ so $b - g = b$, but then we have a contradiction:

$$y_1 \leq b - g \leq b - x_1 = \bigvee_{i=2}^{r} y_i$$

This proves that either $\mathcal{I}^\vee(L_0) \subseteq \mathcal{I}^\vee(L)$ or $\mathcal{I}^\vee(L_0) \setminus \mathcal{I}^\vee(L) = \{g\}$, hence (3) holds true in any case. ⊣

Corollary 3.4 *Let L be a finite extension of a co-Heyting algebra L_0. The following assertions are equivalent.*

1. *L is a minimal proper extension of L_0.*

2. *L is a primitive extension of* L_0.

3. $\mathrm{Card}(\mathcal{I}^\vee(L)) = \mathrm{Card}(\mathcal{I}^\vee(L_0)) + 1$.

As a consequence every finite extension $L_0 \subset L$ *is the union of a tower of primitive extensions* $L_0 \subset L_1 \subset \cdots \subset L_n = L$ *with* $n = \mathrm{Card}(\mathcal{I}^\vee(L)) - \mathrm{Card}(\mathcal{I}^\vee(L_0))$.

Proof: (2)⇒(3): This follows directly from theorem 3.3.

(3)⇒(1): Let L_1 be a proper extension of L_0 contained in L. We have to prove that $L = L_1$. The inclusion maps $\varphi_0 : L_0 \to L_1$ and $\varphi_1 : L_1 \to L$ induce surjective increasing maps $[\varphi_1] : \mathcal{I}^\vee(L) \to \mathcal{I}^\vee(L_1)$ and $[\varphi_0] : \mathcal{I}^\vee(L) \to \mathcal{I}^\vee(L_0)$. This implies that $\mathrm{Card}(\mathcal{I}^\vee(L)) \geq \mathrm{Card}(\mathcal{I}^\vee(L_1)) \geq \mathrm{Card}(\mathcal{I}^\vee(L_0))$. The second inequality is strict, otherwise $[\varphi_0]$ is a bijection hence so is φ_0, contrary to our assumption that L_1 is a proper extension of L_0. But (2) then implies that

$$\mathrm{Card}(\mathcal{I}^\vee(L)) = \mathrm{Card}(\mathcal{I}^\vee(L_1)) = \mathrm{Card}(\mathcal{I}^\vee(L_0)) + 1$$

Thus $[\varphi_1]$ is a bijection, hence so is φ_1, that is $L = L_1$.

(1)⇒(2): By minimality of L it suffices to prove that L contains a primitive extension of L_0, that is to find in L a primitive tuple (x_1, x_2) over L_0. In order to do so, let us take any element x minimal in $\mathcal{I}^\vee(L) \setminus L_0$. Observe that if y is any element of L strictly smaller than x, then all the $\vee$-irreducible components of y in L actually belong to L_0 (by minimality of x) so $y \in L_0$.

Let $g = g(x, L_0)$. For every $a \in L_0$, if $a < g$ then $x \not\leq a$ hence $a \wedge x < x$, so $a \wedge x \in L_0$. It follows that $g^- \wedge x \in L_0$. In particular $g^- \neq g$ hence $g \in \mathcal{I}^\vee(L_0)$.

Moreover $g^- \wedge x < x$ since $x \notin L_0$, hence $g^- \wedge x \ll x$ because x is join irreducible in L. So in the case when $x \ll g$ we have proved that (x, x) is primitive over L_0.

On the other hand, when $x \not\ll g$ then $g - (g - x) = x$, indeed:

$$g - (g - x) = \big(x \vee (g - x)\big) - (g - x) = x - (g - x)$$

The last term is either $\mathbf{0}$ or x due to the join irreducibility of x. But it cannot be $\mathbf{0}$ since $x \leq g - x$ would imply that $g = g - x$ hence $x \ll g$, a contradiction.

Note that $x \not\leq g - x$ implies also that $x \wedge (g - x) < x$ hence $x \wedge (g - x) \in L_0$. So when $x \not\ll g$ we have proved that $(x, g - x)$ is primitive over L_0. ⊣

Corollary 3.5 *Let* L_1, L_2 *be two finite co-Heyting algebras both generated over a common subalgebra by a primitive tuple. If these tuples have the same signature in* L_0 *then they are isomorphic over* L_0 *(there exists an* $\mathcal{L}_{\mathrm{HA}^*}$*-isomorphism from* L_1 *to* L_2 *which fixes* L_0 *pointwise).*

Proof: Assume that L_i is generated over L_0 by a primitive tuple $(x_{i,1}, x_{i,2})$ for $i = 1, 2$ having the same signature (g, H, r) in L_0. By definition of H, changing if necessary the numbering of the $x_{2,j}$'s we can assume that for $i = 1, 2$:

$$g^- \wedge x_{1,j} = g^- \wedge x_{2,j} \tag{4}$$

By definition of r, $x_{1,1} \neq x_{1,2}$ if and only if $x_{2,1} \neq x_{2,2}$. By definition of g and by theorem 3.3 there exists a (unique) bijection σ from $\mathcal{I}^\vee(L_1)$ to $\mathcal{I}^\vee(L_2)$ which fixes $\mathcal{I}^\vee(L_0)$ pointwise and maps each $x_{1,j}$ to $x_{2,j}$. Condition (4) ensures that σ preserves the order. This determines an $\mathcal{L}_{\mathrm{HA}^*}$-isomorphism $[\sigma] : \mathcal{L}^\downarrow(\mathcal{I}^\vee(L_2)) \to$

$\mathcal{L}^{\downarrow}(\mathcal{I}^{\vee}(L_1))$ which fixes $\mathcal{L}^{\downarrow}(\mathcal{I}^{\vee}(L_0))$ pointwise. Recall that for every co-Heyting algebra L we let $\iota_L : L \to \mathcal{L}^{\downarrow}(\mathcal{I}^{\vee}(L))$ denote the canonical isomorphism defined by $\iota(a) = a^{\downarrow} \cap \mathcal{I}^{\vee}(L)$. We finally get that $\iota_{L_1}^{-1} \circ [\sigma] \circ \iota_{L_2}$ is an isomorphism from L_2 to L_1 over L_0. $\dashv$

Remark 3.6 Corollary 3.4 shows that every minimal finite proper extension of L_0 is generated by a primitive tuple (x_1, x_2), which is unique (up to permutation) by theorem 3.3. This tuple has a signature, which is then entirely determined by the embedding of L_0 into L_1. So we will call it the **signature of** L_1 **in** L_0. Of course every other extension of L_0 isomorphic to L_1 over L_0 will have the same signature in L_0. Conversely, corollary 3.5 shows that this signature determines L_1 up to isomorphism over L_0. Finally, it will be shown in the next section 4 that every signature in L_0 is the signature of a primitive extension of L_0 (see remark 4.5). Altogether this proves that the minimal finite proper extensions of L_0 up to isomorphism over L_0 are in one-to-one correspondence with the signatures in L_0.

4 Density and splitting in $\mathcal{V}_1$

For the variety $\mathcal{V}_1$ of all co-Heyting algebras we introduce the following axioms D1 and S1.

[**Density D1**] For every a, c such that $c \ll a \neq \mathbf{0}$ there exists a non zero element b such that:

$$c \ll b \ll a$$

[**Splitting S1**] For every a, b_1, b_2 such that $b_1 \vee b_2 \ll a \neq \mathbf{0}$ there exists non zero elements a_1 and a_2 such that:

$$\begin{aligned} a - a_2 &= a_1 \geq b_1 \\ a - a_1 &= a_2 \geq b_2 \\ a_1 \wedge a_2 &= b_1 \wedge b_2 \end{aligned}$$

Note that $a = a_1 \vee a_2$, so the second axioms allows to split a in two pieces a_1, a_2 along $b_1 \wedge b_2$ (so the name of "splitting").

