Codimension and pseudometric in co-Heyting algebras

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Abstract

In this paper we introduce a notion of dimension and codimension for every element of a distributive bounded lattice L. These notions prove to have a good behavior when L is a co-Heyting algebra. In this case the codimension gives rise to a pseudometric on L which satisfies the ultrametric triangle inequality. We prove that the Hausdorff completion of L with respect to this pseudometric is precisely the projective limit of all its finite dimensional quotients. This completion has some familiar metric properties, such as the convergence of every monotonic sequence in a compact subset. It coincides with the profinite completion of L if and only if it is compact or equivalently if every finite dimensional quotient of L is finite. In this case we say that L is precompact. If L is precompact and Hausdorff, it inherits many of the remarkable properties of its completion, specially those regarding the join/meet irreducible elements. Since every finitely presented co-Heyting algebra is precompact Hausdorff, all the results we prove on the algebraic structure of the latter apply in particular to the former. As an application, we obtain the existence for every positive integers n, d of a term $t_{n,d}$ such that in every co-Heyting algebra generated by an n-tuple a, $t_{n,d}(a)$ is precisely the maximal element of codimension d.

1 Introduction

We attach to every element of a distributive bounded lattice L a (possibly infinite) dimension and codimension, by copying analogous definitions in algebraic geometry. The definitions are second order, in terms of chains of prime filters of L ordered by inclusion, but yield geometric intuition on the elements of L. In the meantime we introduce a first order notion of rank and corank for the elements of L. When the dual of L (that is the same lattice with the reverse order) is a Heyting algebra, we prove in section 3 that the rank and dimension coincide, as well as the finite corank and finite codimension. This ensures a much better behaviour for the dimension and codimension (and for the rank and corank) than in general lattices. Hence we restrict ourselves to this class, known as the variety of **co-Heyting algebras** or **Brouwerian lattices**.

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By defining the dimension of L itself as the dimension of its greatest element, the connection is made with the so-called "slices" of Heyting algebras, studied by Hosoi [Hos67], Komori [Kom75] and Kuznetsov [Kuz75], among others. More precisely, a co-Heyting algebra L has dimension d if and only if its dual belongs to the $(d+1)$-th slice of Hosoi. On the other hand, the (co)dimension of an *element of* L seems to be a new concept in this area.

In section 4 we introduce a pseudometric on co-Heyting algebras based on the codimension, but delay until section 7 the study of complete co-Heyting algebras. By elementary use of Kripke models and the finite model property of intuitionistic propositional calculus, we check in section 5 that the filtration by finite codimensions has several nice properties in any finitely generated co-Heyting algebra L:

1. For every positive integer d, the set dL of elements of L of codimension $\geq d$ is a principal ideal.
2. For every positive integer d, the quotient L/dL is finite.
3. If moreover L is finitely presented, then $\bigcap_{d<\omega} dL = \{\mathbf{0}\}$.

Property (3) asserts that L is Hausdorff (with respect to the topology of the pseudometric we introduce). Property (2) shows that L is precompact (in the sense that its Hausdorff completion is compact). More generally we prove that a variety $\mathcal{V}$ of co-Heyting algebras has the finite model property if and only if every algebra free in $\mathcal{V}$ is Hausdorff. In such a variety we have the following relations:

$$\begin{array}{lclcl} \text{finitely generated} & \Longrightarrow & \text{precompact} & & \\ \text{finitely presented} & \Longrightarrow & \text{precompact Hausdorff} & \Longrightarrow & \text{residually finite} \end{array}$$

Many algebraic properties probably known for finitely presented co-Heyting algebras (but hard to find in the literature) generalize to precompact Hausdorff co-Heyting algebras, as we show in section 6. We prove in particular that L and its completion have the same join irreducible elements, that all of them are completely join irreducible and that every element $a \in L$ is the complete join of its join irreducible components (the maximal join irreducible elements smaller than a). We prove similar (but not completely identical) results for the completely meet irreducible elements. A characterisation of meet irreducible elements which are not completely meet irreducible is also given.

Finally we prove in section 7 that the Hausdorff completion of every co-Heyting algebra L is also its pro-finite-dimensional completion, that is the projective limit of all its finite dimensional quotients. This completion has some nice metric properties, such as the convergence of every monotonic sequence in a compact subset. It coincides with the profinite completion of L if and only if it is compact or equivalently if every finite dimensional quotient of L is finite.

So in the Hausdorff precompact case, our completion is nothing but the classical profinite completion studied in [BGG+06]. But there is an important difference: in our situation every precompact co-Heyting algebra inherits many of the nice properties of its completion, while in general the properties of profinite co-Heyting algebras do not pass to their dense subalgebras (which are exactly all residually finite co-Heyting algebras, a much wider class than the class of precompact Hausdorff ones).

In the appendix we derive from (1) a surprising application: for all positive integers n, d there exists a term $t_{n,d}(x)$ with n variables such that if L is any co-Heyting algebra generated by a tuple $a \in L^n$ then $t_{n,d}(a)$ is the generator of dL. Possible connections with locally finite varieties of co-Heyting algebras are discussed.

Remark 1.1 The results of section 6 on precompact co-Heyting algebras are closely related to those that we derived in [DJ08] from Bellissima's construction of a Kripke model for each finitely generated free Heyting algebra. Actually the approaches that we have developed here and in [DJ08] are quite complementary. The general methods of the present paper do not seem to be helpful for certain results, which are proper to finitely generated co-Heyting algebras (in particular those which concern the generators). On the other hand they allow us to recover with simple proofs many of the remarkable algebraic properties of finitely presented co-Heyting algebras, widely generalised to precompact Hausdorff co-Heyting algebras, without requiring any sophisticated tool of universal algebra nor the intricate construction of Bellissima.

Remark 1.2 The reader accustomed to Heyting algebras will certainly find very annoying to reverse by dualisation all his/her habits. We apologise for this, but there were pretty good reasons for doing so. Indeed we have not invented the (co)dimension: we simply borrowed it from algebraic geometry *via* the Stone-Priestley duality (see example 2.2). So we could not define in a different way the (co)codimension for the elements of a general lattice. Then it turns out that only in co-Heyting algebras we were able to prove that the codimension and the corank coincide when they are finite. Since all the results of this papers require the basic properties that we derive from this coincidence, we had actually no other choice than to focus on these algebras.

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2 Prerequisites

Distributive bounded lattice. The language of distributive bounded lattices is $\mathcal{L}_{\text{lat}} = \{\mathbf{0}, \mathbf{1}, \vee, \wedge\}$, the order being defined by $a \leq b$ iff $a = a \wedge b$. We will denote by $\bigvee$ the join and by $\bigwedge$ the meet of any family of elements of a lattice. We write $\bigwedge$, $\bigvee$ for the logical connectives 'and', 'or' and $\bigwedge\!\!\!\bigwedge$, $\bigvee\!\!\!\bigvee$ for their iterated forms.

We refer the reader to any book on lattices for the notions of (prime) ideals and (prime) filter of L. We denote by $\operatorname{Spec} L$ the **prime filter spectrum**, that is the set of all prime filters of L. For every a in L let:

$$F(a) = \{\mathfrak{p} \in \operatorname{Spec} L \,/\, a \in \mathfrak{p}\}$$

As a ranges over L the family of all the $F(a)$'s forms a basis of closed sets for the **Zariski topology** on $\operatorname{Spec} L$. It also forms a lattice of subsets of $\operatorname{Spec} L$ which is isomorphic to L (Stone-Priestley duality).

Dualizing ordered sets. An ordered set is a pair $(E, \leq)$ where E is a set and $\leq$ a reflexive, symmetric and transitive binary relation. We do not require the order to be linear. For every $x \in E$ we denote:

$$\begin{aligned} x{\uparrow} &= \{y \in E \,/\, x \leq y\}, \quad & x{\downarrow} &= \{y \in E \,/\, y \leq x\},\\ x{\Uparrow} &= \{y \in E \,/\, x < y\}, \quad & x{\Downarrow} &= \{y \in E \,/\, y < x\}.\end{aligned}$$

The **dual of** E, in notation E^*, is simply the set E *with the reverse order*. For any $x \in E$ we will denote by x^* the element x itself *seen as an element of* E^*, so that:

$$y^* \leq x^* \iff x \leq y$$

The stars indicate that the first symbol $\leq$ refers to the order of E^*, while the second one refers to the order of E. Similarly $X^* = \{x^* \,/\, x \in X\}$ for every $X \subseteq E$ hence for instance $x{\downarrow} = (x^*{\uparrow})^*$.

This apparently odd notation is specially convenient when E carries an additional structure. For example the dual L^* of a distributive bounded lattice L is obviously a distributive bounded lattice and for every $a, b \in L$:

$$\begin{aligned} \mathbf{0}^* = \mathbf{1} \quad &\text{and} \quad \mathbf{1}^* = \mathbf{0}\\ (a \vee b)^* = a^* \wedge b^* \quad &\text{and} \quad (a \wedge b)^* = a^* \vee b^*\end{aligned}$$

(Co)foundation rank and ordered sets. The appropriate generalisations to arbitrary ordinals of "the length of the longest chain" of elements in $(E, \leq)$ are the foundation rank and cofoundation rank of an element x of E. The **foundation rank** is inductively defined as follows:

$$\begin{aligned} &\operatorname{rk} x \geq 0, && \\ &\operatorname{rk} x \geq \alpha = \beta + 1 && \iff \exists y \in E,\ x > y \text{ and } \operatorname{rk} y \geq \beta,\\ &\operatorname{rk} x \geq \alpha = \bigcup_{\beta < \alpha} \beta && \iff \forall \beta < \alpha,\ \operatorname{rk} x \geq \beta.\end{aligned}$$

If there exists an ordinal α such that $\operatorname{rk} x \geq \alpha$ and $\operatorname{rk} x \ngeq \alpha + 1$ then $\operatorname{rk} x = \alpha$ otherwise $\operatorname{rk} x = +\infty$. The **cofoundation rank** is the foundation rank with respect to the reverse order, that is:

$$\forall x \in E,\ \operatorname{cork} x = \operatorname{rk} x^*$$

(Co)dimension and lattices. For every element a and every prime filter $\mathfrak{p}$ of a distributive bounded lattice L we let:

- $\operatorname{height}\mathfrak{p}$ = the foundation rank of $\mathfrak{p}$ in $\operatorname{Spec} L$ (ordered by inclusion)
- $\operatorname{coheight}\mathfrak{p}$ = the cofoundation rank of $\mathfrak{p}$ in $\operatorname{Spec} L$
- $\dim_L a = \sup\{\operatorname{coheight}\mathfrak{p} \,/\, \mathfrak{p} \in \operatorname{Spec} L,\ a \in \mathfrak{p}\}$
- $\operatorname{codim}_L a = \min\{\operatorname{height}\mathfrak{p} \,/\, \mathfrak{p} \in \operatorname{Spec} L,\ a \in \mathfrak{p}\}$

Here we use the convention that the supremum (resp. minimum) of an empty set of ordinals is $-\infty$ (resp. $+\infty$). Hence $\mathbf{0}$ has codimension $+\infty$ and is the only element of L with dimension $-\infty$. The subscript L is omitted whenever it is clear from the context.

Remark 2.1 The following fundamental (and intuitive) identities follow immediately from the above definitions, and the fact that $F(a \vee b) = F(a) \cup F(b)$:

$$\begin{aligned} \dim a \vee b &= \max(\dim a, \dim b) \\ \operatorname{codim} a \vee b &= \min(\operatorname{codim} a, \operatorname{codim} b) \end{aligned}$$

Finally we define the **dimension of the lattice** L, in notation $\dim L$, as the dimension of $\mathbf{1}_L$. Observe that:

$$\begin{aligned} \dim L \leq d &\iff \forall a \in L,\ \dim a \leq d \\ &\iff \forall a \in L \setminus \{\mathbf{0}\},\ \operatorname{codim} a \leq d \end{aligned}$$

Example 2.2 Consider the lattice[1] $L(k^n)$ of all algebraic varieties in the affine n-space over an algebraically closed field k. The prime filter spectrum of $L(k^n)$ is homeomorphic to the usual spectrum of the ring $k[X_1, \ldots, X_n]$. For any algebraic variety $V \subseteq k^n$, the (co)dimension of V as an element of $L(k^n)$ is nothing but its geometric (co)dimension, that algebraic geometers define in terms of length of chains in $\operatorname{Spec} k[X_1, \ldots, X_n]$. In particular $\dim L(k^n) = \dim k^n = n$.

(Co)rank and the strong order. For every a, b in a distributive bounded lattice L we let $b \ll a$ if and only if $F(b)$ is "much smaller" than $F(a)$, in the sense that $F(b)$ is contained in $F(a)$ and has empty interior inside $F(a)$ (with other words $F(a) \setminus F(b)$ is dense in $F(a)$). This is a definable relation in L:

$$b \ll a \iff \forall c,\ (a \leq b \vee c \Rightarrow a \leq c)$$

This is a strict order on $L \setminus \{\mathbf{0}\}$ (but not on L because $\mathbf{0} \ll \mathbf{0}$). Nevertheless we call it the **strong order** on L. Obviously $b \ll a$ implies that $b < a$ whenever a or b is non-zero. From now on, except if otherwise specified, when we will

[1] Note that this is the lattice of all Zariski closed subsets of k^n hence a co-Heyting algebra, not a Heyting algebra.

speak of the **rank** and **corank** of an element a of $L \setminus \{\mathbf{0}\}$, in notation $\mathrm{rk}_L\, a$ and $\mathrm{cork}_L\, a$, we will refer to the foundation rank and cofoundation rank of a in $L \setminus \{\mathbf{0}\}$ *with respect to the strong order* $\ll$. As usually the subscript L will often be omitted.

Co-Heyting algebras. Let $\mathcal{L}_{\mathrm{HA}^*} = \mathcal{L}_{\mathrm{lat}} \cup \{-\}$ be the language of co-Heyting algebras and $\mathcal{L}_{\mathrm{HA}} = \mathcal{L}_{\mathrm{lat}} \cup \{\rightarrow\}$ the language of Heyting algebras. The additional operations are defined by:

$$a - b = (b^* \rightarrow a^*)^* = \min\{c \,/\, a \le b \vee c\}$$

So the strong order is *quantifier-free* definable in co-Heyting algebras:

$$b \ll a \quad\iff\quad b \le a = a - b$$

Either by dualizing known results on Heyting algebras or by straightforward calculation using Stone-Priestley duality (see footnote 2) the following rules are easily seen to be valid in every co-Heyting algebra:

- $a = (a - b) \vee (a \wedge b)$.
- $(a \vee b) - c = (a - c) \vee (b - c)$.
- $a - (b \vee c) = (a - b) - c$.
- $a - (a - b) = (a \wedge b) - (a - b)$.
- $(a - b) \wedge b \ll a$.

Note in particular that $b \le a$ if and only if $b - a = \mathbf{0}$, and that $a - (a - b) \le b$. We will use these rules in several calculations without further mention.

In a co-Heyting algebra L we denote by $a \bigtriangleup b$ the **topological symmetric difference**[2]:

$$a \bigtriangleup b = (a - b) \vee (b - a) = (a^* \leftrightarrow b^*)^*$$

This is a commutative, non-associative operation. Note that $a \bigtriangleup b = \mathbf{0}$ if and only if $a = b$. Moreover the following "triangle inequality" for $\bigtriangleup$ will be useful:

$$a \bigtriangleup c \le (a \bigtriangleup b) \vee (b \bigtriangleup c)$$

We remind the reader (dualizing basic properties of Heyting algebras) that each ideal I of L defines a congruence $\equiv_I$ on L:

$$a \equiv_I b \iff a \bigtriangleup b \in I$$

So the quotient L/I carries a natural structure of co-Heyting algebra which makes the canonical projection $\pi_I : L \rightarrow L/I$ an $\mathcal{L}_{\mathrm{HA}^*}$-morphism.