Lemma 4.1 *Let a, c be two elements of a finite co-Heyting algebra L. If $c \ll a \neq \mathbf{0}$ then there exists a finite co-Heyting algebra L' containing L and a non zero element b in L' such that:*

$$c \ll b \ll a$$

Proof: Let $a_1, \dots, a_r$ be the join irreducible components of a. The idea of the proof is to add a new $\vee$-irreducible element α_i immediately below each a_i. Let $\mathcal{I}$ be the set $\mathcal{I}^{\vee}(L)$ augmented by r new elements $\alpha_1, \dots, \alpha_r$. Extend the order of $\mathcal{I}^{\vee}(L)$ to $\mathcal{I}$ as follows. The α_i's are two by two incomparable, and for every $x \in \mathcal{I}^{\vee}(L)$ and every $i \leq r$:

$$x < \alpha_i \quad \Leftrightarrow \quad x < a_i$$

$$\alpha_i < x \quad \Leftrightarrow \quad a_i \leq x$$

For every $\xi \in \mathcal{I}$ let:

$$\pi(\xi) = \begin{cases} x & \text{if } \xi = x \text{ for some } x \in L \\ a_i & \text{if } \xi = \alpha_i \text{ for some } i \leq r \end{cases}$$

This is an increasing projection of $\mathcal{I}$ onto $\mathcal{I}^\vee(L)$. For every $\zeta \in \mathcal{I}$ and every $x \in \mathcal{I}^\vee(L)$ such that $\pi(\zeta) \leq x$ there exists $\xi \in \mathcal{I}$ such that $\pi(\xi) = x$ and $\zeta \leq \xi$: simply take $\xi = x$. Thus proposition 2.7 gives an $\mathcal{L}_{\mathrm{HA}^*}$-embedding φ of L into $\mathcal{L}^\downarrow(\mathcal{I})$.

Each join irreducible element x of L smaller than c is strictly smaller than some join irreducible component a_i of a because $c \ll a$. By construction $x < \alpha_i < a_i$ in $\mathcal{I}$ hence $\varphi(x) < \alpha_i^\downarrow < \varphi(a_i)$. These three elements of L' are join irreducible hence $\varphi(x) \ll \alpha_i^\downarrow \ll \varphi(a_i)$. It follows that:

$$\varphi(c) = \bigvee\{\varphi(x) : x \in \mathcal{I}^\vee(L),\ x \leq c\} \ll \bigvee_{1\leq i\leq r} \alpha_i^\downarrow \ll \bigvee_{1\leq i\leq r} \varphi(a_i) = \varphi(a)$$

So we can take $L' = \mathcal{L}^\downarrow(\mathcal{I})$ and $b = \bigvee_{1\leq i\leq r} \alpha_i^\downarrow$. $\dashv$

Lemma 4.2 *Let a, b_1, b_2 be elements of a finite co-Heyting algebra L. If $b_1 \vee b_2 \ll a \neq \mathbf{0}$ then there exists a finite co-Heyting algebra L' containing L and non zero elements a_1, a_2 in L' such that:*

$$\begin{aligned} a - a_2 = a_1 \geq b_1 \\ a - a_1 = a_2 \geq b_2 \\ a_1 \wedge a_2 = b_1 \wedge b_2 \end{aligned}$$

The idea of the proof uses geometric intuition. Imagine that there exists an $\mathcal{L}_{\mathrm{HA}^*}$-embedding φ of L into the co-Heyting algebra $L(X)$ of all semi-algebraic closed subsets of some real semi-algebraic set X. It can be proved actually that such an embedding exists, and that moreover we can reduce to the case when $\varphi(a)$ is equidimensional (that is its local dimension is the same at every point). So $A = \varphi(a)$, $B_1 = \varphi(b_1)$ and $B_2 = \varphi(b_2)$ are closed semi-algebraic subsets of X. Let $X_1 = X \setminus (B_2 \setminus B_1)$ and $X_2 = X \setminus (B_1 \setminus B_2)$. Glue two copies X_1', X_2' of X_1 and X_2 along $B_1 \cap B_2$. The result X' of this glueing is a real semi-algebraic set which projects onto X in an obvious way. Figure 4 shows this construction when $A = X$.

This defines an embedding $L(X) \hookrightarrow L(X')$ which maps any semi-algebraic subset Y of X closed in X to the preimage Y' of Y via this projection. Then A' is the union of a copy of $A_1 = A \cap X_1$ and $A_2 = A \cap X_2$ glued along $B_1 \cap B_2$. These copies A_1', A_2' of A_1 and A_2 are non empty semi-algebraic subsets of X', closed in X', containing B_1' and B_2' respectively, such that:

$$A_1' \cap A_2' = B_1' \cap B_2'$$

The additional property that A_1' (resp. A_2') is the topological closure in X' of $A' \setminus A_2'$ (resp. $A' \setminus A_1'$) then follows from the assumption that $B_1 \cup B_2 \ll A$ and the fact that we reduced to the case when A_1' and A_2' are equidimensional.

Proof: The above geometric construction *is* a proof, provided an appropriate dictionary between real semi-algebraic sets and elements of co-Heyting algebras is given. However it would be longer to set explicitly this dictionary than to

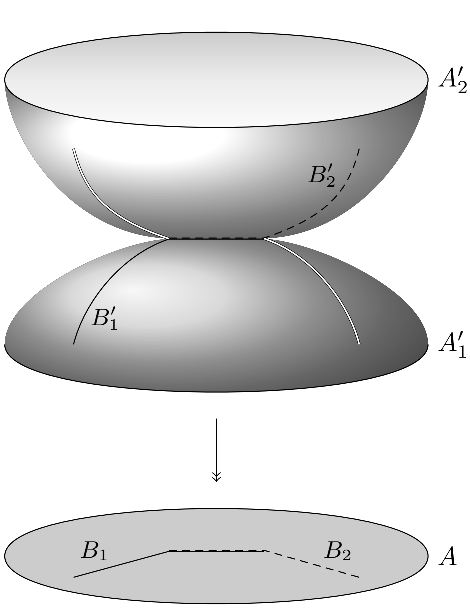


Figure 4: An example of glueing when $A = X$.
The white curves represent cuts in A'_1 and A'_2.

hide the geometric intuition in a shorter combinatorial proof. This is what we do now.

For each $x \in \mathcal{I}^\vee(L)$ such that $x \not\leq b_2$ (resp $x \not\leq b_1$) let $\xi_{x,1}$ (resp. $\xi_{x,2}$) be a new symbol. For each $x \in \mathcal{I}^\vee(L)$ such that $x \leq b_1 \wedge b_2$ let $\xi_{x,0}$ be a new symbol. Let $\mathcal{I}$ be the set of all these symbols and define an order on $\mathcal{I}$ as follows:

$$\xi_{y,j} \leq \xi_{x,i} \quad \Leftrightarrow \quad y \leq x \text{ and } \{i,j\} \neq \{1,2\}$$

The map $\pi : \xi_{x,i} \mapsto x$ defines an increasing projection of $\mathcal{I}$ onto $\mathcal{I}^\vee(L)$. For every $\zeta \in \mathcal{I}$ and every $x \in \mathcal{I}^\vee(L)$ such that $\pi(\zeta) \leq x$ there exists ξ such that $\pi(\xi) = x$ and $\zeta \leq \xi$. Indeed if $\zeta = \xi_{y,1}$ then $\pi(\zeta) = y \not\leq b_2$ so $x \not\leq b_2$, hence $\xi_{x,1}$ exists and does the job. A symmetric argument applies if $\zeta = \xi_{y,2}$. On the other hand if $\zeta = \xi_{y,0}$ then the existence of $\xi_{x,0}$ is not guaranteed, but the existence of at least one among $\xi_{x,0}, \xi_{x,1}, \xi_{x,2}$ is. Just take one of them. Thus proposition 2.7 gives an $\mathcal{L}_{\mathrm{HA}^*}$-embedding φ of L into $\mathcal{L}^\downarrow(\mathcal{I})$. For any $x \in \mathcal{I}^\vee(L)$ we have:

$$\varphi(x) = \begin{cases} \xi_{x,0}^\downarrow & \text{if } x \leq b_1 \wedge b_2 \\ \xi_{x,1}^\downarrow & \text{if } x \leq b_1 \text{ and } x \not\leq b_2 \\ \xi_{x,2}^\downarrow & \text{if } x \leq b_2 \text{ and } x \not\leq b_1 \\ \xi_{x,1}^\downarrow \cup \xi_{x,2}^\downarrow & \text{otherwise.} \end{cases}$$