[2]Note that $F(a - b)$ is the topological closure of $F(a) \setminus F(b)$ in $\mathrm{Spec}\, L$. So $F(a \bigtriangleup b)$ is the topological closure of the usual symmetric difference $(F(a) \setminus F(b)) \cup (F(b) \setminus F(a))$.

Conversely every congruence $\equiv$ on L is of that kind. Indeed $I_\equiv = \{a \in L \,/\, a \equiv \mathbf{0}\}$ is an ideal of L and $\equiv_{I_\equiv}$ is precisely $\equiv$.

The **kernel** $\operatorname{Ker} f = f^{-1}(\{\mathbf{0}\})$ of any morphism $f : L \to L'$ of co-Heyting algebra is an ideal of L. Given an ideal I of L there is a unique morphism $g : L/I \to L'$ such that $f = g \circ \pi_I$ if and only if $I \subseteq \operatorname{Ker} f$. If f is onto, then so is g. If moreover $I = \operatorname{Ker} f$ then g is an isomorphism and we will identify L/I with L' and f with π_I.

For every ordinal d we set:

$$dL = \{a \in L \,/\, \operatorname{codim} a \geq d\}$$

By remark 2.1 this is an ideal of L. The **generator of** dL, whenever it exists, will be denoted $\varepsilon_d(L)$. The canonical projection $\pi_{dL} : L \to L/dL$ will simply be denoted π_d when the context makes it unambiguous.

Remark 2.3 Given a *surjective* $\mathcal{L}_{\mathrm{HA}^*}$-morphism $\varphi : L \to L'$, if $\varphi^{-1}(dL') = dL$ then there exists a unique isomorphism $d\varphi : L/dL \to L'/dL'$ such that $\pi_{dL'} \circ \varphi = d\varphi \circ \pi_{dL}$. In this situation we will identify L'/dL' with L/dL and say that:

$$\varphi^{-1}(dL') = dL \quad \Longrightarrow \quad L'/dL' = L/dL \ \text{ and } \ \pi_{dL'} \circ \varphi = \pi_{dL}$$

Pseudometric spaces. A map $\delta : X \times X \to \mathbf{R}$ such that for every x, y, z in X, $\delta(x,x) = 0$, $\delta(x,y) = \delta(y,x) \geq 0$ and $\delta(x,z) \leq \delta(x,y) + \delta(y,z)$ (triangle inequality), is called a **pseudometric** on the set X. It is a **metric** if and only if moreover $\delta(x,y) \neq 0$ whenever $x \neq y$. For example, if (Y, δ_Y) is a metric space and $f : X \to Y$ a surjective map then $\delta_Y(f(x), f(y))$ defines a pseudometric on X. Every pseudometric on X is of that kind. Indeed δ induces a metric δ' on the quotient X' of X by the equivalence relation:

$$x \sim y \iff \delta(x,y) = 0$$

Lipschitzian maps between pseudometric spaces are defined as in the metric case. So are the open balls and the topology determined by a pseudometric. Lipschitzian functions are obviously continuous. Note also that a pseudometric is a metric if and only its topology is Hausdorff. So $X/\sim$ defined above is the largest Hausdorff quotient of X.

The **Hausdorff completion** of a pseudometric space X is a complete *metric* space X' together with a continuous map $\iota_X : X \to X'$ such that $\iota_X(X)$ is dense in X', and for every continuous map f from X to a complete metric space X'' there is a unique continuous map $g : X' \to X''$ such that $f = g \circ \iota_X$. Note that if f is λ-Lipschitzian then so is g. The Hausdorff completion of X, which is unique up to isomorphism by the above universal property, is also the completion of the largest Hausdorff quotient of X.

3 Axiomatization

In this section we prove that the (co)dimension and (co)rank coincide, at least when they are finite, in every co-Heyting algebra. One can show that this in not true in every distributive bounded lattices. Only the inequalities of proposition 3.4 below are completely general.

Example 3.1 Even in co-Heyting algebras non finite codimensions and coranks do not coincide in general. Here is a counter-example:

$$\mathbf{0} < x_\omega < \cdots < x_2 < x_1 < x_0 = \mathbf{1}$$

Since this is a chain, it is a co-Heyting algebra L in which $\ll$ coincides with $<$ on $L \setminus \{\mathbf{0}\}$, hence $\operatorname{cork} x_\alpha = \alpha$ for every $\alpha \leq \omega$. On the other hand each element x_α generates a prime filter $\mathfrak{p}_\alpha$. There is only one more prime filter which is $\mathfrak{p} = \{x_\alpha\}_{\alpha<\omega}$. Clearly $\operatorname{height} \mathfrak{p} = \omega$ hence $\operatorname{height} \mathfrak{p}_\omega = \omega + 1$. It follows that:

$$\operatorname{codim} x_\omega = \omega + 1 > \operatorname{cork} x_\omega$$

In this section we will make extensive use of the following facts, proved for example in [Hoc69], theorem 1 and its first corollary. A subset of $\operatorname{Spec} L$ which is a boolean combination of basic closed sets $(F(a))_{a\in L}$ is called a **constructible set** (a patch in [Hoc69]). They form a basis of open sets for another topology on $\operatorname{Spec} L$ usually called the **constructible topology**. Recall that a topological space X is compact if and only if every open cover has a finite subcover.

Fact 3.2 $\operatorname{Spec} L$ *is compact with respect to the constructible topology. Consequently every constructible subset of* $\operatorname{Spec} L$ *is compact with respect to this topology since it is closed in* $\operatorname{Spec} L$.

Fact 3.3 *If a prime filter* $\mathfrak{p}$ *belongs to the closure (with respect to the Zariski topology) of a constructible subset* S *of* $\operatorname{Spec} L$ *then it belongs to the closure of a point of* S*, that is* $\mathfrak{p} \supseteq \mathfrak{q}$ *for some* $\mathfrak{q} \in S$.

Proposition 3.4 *For every nonzero element a in a distributive bounded lattice:*

$$\dim a \geq \operatorname{rk} a \quad \textit{and} \quad \operatorname{codim} a \geq \operatorname{cork} a$$

Proof: By induction on the ordinal α we prove that if $(\text{co})\text{rk}\, a \geq \alpha$ then $(\text{co})\text{dim}\, a \geq \alpha$. This is trivial if $\alpha = 0$ because $a \neq \mathbf{0}$.

Assume $\alpha = \beta+1$. Let $b \ll a$ in $L\setminus\{\mathbf{0}\}$ be such that $\operatorname{rk} b \geq \beta$. The induction hypothesis gives a prime filter $\mathfrak{q}$ of coheight at least β containing b. Then $\mathfrak{q}$ also contains a, and since $b \ll a$, $\mathfrak{q}$ belongs to the Zariski closure of $F(a) \setminus F(b)$. It follows that $\mathfrak{q} \supset \mathfrak{p}$ for some $\mathfrak{p}$ in $F(a)\setminus F(b)$ by fact 3.3. Then $\operatorname{coheight} \mathfrak{p} \geq \beta+1$, hence $\dim a \geq \beta + 1$.

Assume α is a limit ordinal. For every $\beta < \alpha$, $\operatorname{rk} a \geq \beta$ hence $\dim a \geq \beta$ by the induction hypothesis, so $\dim a \geq \alpha$.

We turn now to the codimension. Let b in $L \setminus \{\mathbf{0}\}$ be such that $\operatorname{cork} b \geq \alpha$. Assume that $\alpha = \beta + 1$ and let a in L be such that $b \ll a$ and $\operatorname{cork} a \geq \beta$. Choose any prime filter $\mathfrak{q}$ containing b. Then $\mathfrak{q}$ also contains $a - b$ because $b < a = a - b$. So $\mathfrak{q}$ belongs to the closure of $F(a) \setminus F(b)$ hence to the closure of some $\mathfrak{p}$ in $F(a) \setminus F(b)$ by fact 3.3. By induction hypothesis $\operatorname{codim} a \geq \beta$ hence height $\mathfrak{p} \geq \beta$ and thus height $\mathfrak{q} \geq \beta + 1$. Since this is true for every $\mathfrak{q} \in F(b)$ it follows that $\operatorname{codim} b \geq \beta + 1$.

The limit case is as above.

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Proposition 3.5 *In co-Heyting algebras the dimension coincides with the foundation rank with respect to $\ll$ for every nonzero element.*

Proof: It suffices to prove, by induction on the ordinal α, that if $\dim a \geq \alpha$ then $\operatorname{rk} a \geq \alpha$. This is obvious if $\alpha = 0$ since $a \neq \mathbf{0}$. The limit case is clear as well.

Assume that $\alpha = \beta + 1$, let $\mathfrak{p}$ be a prime filter of coheight at least $\beta + 1$ containing a. Let $\mathfrak{q} \supset \mathfrak{p}$ be a prime filter of coheight β and a' an element of $\mathfrak{q} \setminus \mathfrak{p}$. Then $\mathfrak{p}$ belongs to $F(a) \setminus F(a')$, hence to $F(a - a')$. In other words $a - a'$ belongs to $\mathfrak{p}$, hence to $\mathfrak{q}$. Let $b = a' \wedge (a - a')$, then $b \ll a$ by construction. Moreover $b \in \mathfrak{q}$ hence $\dim b \geq \beta$. By induction hypothesis it follows that $\operatorname{rk} b \geq \beta$, hence $\operatorname{rk} a \geq \beta + 1$.

■

For every element a in a distributive bounded lattice L let $mF(a)$ denote the set of minimal elements of $F(a)$, that is the prime filters which are minimal with respect to the inclusion among those containing a.

Lemma 3.6 *Let L be a co-Heyting algebra and $a, b \in L$.*

$$mF(a - b) = mF(a) \cap F(b)^c = mF(a) \cap F(a - b)$$

So the Zariski and the constructible topologies induce the same topology on $mF(A)$. It follows that $mF(a)$ is a Boolean space, and in particular a compact space.

Proof: The two last statements follow immediately from the second equality, so let us prove these two equalities.

We already mentioned in footnote 2 that $F(a - b) = \overline{F(a) \setminus F(b)}$, where the line stands for the Zariski closure in $\operatorname{Spec} L$. The set of minimal elements of $F(a) \setminus F(b)$ is clearly $mF(a) \setminus F(b)$. So by fact 3.3:

$$mF(a - b) = mF(a) \setminus F(b)$$

This proves the first equality. It implies that $mF(a - b) \subseteq mF(a)$ hence:

$$mF(a - b) \subseteq mF(a) \cap F(a - b)$$

Conversely every element of $F(a - b)$ which is minimal in $F(a)$ is *a fortiori* minimal in $F(a - b)$ because $F(a - b) \subseteq F(a)$. So the second equality is proved.

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Proposition 3.7 *Let a be any nonzero element of a co-Heyting algebra L, let $\mathfrak{p}$ a prime filter of L and n a positive integer.*

1. *If* height $\mathfrak{p} \geq n$ *then* $\mathfrak{p}$ *contains an element of codimension at least* n.
2. *If* $\operatorname{codim} a \geq n$ *then* $\operatorname{cork} a \geq n$.

Proof: If $n = 0$ the first statement is trivial. Assume that it has been proved for $n-1$ with $n \geq 1$. Let $\mathfrak{p}' \subset \mathfrak{p}$ be such that height $\mathfrak{p}' \geq n-1$. The induction hypothesis gives $b \in \mathfrak{p}'$ such that $\operatorname{codim} b \geq n-1$. For any $\mathfrak{q} \in mF(b)$, $\mathfrak{p} \not\subseteq \mathfrak{q}$ so we can choose $b_\mathfrak{q} \in \mathfrak{p} \setminus \mathfrak{q}$. The intersection of all the $F(b_\mathfrak{q})$'s with $mF(b)$ is empty. By lemma 3.6, $mF(b)$ is compact hence there exists a finite subfamily $(F(b_{\mathfrak{q}_i}))_{i \leq r}$ whose intersection with $mF(b)$ is empty. Let:

$$a = b \wedge \bigwedge_{i \leq r} b_{\mathfrak{q}_i}$$

By construction $a \in \mathfrak{p}$, $a < b$ and:

$$mF(b) \setminus F(a) \supseteq mF(b) \setminus \bigcap_{i \leq r} F(b_{\mathfrak{q}_i}) = mF(b)$$

So $mF(b) \setminus F(a) = mF(b)$, but $mF(b-a) = mF(b) \setminus F(a)$ by lemma 3.6, so we have proved that $mF(b-a) = mF(b)$. Hence $F(b-a) = F(a)$ by fact 3.3, that is $b - a = b$. It follows that $a \ll b$ hence $\operatorname{cork} b \geq \operatorname{cork} a + 1 = n$.

The second statement is trivial as well if $n = 0$. So let us assume that $n \geq 1$ and the result is proved for $n-1$. For every $\mathfrak{p} \in mF(a)$, height $\mathfrak{p} \geq n$ so we can choose a prime filter $\mathfrak{q} \subset \mathfrak{p}$ such that height $\mathfrak{q} \geq n-1$. The previous point then gives an element $a_\mathfrak{p} \in \mathfrak{q}$ such that $\operatorname{codim} a_\mathfrak{p} \geq n-1$. By construction $a \notin \mathfrak{q}$ because $\mathfrak{p}$ is minimal in $F(a)$, hence $a_\mathfrak{p} - a \in \mathfrak{q}$ and *a fortiori* $a_\mathfrak{p} - a \in \mathfrak{p}$. So $mF(a)$ is covered by $\big(F(a_\mathfrak{p} - a)\big)_{\mathfrak{p} \in mF(a)}$. This is an open cover for the constructible topology, and $mF(a)$ is compact for this topology by lemma 3.6, so there is a finite subfamily $(F(a_{\mathfrak{p}_i} - a))_{i \leq r}$ which covers $mF(a)$. Let $b = \bigvee_{i \leq r}(a_{\mathfrak{p}_i} - a)$. By construction $mF(a)$ is contained in $F(b)$ hence $a \leq b$, and moreover:

$$b - a = \bigvee_{i \leq r}(a_{\mathfrak{p}_i} - a) - a = \bigvee_{i \leq r} a_{\mathfrak{p}_i} - a = b$$

That is $a \ll b$, so $\operatorname{cork} a \geq \operatorname{cork} b + 1$. Finally:

$$\operatorname{codim} b = \min_{i \leq r} \operatorname{codim}(a_{\mathfrak{p}_i} - a) \geq \min_{i \leq r} \operatorname{codim} a_{\mathfrak{p}_i} \geq n - 1$$

By induction hypothesis it follows that $\operatorname{cork} b \geq n-1$, hence $\operatorname{cork} a \geq n$.
∎

Once put together, propositions 3.4, 3.5 and 3.7 imply that $\dim a = \operatorname{rk} a$, and that $\operatorname{codim} a = \operatorname{cork} a$ whenever $\operatorname{cork} a$ is finite, for every non zero element a in a co-Heyting algebra L. This result is the corner stone of this paper. Indeed the (co)dimension has geometrically intuitive properties (remark 2.1) that the (co)rank seems to be lacking. On the other hand the definition of the (co)dimension is not first-order, while the (co)rank is defined only in terms of the strong order which is first order definable. When both coincide the best of the two notions can be put together. Let us emphasize this coincidence.