Let $a_1, \dots, a_r$ be the join irreducible components of a. None of the a_i's is smaller than b_1 or b_2 because by assumption $b_1 \vee b_2 \ll a$, so each $\varphi(a_i) = \xi_{a_i,1}^\downarrow \cup \xi_{a_i,2}^\downarrow$. Define:

$$\alpha_1 = \bigcup_{1 \leq i \leq r} \xi_{a_i,1}^\downarrow \quad \text{and} \quad \alpha_2 = \bigcup_{1 \leq i \leq r} \xi_{a_i,2}^\downarrow$$

By construction $\varphi(a) - \alpha_1 = \alpha_2$ and $\varphi(a) - \alpha_2 = \alpha_1$ and both are non empty since $r \geq 1$ (here we use that $a \neq \mathbf{0}$). Moreover, for any join irreducible element x of L such that $x \leq b_1$, we have $x \leq a_j$ for some $j \leq r$. If $x \not\leq b_2$, by definition of the order on $\mathcal{I}$ it follows that $\xi_{a_j,1}$ exists and $\xi_{x,1} \leq \xi_{a_j,1}$ hence:

$$\varphi(x) = \xi_{x,1}^{\downarrow} \subseteq \xi_{a_j,1}^{\downarrow} \subseteq \alpha_1$$

On the other hand if $x \leq b_2$ then $x \leq b_1 \wedge b_2$ so $\xi_{x,0}$ exists. Since $\xi_{a_j,k}$ exists for some $k \in \{0,1,2\}$ and $\xi_{x,0} \leq \xi_{a_j,k}$ we get:

$$\varphi(x) = \xi_{x,0}^{\downarrow} \subseteq \xi_{a_j,k}^{\downarrow} \subseteq \alpha_1$$

Thus in any case $\varphi(x) \subseteq \alpha_1$. It follows that $\varphi(b_1) \subseteq \alpha_1$, and symmetrically $\varphi(b_2) \subseteq \alpha_2$.

It remains to check that $\alpha_1 \cap \alpha_2 = \varphi(b_1) \cap \varphi(b_2)$. In order to do this, let ξ be any element of $\mathcal{I}$ and $x = \pi(\xi)$. It is sufficient to prove that $\xi^{\downarrow} \subseteq \alpha_1 \cap \alpha_2$ if and only if $\xi^{\downarrow} \subseteq \varphi(b_1) \cap \varphi(b_2)$

If $\xi^{\downarrow} \subseteq \varphi(b_1) \cap \varphi(b_2)$ then $x \leq b_1 \wedge b_2$ hence $\xi = \xi_{x,0}$ and $x \leq a_i$ for some $i \leq r$. It follows that $\xi_{x,0} \leq \xi_{a_i,1}$ so $\xi^{\downarrow} \subseteq \alpha_1$, and $\xi_{x,0} \leq \xi_{a_i,2}$ so $\xi^{\downarrow} \subseteq \alpha_1$. With other words $\xi^{\downarrow} \subseteq \alpha_1 \cap \alpha_2$.

Conversely if $\xi^{\downarrow} \subseteq \alpha_1 \cap \alpha_2$ then there exists $i, j \leq r$ such that $\xi^{\downarrow} \subseteq \xi_{a_i,1}$ and $\xi^{\downarrow} \subseteq \xi_{a_j,2}^{\downarrow}$. Thanks to the definition of the ordering on $\mathcal{I}$ this implies that $\xi = \xi_{x,0}$ hence $x \leq b_1 \wedge b_2$ and so $\xi^{\downarrow} \subseteq \varphi(b_1) \cap \varphi(b_2)$. ⊣

Theorem 4.3 *Every co-Heyting algebra existentially closed in $\mathcal{V}_1$ satisfies the density axiom D1 and the splitting axiom S1.*

Proof: These two axioms can be written under the following form:

$$\forall x \; (\theta(x) \longrightarrow \exists y \; \phi(x, y))$$

where $\theta(x)$ and $\phi(x, y)$ are quantifier-free $\mathcal{L}_{\mathrm{HA}^*}$-formulas. In both cases we have shown in lemmas 4.1 and 4.2 that for every finite co-Heyting algebra L and every tuple a of elements of L such that $L \models \theta(a)$, there exists an extension L' of L which satisfies $\exists y \; \phi(a, y)$. The result follows, by lemma 2.3. ⊣

Here is a partial converse of theorem 4.3.

Theorem 4.4 *Let L be a co-Heyting algebra satisfying the density axiom D1 and the splitting axiom S1. Let L_0 be a finite subalgebra of L. Let L_1 be a finite co-Heyting algebra containing L_0. Then there exists an embedding of L_1 into L which fixes every point of L_0.*

Proof: By an immediate induction based on corollary 3.4, we reduce to the case when L_1 is generated over L_0 by a primitive tuple. Let $\sigma = (g, \{h_1, h_2\}, r)$ be the signature of L_1 in L_0. By corollary 3.5 it is sufficient to prove that σ is the signature of a primitive tuple of elements $x_1, x_2 \in L$.

Case 1: $r = 1$ so $h_1 = h_2$. Since $h_1 \leq g^- \ll g$, the splitting property $S1$ applied to g, g^-, h_1 gives non zero elements y_1, y_2 in L such that:

$$g - y_1 = y_2 \geq g^-$$
$$g - y_2 = y_1 \geq h_1$$
$$y_1 \wedge y_2 = h_1$$

We have $y_1 - h_1 = (g - y_2) - h_1 = (g - h_1) - y_2 = g - y_2 = y_1$ hence $h_1 \ll y_1$. The density axiom D1 then gives $x \in L \setminus \{\mathbf{0}\}$ such that $h_1 \ll x \ll y_1$. By construction:

$$h_1 \leq g^- \wedge x \leq y_2 \wedge y_1 = h_1$$

So $g^- \wedge x = h_1 \in L_0$ and $g^- \wedge x \ll x \ll g$, from which it follows that (x, x) is a primitive tuple with signature $(g, \{h_1\}, 1) = \sigma$ in L_0.

Case 2: $r = 2$ so $h_1 \vee h_2 = g^-$. Since $g^- \ll g$ the splitting property S1 applied to g, h_1, h_2 gives non zero elements y_1, y_2 in L such that:

$$\begin{aligned} g - y_1 = y_2 \geq h_2 \\ g - y_2 = y_1 \geq h_1 \\ y_1 \wedge y_2 = h_1 \wedge h_2 \end{aligned}$$

We have $h_1 \leq g^- \wedge y_1 = (h_1 \wedge y_1) \vee (h_2 \wedge y_1) = h_1 \vee (h_2 \wedge y_1)$. On the other hand $h_2 \wedge y_1 \leq y_2 \wedge y_1 = h_1 \wedge h_2 \leq h_1$. Therefore $g^- \wedge y_1 = h_1$ and symmetrically $g^- \wedge y_2 = h_2$ so both of them belong to L_0. It follows that (y_1, y_2) is a primitive tuple with signature $(g, \{h_1, h_2\}, 2) = \sigma$ in L_0. $\dashv$

Remark 4.5 The above proof shows, incidentally, that any given signature in a finite co-Heyting algebra L_0 is the signature of an extension of L_0 generated by a primitive tuple (inside an existentially closed extension of L_0).

Corollary 4.6 *If L is a non-trivial co-Heyting algebra satisfying the axioms D1 and S1 then any finite non-trivial co-Heyting algebra embeds into L.*

Proof: $\mathbf{L}_2$ is a common subalgebra of L and any co-Heyting algebra L_1. If L_1 is finite, theorem 4.4 applies to $L_0 = \mathbf{L}_2$, L_1 and L. $\dashv$

Corollary 4.7 *If L is a co-Heyting algebra satisfying the axioms D1 and S1, L_0 a finite subalgebra of L, and L' any extension of L_0, then L' embeds over L_0 into an elementary extension of L (or in L itself if L is sufficiently saturated).*

Proof: By standard model-theoretic argument, it suffices to show that any existential formula with parameters in L_0 satisfied in L' is satisfied in L. Let a be the list of all elements of L_0 and $\Delta(a)$ be the conjunction of the quantifier free diagram of L_0, so that a co-Heyting algebra is a model of the formula $\Delta(a)$ if and only if a enumerates a substructure isomorphic to L_0. Let $\exists x\, \theta(x, a)$ be any existential formula with parameters in L_0 satisfied in L' (where x is a tuple of variables). By proposition 2.2 there is a finite co-Heyting algebra L_1 satisfying $\exists x\, \theta(x, a) \wedge \Delta(a)$. Since L_1 models $\Delta(a)$ it contains a subalgebra isomorphic to L_0, which we can then identify to L_0. By corollary 4.6, L_1 embeds into L over L_0 hence L itself models $\exists x\, \theta(x, a)$ and the conclusion follows. $\dashv$

Remark 4.8 If L is a non-trivial co-Heyting algebra satisfying the axioms D1 and S1, then every non-trivial 'co-Heyting algebra L' embeds into an elementary extension of L by corollary 4.7 since $\mathbf{L}_2$ is a finite common subalgebra of L' and L.