Theorem 3.8 *For every co-Heyting algebra L, every element a of L and every positive integer d:*

$$\begin{aligned} \dim a \geq d &\iff \exists x_0, \dots, x_d,\ \mathbf{0} \neq x_d \ll \cdots \ll x_0 \leq a \\ \operatorname{codim} a \geq d &\iff \exists x_0, \dots, x_d,\ a \leq x_d \ll \cdots \ll x_0 \end{aligned}$$

In particular[3] *dL is uniformly definable by a positive existential $\mathcal{L}_{\mathrm{HA}^*}$-formula.*

Proof: The two equivalences have already been proved. The last statement follows since $\ll$ is definable by a positive quantifier free formula: $b \ll a$ iff $b \leq a \bigwedge a - b = a$.

■

Corollary 3.9 *Let $\varphi : L \to L'$ be an $\mathcal{L}_{\mathrm{HA}^*}$-morphism and d a positive integer.*

1. *$\varphi(dL) \subseteq dL'$.*
2. *If φ is surjective then:*
   (a) *$\varphi(dL) = dL'$.*
   (b) $\dim L' < d \iff dL \subseteq \operatorname{Ker} \varphi$
   (c) $\operatorname{Ker} \varphi \subseteq dL \iff \varphi^{-1}(dL') = dL$.
3. *If φ is surjective and dL is principal then dL' is principal and $\varphi(\varepsilon_d(L)) = \varepsilon_d(L')$.*

Proof: (1) By theorem 3.8, dL and dL' are both defined by the same positive existential $\mathcal{L}_{\mathrm{HA}^*}$-formula, hence $\varphi(dL) \subseteq dL'$.

(2) For the first point it is sufficient to check that $dL' \subseteq \varphi(dL)$. If $d = 0$ then $dL = L$ and $\varphi(L) = L'$ because φ is surjective. Now assume that $d \geq 1$. For any $b' \in dL' \setminus \{\mathbf{0}\}$ theorem 3.8 gives $a' \in L'$ such that $b' \ll a'$ and $\operatorname{codim} a' \geq d - 1$. Let $a, b \in L$ be such that $\varphi(a) = a'$ and $\varphi(b) = b'$. By induction hypothesis a can be chosen in $(d-1)L$. Lastly let $x = a - b$ and $y = x \wedge b$. Then:
$\varphi(x) = \varphi(a) - \varphi(b) = a' - b' = a'$.
$\varphi(y) = \varphi(x) \wedge \varphi(b) = a' \wedge b' = b'$.
$x - y = x - (x \wedge b) = x - b = (a - b) - b = a - b = x$.

Moreover $y \leq x$ so $y \ll x$. Note that $x, y \neq \mathbf{0}$ since their respective images are non zero. By theorem 3.8 again it follows that $\operatorname{codim} y > \operatorname{codim} x \geq \operatorname{codim} a$ hence $y \in dL$.

Equivalence (2b) follows since $\dim L' < d$ if and only if every non zero element of L' has codimension at most $d - 1$, that is $dL' = \{\mathbf{0}\}$.

Finally $dL = \varphi^{-1}(\varphi(dL))$ if and only if $\operatorname{Ker} \varphi \subseteq dL$. But $\varphi(dL) = dL'$ so we are done.

(3) We already know that $\varphi(\varepsilon_d(L)) \in dL'$. For any $a' \in dL'$ let $a \in dL$ such that $\varphi(a) = a'$. Then $a \leq \varepsilon_d(L)$ hence $a' \leq \varphi(\varepsilon_d(L))$.

■

[3] Recall that we defined $dL = \{a \in L \,/\, \operatorname{codim} a \geq d\}$.

Remark 3.10 Corollary 3.9(2) implies that $\dim L/dL < d$ for every positive integer d and every co-Heyting algebra L.

Corollary 3.11 *Let L be a co-Heyting algebra such that dL and $(d+1)L$ are principal for some d. Then $\varepsilon_{d+1}(L) \ll \varepsilon_d(L)$.*

Proof: If $\varepsilon_{d+1}(L) = \mathbf{0}$ this is obvious. Otherwise by theorem 3.8 there an element a in L such that $\varepsilon_{d+1}(L) \ll a$ and $\operatorname{codim} a = d$. Then $a \leq \varepsilon_d(L)$ by definition hence $\varepsilon_{d+1}(L) \ll \varepsilon_d(L)$.
■

Codimension and slices

The dimension of a co-Heyting algebra should be a familiar notion to the specialists in Heyting algebras, since it coincides after dualisation with the notion of "slice", which can be defined as follows. Let $P_n(x_1, \dots, x_n)$ be a term defined inductively by $P_0 = \mathbf{1}$ and:

$$P_{n+1} = (P_n - x_{n+1}) \wedge x_{n+1}$$

Let $\mathcal{S}_n$ denote the variety of co-Heyting algebras satisfying the equation $P_n = \mathbf{0}$, and $\mathcal{S}_n^*$ the corresponding variety of Heyting algebras. The variety $\mathcal{S}_n^*$ appears for example in [Kom75]. The above axiomatization is mentioned in [Bez01]. It is folklore that a Heyting algebra L^* belongs to $\mathcal{S}_n^*$ if and only if its prime filter spectrum does not contain any chain of length n, or equivalently is prime ideal spectrum has this property. So dually L belongs to $\mathcal{S}_n$ if and only if its prime filter spectrum does not contain any chain of length n, that is $\dim L < n$. For lack of a reference, we give here an elementary proof.

Proposition 3.12 *A co-Heyting algebra L has dimension $\leq d$ if and only if it belongs to the $\mathcal{S}_{d+1}$.*

Proof: We mentioned in section 2 that $(a - b) \wedge b \ll b$ for every $a, b \in L$. Then for every $a_1, \dots, a_{d+1} \in L$:

$$P_{d+1}(a_1, \dots, a_{d+1}) \ll P_d(a_1, \dots, a_d) \ll \cdots \ll P_1(a_1) \ll \mathbf{1}$$

So if L does not belong to $\mathcal{S}_{d+1}$ there is a tuple a in L^{d+1} such that $P_{d+1}(a) \neq \mathbf{0}$. Then by the above property (and theorem 3.8) $\dim L = \dim_L \mathbf{1} \geq d + 1$.

Conversely if $\dim L \geq d + 1$ there are $b_1, \dots, b_{d+1} \in L$ such that:

$$\mathbf{0} \neq b_{d+1} \ll b_d \ll \cdots \ll b_1 \ll \mathbf{1}$$

Then $\mathbf{1} - b_1 = \mathbf{1}$ hence $P_1(b_1) = b_1$, and inductively $P_{d+1}(b_1, \dots, b_{d+1}) = b_{d+1}$. Since $b_{d+1} \neq \mathbf{0}$ it follows that L does not belong to $\mathcal{S}_{d+1}$.
■

4 Pseudometric induced by the codimension

The "triangle inequality" for $\triangle$ (see section 2) and the fundamental property of the codimension (see remark 2.1) prove that the codimension defines a **pseudometric δ_L on L** as follows:

$$\delta_L(a,b) = \begin{cases} 2^{-\operatorname{codim} a \triangle b} & \text{if } \operatorname{codim} a \triangle b < \omega, \\ 0 & \text{otherwise.} \end{cases}$$

As usually the index L will be omitted whenever it is clear from the context. The topology determined by this pseudometric will be called the **codimetric topology**. In the remaining of this paper, every metric or topological notion, when applied to a co-Heyting algebra, will refer to its pseudometric, except if otherwise specified.

Note that ωL is the topological closure of $\{\mathbf{0}\}$ and that a basis of neighborhood for any $x \in L$ is given, as d ranges over the positive integers, by:

$$U(x,d) = \{y \in L \big/ x \triangle y \in dL\} \tag{1}$$

It follows that L is a **Hausdorff co-Heyting algebra** (with other words its codimetric topology is Hausdorff, or equivalently δ_L is a metric) if and only if $\omega L = \{\mathbf{0}\}$, that is if every non-zero element of L has finite codimension. Note that the largest Hausdorff quotient of L (as a pseudometric space) is exactly $L/\omega L$.

A pseudometric space is called precompact if and only if its Hausdorff completion is compact. It will be shown in section 7 that L is a **precompact co-Heyting algebra** if and only if L/dL is finite for every positive integer d (corollary 7.5). Until then we simply take this characterisation as a definition.

Remark 4.1 If L has finite dimension d then $(d+1)L = \{\mathbf{0}\}$ (see section 2) hence the codimetric topology boils down to the discrete topology. In particular for every co-Heyting algebra L, the codimetric topology in L/dL is discrete (see remark 3.10).

Proposition 4.2 *Every $\mathcal{L}_{\mathrm{HA}^*}$-morphism $\varphi : L \to L'$ is 1-Lipschitzian. In particular φ is continuous.*

Proof: For every positive integer d such that $\delta(a,b) \leq 2^{-d}$ we have by corollary 3.9(1):

$$a \triangle b \in dL \Rightarrow \varphi(a) \triangle \varphi(b) = \varphi(a \triangle b) \in dL'$$

that is $\delta(\varphi(a), \varphi(b)) \leq 2^{-d}$.

∎

We extend δ_L to L^n by setting:

$$\delta_L\big((a_1, \dots, a_n), (b_1, \dots, b_n)\big) = \max_{1 \leq i \leq n} \delta_L(a_i, b_i)$$

This is again a pseudometric on L^n. Clearly the topology that it defines on L^n is the product topology of the codimetric topology of L.

Proposition 4.3 *The function $t : L^n \to L$ defined in the obvious way by an arbitrary $\mathcal{L}_{\mathrm{HA}^*}$-term $t(x)$ with n variables (and parameters in L) is 1-Lipshitzian. As a consequence if L is Hausdorff then the set of solutions of any system of equations (with parameters in L) is closed.*

Proof: For every $a, b \in L^n$ and every positive integer d, $\delta(a,b) \leq 2^{-d}$ if and only if $\pi_d^n(a) = \pi_d^n(b)$, where $\pi_d^n : L^n \to (L/dL)^n$ is the product map induced by π_d in the obvious way. In this case:

$$\pi_d(t(a)) = t(\pi_d^n(a)) = t(\pi_d^n(b)) = \pi_d(t(b))$$

Hence $\pi_d(t(a) \bigtriangleup t(b)) = \pi_d(t(a)) \bigtriangleup \pi_d(t(b)) = \mathbf{0}$ that is $\delta(t(a), t(b)) \leq 2^{-d}$. This proves the first point.

Now given any set $(t_i)_{i \in I}$ of $\mathcal{L}_{\mathrm{HA}^*}$-terms with n variables (and parameters in L):

$$\{a \in L^n \,/\, \forall i \in I,\ t_i(a) = \mathbf{0}\} = \bigcap_{i \in I} t_i^{-1}(\{\mathbf{0}\})$$

If L is Hausdorff then $\{\mathbf{0}\}$ is closed. So each $t_i^{-1}(\{\mathbf{0}\})$ is closed by continuity of t_i hence so is their intersection.

■

Proposition 4.4 *The quotient of a Hausdorff co-Heyting algebra L by an ideal I is Hausdorff if and only if I is closed. In particular the quotient of any Hausdorff co-Heyting algebra by a principal ideal is Hausdorff.*

Note that closed ideals need not to be principal, see example 5.6.

Proof: Let $\pi : L \to L/I$ denote that canonical projection. If the codimetric topology on L/I is Hausdorff then $\{\mathbf{0}_{L/I}\}$ is closed hence $I = \pi^{-1}(\{\mathbf{0}_{L/I}\})$ is closed because π is continuous.

Conversely if the codimetric topology on L/I is not Hausdorff then there exists a non zero element $a' \in L/I$ whose codimension is not finite. Let $a \in L$ such that $\pi(a) = a'$. Note that $a \notin I$ because $a' \neq \mathbf{0}$. For every d, corollary 3.9(2) gives an $a_d \in dL$ such that $\pi(a_d) = a'$. The sequence $(a_d)_{d<\omega}$ is convergent to $\mathbf{0}$ hence $a_d \bigtriangleup a$ is convergent to a. But $a_d \bigtriangleup a \in I$ for every d since $\pi(a_d) = a' = \pi(a)$ so I is not closed.

The last statement follows since an ideal generated by a single element a is obviously closed: it is the inverse image of the closed set $\{\mathbf{0}\}$ by the continuous map $x \mapsto x - a$.

■

5 The finitely generated case

We prove in this section that finitely generated co-Heyting algebras are precompact, and Hausdorff if moreover they are finitely presented. This mostly a rephrasing of known facts. It can be derived for example from Bellissima's construction [Bel86], see [DJ08]. We provide here a proof using only the most basic properties of Kripke models, and the finite model property.

Given a language $\mathcal{L}$ and a set Var of variables, an $\mathcal{L}$-term whose variables belong to Var is called an $\mathcal{L}(\mathrm{Var})$-term. Remember that Heyting algebras are the algebraic models of IPC, the intuitionistic propositional calculus. So $\mathcal{L}_{\mathrm{HA}}(\mathrm{Var})$-terms are nothing but formulas of IPC with propositional variables in Var, the function symbols of $\mathcal{L}_{\mathrm{HA}}$ being interpreted as logical connectives in the obvious way, and the constant symbols $\mathbf{0}$, $\mathbf{1}$ as $\perp$, $\top$ respectively.

A **Kripke model** is a map $u : P \to \mathcal{P}(\mathrm{Var})$ where Var is a set of variables, P is an ordered set, and u obeys the following monotonicity condition[4]:

$$q \leq p \implies u(q) \supseteq u(p)$$

The Kripke model $u : P \to \mathcal{P}(\mathrm{Var})$ is **finite** if P is a finite set. An **isomorphism** with another Kripke model $u' : P' \to \mathcal{P}(\mathrm{Var}')$ is an order preserving bijection $\sigma : P \to P'$ such that $u = u' \circ \sigma$. The notion of an $\mathcal{L}_{\mathrm{HA}}(\mathrm{Var})$-term (or IPC formula) t being **true at a point** p **in** u, which is denoted $u \Vdash_p t$, is defined by induction on t:

$$\begin{array}{lcl}
u \Vdash_p \top \ \text{ and } \ u \nVdash_p \perp, & & \\
u \Vdash_p x & \iff & x \in u(p), \text{ for } x \in \mathrm{Var}, \\
u \Vdash t_1 \wedge t_2 & \iff & u \Vdash t_1 \text{ and } u \Vdash t_2, \\
u \Vdash t_1 \vee t_2 & \iff & u \Vdash t_1 \text{ or } u \Vdash t_2, \\
u \Vdash t_1 \to t_2 & \iff & \forall q \leq p \ \big(u \Vdash t_1 \Rightarrow u \Vdash t_2\big).
\end{array}$$

We denote by $\mathrm{Th}(p, u)$ the **theory of** p **in** u, that is the set of $\mathcal{L}_{\mathrm{HA}}(\mathrm{Var})$-terms true at p in u. If t is true at every point in u we say that t is **true in** u and note it $u \Vdash t$. The set of $\mathcal{L}_{\mathrm{HA}}(\mathrm{Var})$-terms true in u is denoted $\mathrm{Th}(u)$. Here is the fundamental theorem on Kripke models and IPC (see for example [Pop94]):

Theorem 5.1 *Let t be an $\mathcal{L}_{\mathrm{HA}}$-term and* Var *be the (finite) set of its variables. Then the following are equivalent:*

1. *t is a theorem of IPC.*
2. *t is true in every Kripke model $u : P \to \mathcal{P}(\mathrm{Var})$.*
3. *t is true in every finite Kripke model $u : P \to \mathcal{P}(\mathrm{Var})$.*

The classical duality between finite Kripke models and finite Heyting algebras (see for example chapter 1 of [Fit69]) provides an algebraic translation of the finite model property. We need to make a couple of precise observations on this duality, so let us recall it now in detail.