5 Density and splitting in $\mathcal{V}_2$

We introduce the following axioms:

[**Density D2**] Same as D1.

[**Splitting S2**] Same as S1 with the additional assumption that $b_1 \wedge b_2 \wedge (\mathbf{1} - (\mathbf{1} - a)) = \mathbf{0}$

Fact 5.1 *Let L_0 be a* finite *co-Heyting algebra. Let $x_1, \cdots, x_r$ be the join irreducible components of $\mathbf{1}$ in L_0 (that is the maximal elements of $\mathcal{I}^\vee(L_0)$). The following conditions are equivalent:*

1. *L_0 belongs to $\mathcal{V}_2$.*
2. *$x_i \wedge x_j = \mathbf{0}$ whenever $i \neq j$.*
3. *L_0 is isomorphic to a product of co-Heyting algebras $L_1, \ldots, L_r$ such $\mathbf{1}_{L_i}$ is join irreducible.*

This is folklore, but let us recall the argument.

Clearly $\mathbf{1}_{L_0} = \mathbf{0}_{L_0}$ if and only if $r = 0$, in which case the whole fact is trivial. So let's assume that $r \geq 1$.

(1)⇒(2)⇐(3) is clear. (1)⇐(2) is an easy computation using that $1 - x$ is the join of all the join-irreducible components of $\mathbf{1}$ which are not in $x^\downarrow$. (2)⇒(3) is true because if we let $y_i = \bigvee_{j \neq i} x_j$ and $L_i = L/y_i^\downarrow$ for every $i \leq r$, then it is an easy exercise to check that each $\mathbf{1}_{L_i}$ is join irreducible and to derive from (2) that the natural map from L to the product $L_1 \times \cdots \times L_r$ is an isomorphism.

Lemma 5.2 *Let L be a finite algebra in $\mathcal{V}_2$ such that $\mathbf{1}$ is join irreducible. Let a, c be any two elements of L such that $c \ll a$. Then there exists an extension L' of L in $\mathcal{V}_2$ and an element b in L' such that:*

$$c \ll b \ll a$$

If moreover $a \neq \mathbf{0}$ then one can require that $b \neq \mathbf{0}$.

Proof: By assumption $\mathbf{1}$ has a unique predecessor x, thus $L_0 = x^\downarrow$ has a natural structure of co-Heyting algebra.

If $a = \mathbf{0}$ one can take $b = \mathbf{0}$.

If $a = \mathbf{1}$ then $c \leq x$. Let L' be the co-Heyting algebra obtained by inserting one new element b between x and $\mathbf{1}$. Then a and b are join irreducible in L' and $c < b < a$ hence we are done.

Otherwise $\mathbf{0} \neq a \leq x$ thus lemma 4.1 gives an $\mathcal{L}_{\mathrm{HA}^*}$-embedding φ of L_0 into a co-Heyting algebra L_1 containing a non zero element b such that $c \ll b \ll a$. Let L' be the co-Heyting algebra obtained by adding to L_1 a new element on the top. The embedding φ extends uniquely to an $\mathcal{L}_{\mathrm{HA}^*}$-embedding of L into L' and we are done. ⊣

Lemma 5.3 *Let L be a finite algebra in $\mathcal{V}_2$ such that $\mathbf{1}$ is join irreducible. Let a, b_1, b_2 in L be such that $b_1 \vee b_2 \ll a$ and $b_1 \wedge b_2 \wedge (\mathbf{1} - (\mathbf{1} - a)) = \mathbf{0}$. Then there exists an extension L' of L in $\mathcal{V}_2$ and elements a_1, a_2 such that:*

$$a - a_2 = a_1 \geq b_1$$
$$a - a_1 = a_2 \geq b_2$$
$$a_1 \wedge a_2 = b_1 \wedge b_2$$

If $a \neq \mathbf{0}$ one can require that a_1, a_2 are both non zero.

Proof: By assumption $\mathbf{1}$ has a unique predecessor x, thus $L_0 = x^{\downarrow}$ has a natural structure of co-Heyting algebra.

Case 1: $a = \mathbf{0}$. One can take $a_1 = a_2 = \mathbf{0}$.

Case 2: $\mathbf{0} \neq a \leq x$. Lemma 4.2 gives an $\mathcal{L}_{\mathrm{HA}^*}$-embedding φ of L_0 into a co-Heyting algebra L_1 containing non zero elements a_1, a_2 with the required properties. Let L' be the co-Heyting algebra obtained by adding to L_1 a new element on the top. Clearly L' belongs to $\mathcal{V}_2$ by fact 5.1 and the embedding φ extends uniquely to an $\mathcal{L}_{\mathrm{HA}^*}$-embedding of L into L', so we are done.

Case 3: $a = \mathbf{1}$ and $x = \mathbf{0}$. Then $L_0 = \mathbf{L}_2$ embeds into $\mathbf{L}_2 \times \mathbf{L}_2 \in \mathcal{V}_2$ which gives the conclusion.

Case 4: $a = \mathbf{1}$ and $x \neq \mathbf{0}$. Then by assumption $b_1 \wedge b_2 = \mathbf{0}$. Lemma 4.2 applied to x, b_1, b_2 gives an $\mathcal{L}_{\mathrm{HA}^*}$-embedding φ of L_0 into a finite co-Heyting algebra L_1 containing non-zero elements x_1, x_2 such that $x - x_1 = x_2 \geq b_2$, $x - x_2 = x_1 \geq b_1$ and $x_1 \wedge x_2 = \mathbf{0}$. Just as in case 2, the co-Heyting algebra $L^{\dagger}$ obtained by adding to L_1 a new element on the top belongs to $\mathcal{V}_2$, and φ extends to an embedding of L into $L^{\dagger}$. Now $\{a, \{x_1, x_2\}, 2\}$ is a signature in $L^{\dagger}$ since $x_1 \vee x_2 = x$ is the predecessor of $a = \mathbf{1}$ in $L^{\dagger}$. Let L' be an extension generated over $L^{\dagger}$ by a primitive tuple (a_1, a_2) with signature $(a, \{x_1, x_2\}, 2)$ (see remark 4.5). By construction $a_1 \geq a_1 \wedge x = x_1 \geq b_1$ and symmetrically $a_2 \geq b_2$. We also have $a = a_1 \vee a_2$ and $a_1 \wedge a_2 = x_1 \wedge x_2 = \mathbf{0}$. By theorem 3.3 a_1, a_2 are exactly the two join irreducible components of $\mathbf{1}$ in L' hence L' belongs to $\mathcal{V}_2$ by fact 5.1, $a - a_1 = a_2$ and $a - a_2 = a_1$, so we are done. ⊣

Theorem 5.4 *Every co-Heyting algebra existentially closed in $\mathcal{V}_2$ satisfies the density axiom D2 and the splitting axiom S2.*

Proof: By fact 5.1 and lemma 2.4 this follows directly from lemmas 5.2 and 5.3. ⊣

Here is a partial converse of theorem 5.4.

Theorem 5.5 *Let L be an algebra in $\mathcal{V}_2$ satisfying the density axiom D2 and the splitting axiom S2. Let L_0 be a finite subalgebra of L and L_1 be a finite algebra in $\mathcal{V}_2$ containing L_0. Then there exists an embedding of L_1 into L which fixes every point of L_0.*

Proof: By corollary 3.4 we can assume that L_1 is generated over L_0 by a primitive tuple. Let $\sigma = (g, \{h_1, h_2\}, r)$ be the signature of L_1 in L_0. By corollary 3.5 it is sufficient to prove that σ is the signature of a primitive tuple of elements $x_1, x_2 \in L$.

Case 1: $r = 1$ so $h_1 = h_2$. Since $h_1 \leq g^- \ll g$ we have $\mathbf{1} - g^- = \mathbf{1}$ hence obviously $h_1 \wedge g^- \wedge (\mathbf{1} - (\mathbf{1} - g^-)) = \mathbf{0}$. The splitting property $S2$ then applies to the elements g, g^-, h_1 in L. Then continue like in case 1 of the proof of theorem 4.4.

Case 2: $r = 2$ so $h_1 \vee h_2 = g^-$. If $\mathbf{1} - g < \mathbf{1}$ then g is one of the join irreducible components of $\mathbf{1}$ in L_0. By theorem 3.3 x_1, x_2 are then distinct join irreducible components of $\mathbf{1}$ in L_1, and since L_1 belongs to $\mathcal{V}_2$ it follows that $x_1 \wedge x_2 = \mathbf{0}$ and *a fortiori* $h_1 \wedge h_2 = \mathbf{0}$. On the other hand if $\mathbf{1} - g = \mathbf{1}$ then obviously $\mathbf{1} - (\mathbf{1} - g) = \mathbf{0}$. So in any case we have:

$$h_1 \wedge h_2 \wedge (\mathbf{1} - (\mathbf{1} - g)) = \mathbf{0} \tag{5}$$

The splitting property S2 then applies in L to the elements g, h_1, h_2. Then continue like in case 2 of the proof of theorem 4.4. $\dashv$

Remark 5.6 The proof shows that the minimal extension of a finite co-Heyting algebra L_0 in $\mathcal{V}_2$ determined by a signature $(g, \{h_1, h_2\}, r)$ belongs to $\mathcal{V}_2$ if and only if either $r = 1$, or $r = 2$ and condition (5) holds. Also the analogues of corollaries 4.6 and 4.7 hold for $\mathcal{V}_2$ as a consequence of theorem 5.5

6 Density and splitting in $\mathcal{V}_3$

We introduce the following axioms:

[Density D3] For every a such that $a = \mathbf{1} - (\mathbf{1} - a) \neq \mathbf{0}$ there exists a non zero element b such that $b \ll a$.

[Splitting S3] Same as S1.

A co-Heyting algebra L belongs to $\mathcal{V}_3$ if and only if it has dimension ≤ 1. If L is finite this is equivalent to say that every join irreducible element of L is either maximal or minimal (or both) in $\mathcal{I}^\vee(L)$.

Lemma 6.1 *Let a be any element of a finite algebra L in $\mathcal{V}_3$. If $a = \mathbf{1} - (\mathbf{1} - a) \neq \mathbf{0}$ then there exists a finite algebra L' in $\mathcal{V}_3$ containing L and a non zero element b in L' such that $b \ll a$.*

Proof: Let $a_1, \dots, a_r$ be the join irreducible components of a in L. The assumption that $a = \mathbf{1} - (\mathbf{1} - a) \neq \mathbf{0}$ means that $r \neq 0$ and all the a_i's are join irreducible components of $\mathbf{1}$, that is maximal elements in $\mathcal{I}^\vee(L_0)$. If there exists $i \leq r$ such that a_i is not in the same time minimal in L (that is a_i is not an atom of L) then we can choose $b \in \mathcal{I}^\vee(L)$ such that $b < a_i$. Then b is non zero and $b \ll a_i$ because a_i is join irreducible, so *a fortiori* $b \ll a$. The conclusion follows, with $L' = L$.

It only remains to deal with the case when all the a_i are both maximal and minimal in $\mathcal{I}^\vee(L)$. But in this case the construction of lemma 4.1 (with $c = \mathbf{0}$) gives an extension L' on L such that:

- $\mathcal{I}^\vee(L') = \mathcal{I}^\vee(L) \cup \{x_1, \dots, x_r\}$.
- For every $i \leq r$ and every $x \in \mathcal{I}^\vee(L)$, $x \not< x_i$ and:

$$x_i < x \iff x = a_i$$

So there are still no chain in $\mathcal{I}^\vee(L')$ containing more than two distinct join irreducible elements, that is L' belongs to $\mathcal{V}_2$, and clearly:

$$\mathbf{0} \neq x_1 \vee \cdots \vee x_r \ll a$$

$\dashv$

Lemma 6.2 *Let a, b_1, b_2 be elements of a finite algebra L in $\mathcal{V}_3$. If $b_1 \vee b_2 \ll a \neq \mathbf{0}$ then there exists a finite algebra L' in $\mathcal{V}_3$ containing L and non zero elements a_1, a_2 in L' such that:*

$$\begin{aligned} a - a_2 = a_1 \geq b_1 \\ a - a_1 = a_2 \geq b_2 \\ a_1 \wedge a_2 = b_1 \wedge b_2 \end{aligned}$$

Proof: Same proof as for lemma 4.2. Indeed, in the extension L' of L constructed in that proof the maximal length of the chains of join irreducible elements is the same as in L. So if L belongs to $\mathcal{V}_3$ then so does L'. ⊣

Theorem 6.3 *The theory of the variety $\mathcal{V}_3$ has a model-completion which is axiomatized by the axioms of co-Heyting algebras augmented by the density and splitting axioms D3 and S3.*

Proof: As for theorem 4.3 it immediately follows from lemmas 6.1 and 6.2, *via* lemma 2.3, that every algebra existentially closed in $\mathcal{V}_3$ satisfies the axioms D3 and S3.

For the converse, by fact 2.1 it is sufficient to show that given an algebra L in $\mathcal{V}_3$ satisfying D3 and S3, a finitely generated subalgebra L_0 and a finitely generated extension L_1 of L_0 in $\mathcal{V}_3$ there exists an embedding of L_1 in L which fixes L_0 pointwise. Since $\mathcal{V}_3$ is locally finite, L_0 and L_1 are finite and by corollary 3.4 we can assume that L_1 is generated by a primitive tuple (x_1, x_2). Let $\sigma = (g, \{h_1, h_2\}, r)$ be the signature of L_1 in L, numbered so that $h_i = x_i \wedge g^-$. By corollary 3.5 we have to find a primitive tuple in L having signature σ.

Case 1: $r = 1$ so $x_1 = x_2$ and $h_1 \ll x_1 \ll g$. Since x_1, g are join irreducible in L_1 and since L_1 belongs to $\mathcal{V}_3$, necessarily g is a join irreducible component of $\mathbf{1}$, x_1 is an atom of L_1, and consequently $h_1 = \mathbf{0}$. The splitting axiom S3 applied to $g, g^-, \mathbf{0}$ gives non zero elements y_1, y_2 in L such that:

$$\begin{aligned} g - y_1 = y_2 \geq g^- \\ g - y_2 = y_1 \\ y_1 \wedge y_2 = \mathbf{0} \end{aligned}$$

By construction (y_1, y_2) is a primitive tuple over L_0 hence by remark 3.1 and theorem 3.3 we have:

$$\mathcal{I}^\vee(L_0\langle y_1\rangle) = \big(\mathcal{I}^\vee(L_0) \setminus \{g\}\big) \cup \{y_1, y_2\}$$

Since g was a join irreducible component of $\mathbf{1}$ in L_0, the same then holds for y_1, y_2 in $L_0\langle y_1\rangle$. It follows that $\mathbf{1} - (\mathbf{1} - y_1) = y_1$ hence the density axiom D3 gives $x \in L \setminus \{\mathbf{0}\}$ such that $x \ll y_1$. *A fortiori* $x \ll g$ and by construction $x \wedge g^- \leq y_1 \wedge y_2 = \mathbf{0}$. It easily follows that (x, x) is a primitive tuple with signature $(g, \{\mathbf{0}\}, 1) = \sigma$ in L_0.

Case 2: $r = 2$ so $h_1 \vee h_2 = g^-$. The same construction as in the case 2 of the proof of theorem 4.4 applies here and gives the conclusion. ⊣

7 Density and splitting in $\mathcal{V}_4$

We introduce the following axioms:

[Density D4] Same as D3.

[Splitting S4] Same as S1 with the additional assumption that $b_1 \wedge b_2 \wedge (\mathbf{1} - a) = \mathbf{0}$.