Given a Kripke model $u : P \to \mathcal{P}(\mathrm{Var})$ and an $\mathcal{L}_{\mathrm{HA}}(\mathrm{Var})$-term t we define $u[t] = \{p \in P \,/\, u \Vdash_p t\}$. The monotonic assumption on u implies by an

[4] In the literature the order on P is often reversed. We follow here the convention of [Ghi99] which suits perfectly well to our purpose.

immediate induction that $u[t]$ is a decreasing subset of P. The family $\mathcal{O}(P)$ of decreasing subsets of P is easily seen to be a topology on P, hence a Heyting algebra. Define:

$$G_u = \{u[x] \,/\, x \in \mathrm{Var}\} \quad \text{and} \quad L_u = \{u[t] \,/\, t \in \mathcal{L}_{\mathrm{HA}}(\mathrm{Var})\}$$

One can show that L_u is an $\mathcal{L}_{\mathrm{HA}}$-substructure of $\mathcal{O}(P)$ hence a Heyting algebra again. For any $\mathcal{L}_{\mathrm{HA}}$-term $t(x_1, \ldots, x_n)$ and any elements $u[t_1], \ldots, u[t_n]$ in L_u:

$$t(u[t_1], \ldots, u[t_n]) = u[t(t_1, \ldots, t_n)].$$

In particular $u[t] = t(u[x_1, \ldots, x_n])$ hence G_u is a set of generators of L_u. Moreover $u \Vdash t(x_1, \ldots, x_n)$ if and only if $t(u[x_1], \ldots, u[x_n]) = \mathbf{1}_{L_u}$.

Conversely, given a Heyting algebra L with a set of generator G we can construct a Kripke model as follows. Let P_L be the set of all prime ideals of L, ordered by inclusion[5]. Let Var_G be any set of variables indexed by G. For every prime ideal $\mathfrak{i} \in P_L$ define:

$$u_{L,G}(\mathfrak{i}) = \{x_g \in \mathrm{Var}_G \,/\, g \notin \mathfrak{i}\}$$

Then $u_{L,G} : P_L \to \mathcal{P}(\mathrm{Var}_G)$ is a Kripke model. Moreover for every $\mathcal{L}_{\mathrm{HA}}(\mathrm{Var}_G)$-term $t = t(x_{g_1}, \ldots, x_{g_n})$ and every prime ideal $\mathfrak{i} \in P_L$:

$$u \Vdash_{\mathfrak{i}} t(x_{g_1}, \ldots, x_{g_n}) \iff t(g_1, \ldots, g_n) \notin \mathfrak{i}$$

In particular t is true in u if and only if $t(g_1, \ldots, g_n) = \mathbf{1}_L$.

Obviously a Kripke model u is finite if and only if L_u is finite, and a Heyting algebra L is finite if and only if it has finitely many prime ideals, that is if P_L is finite. So the contraposition of theorem 5.1 translates algebraically as follows:

Fact 5.2 *Let t be an $\mathcal{L}_{\mathrm{HA}}$-term. The formula $\exists x,\ t(x) \neq \mathbf{1}$ has a model (a Heyting algebra L in which $t(a) \neq \mathbf{1}$ for some tuple a in L) if and only if it has a finite model.*

But there is something more. Observe that for any $\mathfrak{i} \in P_{L,G}$:

$$\mathfrak{i} = \{t(g_1, \ldots, g_n) \,/\, t \in \mathrm{Th}(\mathfrak{i}, u_{L,G})\}$$

So any two points in P_L having the same theory in $u_{L,G}$ are equal. A Kripke model having this property will be called **reduced**.

Define the **length of a Kripke model** $u : P \to \mathcal{P}(\mathrm{Var})$ as the maximal length[6] of a chain of elements of P. Fix a finite set of n variables Var and a positive integer d. In a Kripke model $u : P \to \mathcal{P}(\mathrm{Var})$ of length 0 the theory at any point p is determined by $u(p)$. So if u is reduced it can have at most 2^n

[5]Since $\mathfrak{i} \in P_L$ iff $\mathfrak{i}^* \in \operatorname{Spec} L^*$, P_L as an ordered set is nothing but the prime filter spectrum (ordered by inclusion) of the co-Heyting algebra L^*.

[6]More exactly the length of $u : P \to \mathcal{P}(\mathrm{Var})$, or simply the length of P, is the smallest ordinal α such that every element of P has foundation rank $\leq \alpha$, if such an ordinal exists, and $+\infty$ otherwise.

points. Consequently there exists only finitely many non isomorphic reduced Kripke models of length 0.

Assume that for some positive integers d, ν we have proved that there exists at most ν non isomorphic reduced Kripke models of length at most d. Consider a reduced Kripke model $u : P \to \mathcal{P}(\mathrm{Var})$ of length at most $d+1$. For every point p of rank $d+1$ the restriction $u_{|p\Downarrow}$ of u to $p\Downarrow$ is a reduced Kripke model of length at most d. If q is another element of rank $d+1$ such that $u_{|q\Downarrow}$ and $u_{|p\Downarrow}$ are isomorphic then $u(p) \neq u(q)$, otherwise a straightforward induction would show that $\mathrm{Th}(p, u) = \mathrm{Th}(q, u)$. So u has at most $2^n\nu$ points of rank $d+1$. Consequently there exists only finitely many non isomorphic reduced Kripke models of length at most $d+1$.

Let us say that two $\mathcal{L}_{\mathrm{HA}}$-terms t_1, t_2 with variables in some finite set Var are ***d*-equivalent** if they are true in exactly the same reduced Kripke model $u : P \to \mathcal{P}(\mathrm{Var})$ of length at most d. By the above induction there exists a finite number μ of non isomorphic such models, hence at most 2^μ different classes of d-equivalence. Let us stress this:

Fact 5.3 *For every positive integers n, d there exists finitely many d-equivalence classes of $\mathcal{L}_{\mathrm{HA}}$-terms with n variables.*

We can return now to co-Heyting algebras. Let us say that a variety $\mathcal{V}$ (in the sense of universal algebra) of co-Heyting algebras has the **finite model property** iff for every $\mathcal{L}_{\mathrm{HA}^*}$-term $t(x)$, if there exists an algebra L in $\mathcal{V}$ such that $\exists x,\ t(x) \neq \mathbf{0}$ holds in L then there exists a finite algebra in $\mathcal{V}$ having this property. So fact 5.2 asserts that the variety of all co-Heyting algebras has the finite model property.

Proposition 5.4 *Let $\mathcal{V}$ be a variety of co-Heyting algebras. The following are equivalent:*

1. *$\mathcal{V}$ has the finite model property.*
2. *Every algebra free in $\mathcal{V}$ is residually finite[7].*
3. *Every algebra free in $\mathcal{V}$ is Hausdorff.*
4. *Every algebra finitely presented in $\mathcal{V}$ is precompact Hausdorff.*

Proof: (1)⇒(2) Let $\mathcal{F}$ be an algebra free in $\mathcal{V}$. Every non zero element of $\mathcal{F}$ can be written as $t(X)$ for some $\mathcal{L}_{\mathrm{HA}^*}$-term $t(x)$ and some finite subset X of the free generators of $\mathcal{F}$. Since $\mathcal{V}$ has the finite model property there exists a finite algebra L' in $\mathcal{V}$ such that $t(a') \neq \mathbf{0}$ for some tuple a' of elements of L'. Let $\varphi : \mathcal{F} \to L'$ be the unique $\mathcal{L}_{\mathrm{HA}^*}$-morphism which maps X onto a' and the other generators of $\mathcal{F}$ to $\mathbf{0}$. Then $I = \operatorname{Ker}\varphi$ is an ideal of $\mathcal{F}$ not containing $t(X)$ such that $\mathcal{F}/I$ is finite.