Fact 7.1 *For any finite co-Heyting algebra L the following conditions are equivalent.*

1. *L belongs to $\mathcal{V}_4$.*
2. *L belongs to $\mathcal{V}_3$ (every element of $\mathcal{I}^{\vee}(L)$ is either maximal or minimal) and for any three distinct join irreducible components x_1, x_2, x_3 of $\mathbf{1}$, we have $x_1 \wedge x_2 \wedge x_3 = \mathbf{0}$.*
3. *L $\mathcal{L}_{\mathrm{HA}^*}$-embeds in a product of finitely many copies of $\mathbf{L}_5$.*

This is probably well known. For lack of a reference we give here an elementary (and sketchy) proof. We can assume that $L \neq \mathbf{L}_1$ otherwise everything is trivial.

Proof: (3)⇒(1) is clear.

(1)⇒(2) Since L belongs to $\mathcal{V}_4$, which is generated by $\mathbf{L}_5$, which belongs to $\mathcal{V}_3$, obviously L belongs to $\mathcal{V}_3$. Now assume that $\mathbf{1}$ has at least three distinct join irreducible components x_1, x_2, x_3 in L. The equation defining $\mathcal{V}_4$ gives:

$$(x_1 - x_2) \wedge (x_2 - x_1) \wedge (x_2 \bigtriangleup (\mathbf{1} - x_1)) = \mathbf{0} \tag{6}$$

We have $x_1 - x_2 = x_1$, $x_2 - x_1 = x_2$ and $\mathbf{1} - x_1$ is the join of all join irreducible components of $\mathbf{1}$ except x_1. In particular it is greater than x_2 and x_3 so we get:

$$x_2 \bigtriangleup (\mathbf{1} - x_1) = (\mathbf{1} - x_1) - x_2 \geq x_3$$

Finally (6) becomes $x_1 \wedge x_2 \wedge ((\mathbf{1} - x_1) - x_2) = \mathbf{0}$ hence *a fortiori* $x_1 \wedge x_2 \wedge x_3 = \mathbf{0}$.

(2)⇒(3) We consider:

$$\mathcal{I} = \{(x_1, x_2) \in \mathcal{I}^{\vee}(L) \times \mathcal{I}^{\vee}(L) : x_1 < x_2 \text{ or } x_1 = x_2 \text{ is an atom}\}$$

$\mathcal{I}$ is ordered as follows:

$$(y_1, y_2) < (x_1, x_2) \iff y_1 = y_2 = x_1 < x_2$$

The ordered set $\mathcal{I}$ looks like $\mathcal{I}^{\vee}(L)$ except that every point of $\mathcal{I}^{\vee}(L)$ strictly greater than r atoms has been "split" in r points strictly greater than only one atom. We "collapse" these r points *via* the map π defined for any $\xi = (x_1, x_2) \in \mathcal{I}$ by $\pi(\xi) = x_2$. This defines an $\mathcal{L}_{\mathrm{HA}^*}$-embedding of L into $L' = \mathcal{L}^{\downarrow}(\mathcal{I})$ by means of proposition 2.7. Then (2) implies that $\mathcal{I}$ is a finite disjoint union of copies of sets represented in figure 5. The family of all decreasing subsets of these sets

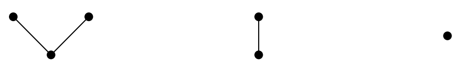


Figure 5: The connected components of $\mathcal{I}$

are respectively isomorphic to $\mathbf{L}_5$, $\mathbf{L}_3$ and $\mathbf{L}_2$. Since $\mathcal{I}^{\vee}(L')$ is order-isomorphic to $\mathcal{I}$, it follows that L' is a direct product of finitely many copies of these three algebras. Each of these copies obviously $\mathcal{L}_{\mathrm{HA}^*}$-embeds into $\mathbf{L}_5$ so we are done. ⊣

Lemma 7.2 *Let a be any element of $\mathbf{L}_5$ such that $a = \mathbf{1} - (\mathbf{1} - a)$. Then there exists an element b in $\mathbf{L}_5$ such that $b \ll a$. If moreover $a \neq \mathbf{0}$ then b can be chosen non zero.*

Proof: The assumption that $\mathbf{1} - (\mathbf{1} - a) = a \neq \mathbf{0}$ implies that a is not the unique atom c of $\mathbf{L}_5$. If $a = \mathbf{0}$ one can take $b = \mathbf{0}$. Otherwise one can take $b = c$. ⊣

Lemma 7.3 *Let a, b_1, b_2 be any elements of $\mathbf{L}_5$ such that $b_1 \vee b_2 \ll a$ and $b_1 \wedge b_2 \wedge (\mathbf{1} - a) = \mathbf{0}$. Then there exists an extension L' of L in $\mathcal{V}_4$ and elements a_1, a_2 in L' such that:*

$$\begin{array}{l} a - a_2 = a_1 \geq b_1 \\ a - a_1 = a_2 \geq b_2 \\ a_1 \wedge a_2 = b_1 \wedge b_2 \end{array}$$

If moreover $a \neq \mathbf{0}$ then a_1, a_2 can be chosen both non zero.

Proof: Let c denote the unique atom of $\mathbf{L}_5$. The first assumption on a, b_1, b_2 implies that $b_1 \vee b_2$ is either $\mathbf{0}$ or c. In particular we can always assume that $b_2 \leq b_1$.

Case 1: $a = \mathbf{0}$. One can take $a_1 = a_2 = \mathbf{0}$.

Case 2: $b_1 = b_2 = c$. By assumption $c \ll a$ and $c \wedge (\mathbf{1} - a) = \mathbf{0}$ hence $a = \mathbf{1}$. So we can take $L' = \mathbf{L}_5$ and for a_1, a_2 the join irreducible components of $\mathbf{1}$.

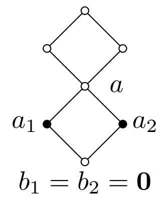


Figure 6: Case 3

Case 3: $a = c$. Then $b_1 = b_2 = \mathbf{0}$ and we can take for a_1, a_2 the atoms of the extension L' of $\mathbf{L}_5$ shown in figure 6 (the white points are the points of $\mathbf{L}_5$). Note that the four join irreducible elements of $L' = \mathbf{L}_5 \times \mathbf{L}_2$ belongs to $\mathcal{V}_4$.

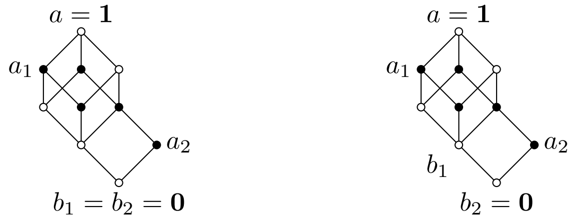


Figure 7: Cases 4 and 5

Cases 4 to 7: The four remaining cases when $a > c$ are summarised in figures 7 and 8. In each case the white points represent the points of $\mathbf{L}_5$ and one can take for a_1, a_2 the points in the extension L' of $\mathbf{L}_5$ shown in the figures. Note that L' is just $\mathbf{L}_5 \times \mathbf{L}_3$ so it belongs to $\mathcal{V}_4$. ⊣

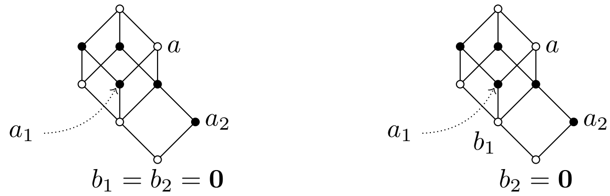


Figure 8: Cases 6 and 7

Theorem 7.4 *The theory of the variety $\mathcal{V}_4$ has a model-completion which is axiomatized by the axioms of co-Heyting algebras augmented by the density and splitting axioms D4 and S4.*

Proof: As for theorem 6.3, the only thing which it remains to prove after lemmas 7.2 and 7.3 is that: given an algebra L in $\mathcal{V}_4$ satisfying D4 and S4, a finitely generated subalgebra L_0 and a finitely generated extension L_1 of L_0 in $\mathcal{V}_4$ generated by a primitive tuple (x_1, x_2) with signature $\sigma = (g, \{h_1, h_2\}, r)$ in L_0 (numbered so that $h_i = x_i \wedge g^-$), there exists a primitive tuple in L having the same signature σ.

Case 1: $r = 1$. The same argument as in the case 1 in the proof of theorem 6.3 applies here (when applying S4 in place of S3 to $g, g^-, \mathbf{0}$ the additional condition $g^- \wedge \mathbf{0} \wedge (\mathbf{1} - g) = \mathbf{0}$ is obviously satisfied).

Case 2: $r = 2$ so $h_1 \vee h_2 = g^-$. In order to apply the splitting axiom S4 to g, h_1, h_2 we have to check that $h_1 \wedge h_2 \wedge (\mathbf{1} - g) = \mathbf{0}$. Assume the contrary. Then h_1, h_2 are non zero so g is not an atom. Since L_0 belongs to $\mathcal{V}_4 \subseteq \mathcal{V}_3$ it follows that g is maximal in $\mathcal{I}^\vee(L_0)$ hence so are x_1, x_2 in $\mathcal{I}^\vee(L_1)$ (see theorem 3.3). With other words x_1, x_2 are two distinct join irreducible components of $\mathcal{I}^\vee(L_1)$ and $\mathbf{1} - g$ is the join of all the other join irreducible components of $\mathbf{1}$ in L_1. But for any such component x_3 we must have $x_1 \wedge x_2 \wedge x_3 = \mathbf{0}$ by fact 7.1 so $x_1 \wedge x_2 \wedge (\mathbf{1} - g) = \mathbf{0}$. Since each $h_i \leq x_i$ this contradicts our assumption.

So we can apply S4 to g, h_1, h_2 and it gives y_1, y_2 in L. Then finish like in the case 2 of the proof of theorem 4.4. ⊣

8 Density and splitting in $\mathcal{V}_5$

The density and splitting axioms for $\mathcal{V}_5$ are respectively D5=D3 and S5=S2.

Fact 8.1 *For a finite co-Heyting algebra L the following conditions are equivalent:*

1. *L belongs to $\mathcal{V}_5$.*

2. *L belongs to $\mathcal{V}_2$ and $\mathcal{V}_3$, that is every join irreducible element of L which is not an atom is a join irreducible component of $\mathbf{1}$, and for any two distinct join irreducible components x_1, x_2 of $\mathbf{1}$ we have $x_1 \wedge x_2 = \mathbf{0}$.*

3. *L $\mathcal{L}_{\mathrm{HA}^*}$-embeds in a direct product of finitely many copies of the three elements co-Heyting algebra $\mathbf{L}_3$.*

This is probably well known, and anyway the adaptation to this context of the proof that we gave for the analogous fact 7.1 is straightforward.

Theorem 8.2 *The theory of the variety $\mathcal{V}_5$ has a model-completion which is axiomatized by the axioms of co-Heyting algebras augmented by the density and splitting axioms D5 and S5.*

Proof: Let c denote the unique atom of $\mathbf{L}_3$.

The only elements a in $\mathbf{L}_3$ such that $a = \mathbf{1} - (\mathbf{1} - a)$ are $\mathbf{0}$ and $\mathbf{1}$. Clearly if $a = \mathbf{0}$ then $b = \mathbf{0}$ satisfies $b \ll a$, and otherwise $b = c$ satisfies $\mathbf{0} \neq b \ll a$. By fact 8.1 and lemma 2.4 it follows that every algebra existentially closed in $\mathcal{V}_5$ satisfies D5.

Now let a, b_1, b_2 in L_3 be such that $b_1 \vee b_2 \ll a$ and $b_1 \wedge b_2 \wedge (\mathbf{1} - (\mathbf{1} - a)) = \mathbf{0}$. If $a = \mathbf{0}$ then one can take $a_1 = a_2 = \mathbf{0}$ as a solution for the conclusion of S5. Otherwise three cases may happen:

Case 1: $a = c$ and $b_1 = b_2 = \mathbf{0}$.

Case 2: $a = \mathbf{1}$ and $b_1 = b_2 = \mathbf{0}$.

Case 3: $a = \mathbf{1}$ and (renaming b_1 and b_2 if necessary) $b_1 = c$ and $b_2 = \mathbf{0}$.

In each of these cases one can take for a_1, a_2 the elements of the extension $\mathbf{L}_9$ of $\mathbf{L}_3$ shown in figure 9 (the white points represent $\mathbf{L}_3$). Note that $\mathbf{L}_9 = \mathbf{L}_3 \times \mathbf{L}_3$ belongs to $\mathcal{V}_5$. By fact 8.1 and lemma 2.4 again, it follows that every algebra existentially closed in $\mathcal{V}_5$ satisfies S5.

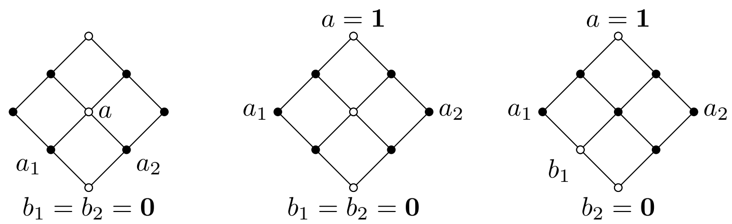


Figure 9: $\mathbf{L}_3 \subset \mathbf{L}_9$ gives solutions for S5 (three possible cases).

Conversely let L in $\mathcal{V}_5$ satisfying D5 and S5, L_0 a finitely generated subalgebra and L_1 a finitely generated extension of L_0 in $\mathcal{V}_5$ generated by a primitive tuple (x_1, x_2) with signature $\sigma = (g, \{h_1, h_2\}, r)$ in L_0 (numbered so that $h_i = x_i \wedge g^-$). As usually it only remains to find a primitive tuple in L having the same signature σ in order to conclude that L_1 embeds into L over L_0 by corollary 3.5, hence to finish the proof by fact 2.1.

Case 1: $r = 1$ so $x_1 = x_2$ and $h_1 \ll x_1 \ll g$. Same as case 1 in the proof of theorem 6.3.

Case 2: $r = 2$ so $h_1 \vee h_2 = g^-$. Same as case 2 in the proof of theorem 5.4 (note that $\mathcal{V}_5$ is contained in $\mathcal{V}_2$ when applying this proof). ⊣

9 Density and splitting in $\mathcal{V}_6$

We introduce our last axioms.

[Density D6] Same as D1.

[Splitting S6] Same as S1 with the additional assumption that $b_1 \wedge b_2 = \mathbf{0}$.

Fact 9.1 *A finite co-Heyting algebra belongs to $\mathcal{V}_6$ if and only if it embeds into a direct product of finitely many finite chains.*

This is certainly well known, and easy to check.

Theorem 9.2 *The theory of the variety $\mathcal{V}_6$ has a model-completion which is axiomatized by the axioms of co-Heyting algebras augmented by the density and splitting axioms D6 and S6.*

Proof: Let a, c be any elements in a finite chain L such that $c \ll a$. If $a = \mathbf{0}$ then $b = \mathbf{0}$ satisfies $c \ll b \ll a$. Otherwise $c < a$ and obviously L embeds into a chain L' containing a new intermediate element b between a and a^-. Then by construction $c \ll b \ll a$ and $b \neq \mathbf{0}$. By fact 9.1 and lemma 2.4 it follows that every algebra existentially closed in $\mathcal{V}_6$ satisfies D6.

Let a, b_1, b_2 be three elements in a finite chain L such that $b_1 \vee b_2 \ll a$ and $b_1 \wedge b_2 = \mathbf{0}$. We may assume that $b_2 \leq b_1$, so by assumption $b_2 = \mathbf{0}$. If $a = \mathbf{0}$ then $a_1 = a_2 = \mathbf{0}$ satisfy the conclusion of S6. Otherwise $b_1 < a$ and one can take for a_1, a_2 the non zero points in the extension L' of L shown in figure 10 (the white points represent L). Note that $L' = L \times \mathbf{L}_2$ belongs to $\mathcal{V}_2$. By fact 9.1 and lemma 2.4 again, it follows that every algebra existentially closed in $\mathcal{V}_6$ satisfies S6.

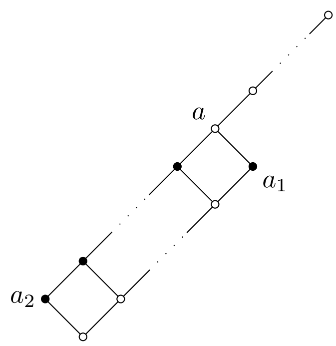


Figure 10: L (in white) inside $L \times \mathbf{L}_2$

Conversely let L in $\mathcal{V}_6$ be satisfying D6 and S6, L_0 a finitely generated subalgebra and L_1 a finitely generated extension of L_0 in $\mathcal{V}_6$ generated by a primitive tuple (x_1, x_2) with signature $\sigma = (g, \{h_1, h_2\}, r)$ in L_0 (numbered so that $h_i = x_i \wedge g^-$). As usually it only remains to find a primitive tuple in L having the same signature σ in order to conclude that L_1 embeds into L over L_0 by corollary 3.5, hence to finish the proof by fact 2.1.

Case 1: $r = 1$ so $x_1 = x_2$ and $h_1 \ll x_1 \ll g$. Same as case 1 in the proof of theorem 4.4.

Case 2: $r = 2$ so $h_1 \vee h_2 = g^-$. Since x_1, x_2 are join irreducible and incomparable, $x_1 - x_2 = x_1$ and $x_2 - x_1 = x_2$. By definition of $\mathcal{V}_6$ it follows that $x_1 \wedge x_2 = \mathbf{0}$, hence *a fortiori* $h_1 \wedge h_2 = \mathbf{0}$. So the splitting axiom S6 applies to g, h_1, h_2. Then finish the proof like in case 2 of theorem 4.4. ⊣

10 Appendix

It is proven in [GZ97], page 44, that for every x, z in an existentially closed algebra L in $\mathcal{H}_1$ there are elements x_1, x_2 such that $x_1 \vee x_2 = \mathbf{1}$, $x_1 \wedge x_2 = x$ and:

$$(z - x_1) \wedge x = (z - x_2) \wedge x$$

Since this axiom asserts the existence of a splitting of $\mathbf{1}$ in two parts x_1 and x_2 intersecting along x with an additional condition, it is very close in spirit to our axiom S1. Is it equivalent to S1? Our guess is no. However it follows from the next proposition (with $w = \mathbf{1}$) that the above axiom is implied by S1.

Proposition 10.1 *Let L be a Heyting algebra satisfying the splitting axiom S1. Then for every w, x, z in L such that $x \leq w$ there are elements x_1, x_2 in L such that $x_1 \vee x_2 = w$, $x_1 \wedge x_2 = x$ and:*

$$(z - x_1) \wedge x = (z - x) \wedge x = (z - x_2) \wedge x$$

Proof: Since $(z - x) \wedge x \ll z - x$, S1 gives[8] elements a_1, a_2 such that:

$$\begin{aligned} (z - x) - a_2 = a_1 &\geq (z - x) \wedge x \\ (z - x) - a_1 = a_2 &\geq (z - x) \wedge x \\ a_1 \wedge a_2 &= (z - x) \wedge x \end{aligned}$$

Let $c = w - (z \vee x)$. Since $a_i \leq z - x \leq z \vee x$ for $i = 1, 2$ we have that $c \wedge a_i \ll c$. Thus S1 again gives[9] elements c_1, c_2 such that:

$$\begin{aligned} c - c_2 = c_1 &\geq c \wedge a_1 \\ c - c_1 = c_2 &\geq c \wedge a_2 \\ c_1 \wedge c_2 &= c \wedge a_1 \wedge a_2 \end{aligned}$$

Let $x_i = x \vee a_i \vee c_i$ for $i = 1, 2$. By construction:

$$x_1 \vee x_2 = x \vee (a_1 \vee a_2) \vee (c_1 \vee c_2) = x \vee (z - x) \vee c = (z \vee x) \vee c = w$$

Moreover $c_1 \wedge a_2 = c_1 \wedge c \wedge a_2 \leq c_1 \wedge c_2$. The latter is smaller than $a_1 \wedge a_2$ which is smaller than x. Symmetrically $c_2 \wedge a_2 \leq x$ so by distributivity we get:

$$(a_1 \vee c_1) \wedge (a_2 \vee c_2) \leq x$$

Thus $x_1 \wedge x_2 = x \vee \big[(a_1 \vee c_1) \wedge (a_2 \vee c_2)\big] = x$. Finally we have by construction:

$$z - x_1 = \big((z - x) - a_1\big) - c_1 = a_2 - c_1$$

We already noticed that $c_1 \wedge a_2 \leq c_1 \wedge c_2$. The latter is smaller than $a_1 \wedge a_2 \ll a_2$ so $a_2 - c_1 = a_2$. Recall that:

$$(z - x) \wedge x \leq a_2 \leq z - x$$

Thus $(z - x_1) \wedge x = a_2 \wedge x = (z - x) \wedge x$ and symmetrically for x_2. $\dashv$

[8] Of course S1 applies only if $z - x \neq \mathbf{0}$ but otherwise it suffices to take $a_1 = a_2 = \mathbf{0}$.

[9] As above, if $c = \mathbf{0}$ we cannot apply S1 but $c_1 = c_2 = \mathbf{0}$ then suits perfectly our needs.

All the other properties of non-zero existentially closed Heyting algebras listed in proposition A2 (i)–(iv) of [GZ97] easily follow from the density axiom D1, except (iv) that we derive from S1 in the next proposition.

Proposition 10.2 *Let x, y be any elements in a co-Heyting algebra L satisfying S1. Then $y \to x$ exists in L if and only if $(\mathbf{1} - y) \wedge y \leq x$.*

Remark 10.3 It is an easy exercise to check that in every co-Heyting algebra, if $(\mathbf{1}-y)\wedge y \leq x$ then $y \to x$ exists and equals $(\mathbf{1}-y)\vee x$. So the above proposition shows that among co-Heyting algebras, those which satisfy the axiom S1 are "the least possibly bi-Heyting".

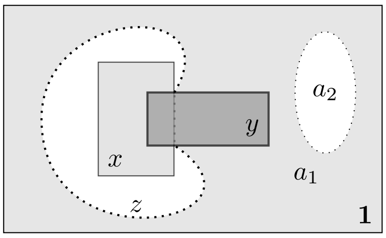


Figure 11: Splitting of $\mathbf{1} - (z \vee y)$ (here $x \subseteq z$)

As we explained at the beginning of this paper, many of our proofs are inspired by the geometric intuition coming from the "co-Heyting" (instead of "Heyting"). As an illustration, we add the "picture of the proof" and how to use it for this last proof.

By the above remark we only have to prove that, assuming $(\mathbf{1} - y) \wedge y \not\leq x$, the set $\mathcal{Z}$ of elements z in L such that $z \wedge y \leq x$ has (thanks to S1) no greatest element. So let z be any element in $\mathcal{Z}$, and let us imagine that x, y, z are semi-algebraic subsets of the real plane in figure 11.

By assumption $z \cap y \subseteq x$, that is z does not contain any point of y which is not in x. The largest possible such set is the complement of $y \setminus x$, but z cannot be so large without meeting the frontier of y, that is $(\mathbf{1} - y) \cap y$, outside x. One sees then in figure 11 how to increase z without changing $z \cap y$: it suffices to split (using S1) the intermediate piece which is the complement of $z \cup y$ into two disjoint pieces, one of which avoids to touch the border, and to add the latter to z.

Proof: Let $x, y \in L$ such that $(\mathbf{1} - y) \wedge y \not\leq x$, and $\mathcal{Z}$ the set of elements z in L such that $z \wedge y \leq x$. We have to prove that $\mathcal{Z}$ has no greatest element. For any element z in $\mathcal{Z}$, let $a = \mathbf{1} - (z \vee y)$. Note that:

$$[(z \vee y) - y] \wedge y = (z - y) \wedge y \leq z \wedge y \leq x$$

Since $(\mathbf{1} - y) \wedge y \not\leq x$ by assumption, it follows that $\mathbf{1} \neq z \vee y$ hence $a \neq \mathbf{0}$. The splitting property S1 then gives non-zero elements a_1, a_2 in L such that:

$$\begin{aligned} a - a_2 = a_1 &\geq a \wedge (z \vee y) \\ a - a_1 = a_2 &\geq \mathbf{0} \\ a_1 \wedge a_2 &= \mathbf{0} \end{aligned}$$

Clearly $a_2 \not\leq z$ since $a - z = a$ and $a - a_2 < a$. On the other hand:

$$a_2 \wedge y = a_2 \wedge a \wedge y \leq a_2 \wedge a_1 = \mathbf{0}$$

Thus $(z \vee a_2) \wedge y = z \wedge y \leq x$, which proves that $z \vee a_2 \in \mathcal{Z}$ and consequently that z is not maximal in $\mathcal{Z}$. $\dashv$

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