[7] A co-Heyting algebra L is residually finite if for every non zero element a there is an ideal I not containing a such that L/I is finite.

(2)⇒(3) Let $\mathcal{F}$ be an algebra free in $\mathcal{V}$ and $t(X)$ a non zero element of $\mathcal{F}$. The assumption (2) gives an ideal I of $\mathcal{F}$ not containing $t(X)$ such that $\mathcal{F}/I$ is finite. Then $\mathcal{F}/I$ has finite dimension, say d. By corollary 3.9(2) it follows that $(d+1)\mathcal{F}$ is contained in $\operatorname{Ker}\varphi$. So $t(X) \notin (d+1)L$ that is $\operatorname{codim} t(X) \leq d$ is finite as required.

(3)⇒(4) By proposition 4.4 it is sufficient to show that every free Heyting algebra $\mathcal{F}$ with a finite set of generators X is precompact. Let $t_1(X), t_2(X)$ be any two elements of $\mathcal{F}$ having different images in $\mathcal{F}/d\mathcal{F}$. Let Z be the image of X in $\mathcal{F}/d\mathcal{F}$ and $Z^* = \{z^*\}_{z\in Z}$ its image in the dual $(\mathcal{F}/d\mathcal{F})^*$. By assumption $t_1(Z) \neq t_2(Z)$ *ie.* $t_1(Z) \bigtriangleup t_2(Z) \neq \mathbf{0}$. Dualizing:

$$(\mathcal{F}/d\mathcal{F})^* \models t_1^*(Z^*) \leftrightarrow t_2^*(Z^*) \neq \mathbf{1}$$

where t_i^* is the $\mathcal{L}_{\mathrm{HA}}$-term obtained from t_i by dualisation. Since $\mathcal{F}/d\mathcal{F}$ has dimension at most d any chain of prime filters of $\mathcal{F}/d\mathcal{F}$ has length at most d. But the prime filters of $\mathcal{F}/d\mathcal{F}$ are exactly the complements of the prime ideals of its dual $(\mathcal{F}/d\mathcal{F})^*$. So the Kripke model $u_{(\mathcal{F}/d\mathcal{F})^*,Z^*} : P_{(\mathcal{F}/d\mathcal{F})^*} \to Z^*$ is a reduced Kripke model of height at most d in which $t_1^* \leftrightarrow t_2^*$ is not true.

This proves that if $t_1(X), t_2(X)$ have different images in $\mathcal{F}/d\mathcal{F}$ then t_1^*, t_2^* are not d-equivalent. By fact 5.3 there is only a finite number of d-equivalence classes of $\mathcal{L}_{\mathrm{HA}}$-terms with variables in the finite set Z^* hence $\mathcal{F}/d\mathcal{F}$ is finite.

(4)⇒(1) Let t be an $\mathcal{L}_{\mathrm{HA}^*}$-term with n variables such that the formula $\exists x,\, t(x) \neq \mathbf{0}$ holds in some algebra L in $\mathcal{V}$. Let $\mathcal{F}$ be a free algebra in $\mathcal{V}$ having an n-tuple X of generators. The assumption on t implies that $t(X) \neq \mathbf{0}$. Since $\mathcal{F}$ is Hausdorff by (4), there is a positive integer d such that $t(X) \notin d\mathcal{F}$ hence the formula $\exists x, t(x) \neq \mathbf{0}$ holds in $\mathcal{F}/d\mathcal{F}$ as well, which is finite by (4).
■

Corollary 5.5 *Every finitely generated co-Heyting algebra is precompact. Every finitely presented co-Heyting algebra is precompact Hausdorff.*

Proof: If I is any ideal of a $\mathcal{V}$-algebra L and $L' = L/I$ then L'/dL' is also the quotient of L/dL by $\pi_d(I)$. So the homomorphic image of any precompact co-Heyting algebra is precompact. Since the variety of all co-Heyting algebras has the finite model property, the result then follows immediately from proposition 5.4.
■

Note that the quotient of a free co-Heyting algebra by any closed ideal is Hausdorff by proposition 4.4, hence a finitely generated co-Heyting algebra can be Hausdorff without being finitely presented.

Example 5.6 Let $\mathcal{F}_n$ be the free co-Heyting algebra with n generators with $n \geq 2$ so that $\mathcal{F}_n \neq \widehat{\mathcal{F}_n}$ (the Hausdorff completion of $\mathcal{F}_n$, see section 6 or the comments after fact 3.6 in [DJ08]). Choose any $\widehat{a}$ in $\widehat{\mathcal{F}_n} \setminus \mathcal{F}_n$. Then $I = \widehat{a}{\downarrow} \cap \mathcal{F}_n$ is a closed ideal of $\mathcal{F}_n$ which is not principal hence $\mathcal{F}_n/I$ is finitely generated and Hausdorff but not finitely presented.

Example 5.7 Let $\mathcal{F}_n$ be as above. For every $n \geq 2$ there are many elements in $\mathcal{F}_n$ which can not be written as the join of finitely many join irreducible elements, such as the meet of any two join irreducible elements (see remark 4.14 in [DJ08]). Given any such element a, the ideal I generated by the join irreducible elements smaller than a is not closed because $a \notin I$ but a belongs to the topological closure of I (here we use that $a = \bigvee\!\!\!\!\bigvee \mathrm{Comp}^{\vee}(a)$, see proposition 6.6). So $\mathcal{F}_n/I$ is finitely generated but not Hausdorff by proposition 4.4.

6 Precompact Hausdorff co-Heyting algebras

We have seen that every finitely presented co-Heyting algebra is precompact Hausdorff, but the latter form a much larger class. It is then remarkable that most of the very nice algebraic properties of finitely presented free Heyting algebras obtained in [DJ08] from [Bel86] actually generalise, after dualisation, to precompact Hausdorff co-Heyting algebras.

Precompactness and profinite completion

Let L be a co-Heyting algebra, d a positive integer and $L' = L/(d+1)L$. Corollary 3.9(2) asserts that $\pi_{dL}^{-1}(dL') = dL$ hence (see remark 2.3) L'/dL' identifies with L/dL and $\pi_{dL'}$ with a surjective map that we denote:

$$\pi_{d,d+1} : L/(d+1)L \to L/dL$$

Similarly, in order to make the reading easier, we let π_d denote π_{dL} for every positive integer d. So $\{\pi_{d,d+1} : L/(d+1)L \to L/dL\}_{d<\omega}$ is a projective system and the following diagram is commutative:

$$\begin{tikzcd}
 & & & & L \arrow[dlll, "\pi_{d+1}"'] \arrow[dll, "\pi_d"'] \arrow[d, "\pi_0"] \\
\cdots \arrow[r] & L/(d+1)L \arrow[r, "\pi_{d,d+1}"'] & L/dL \arrow[r] & \cdots \arrow[r] & L/0L
\end{tikzcd}$$

We denote by $\widehat{L}$ its projective limit. Note that the canonical map from L to $\widehat{L}$ is an embedding if and only if L is Hausdorff. The codimetric topology on each L/dL is the discrete topology. We equip $\widehat{L}$ with the corresponding projective topology. As a projective limit of Hausdorff topologies, this topology on $\widehat{L}$ is Hausdorff and the image of L in $\widehat{L}$ is dense in $\widehat{L}$. It will be shown in section 7 that $\widehat{L}$ is nothing but the Hausdorff completion of L. However, when L is precompact Hausdorff, the proof that we provide below is much simpler.

Assume that L is precompact Hausdorff. Then the projective topology on $\widehat{L}$ is profinite hence compact Hausdorff. We refer the reader to any book of topology for this and the following classical results on projective limits of topological spaces. We identify L with its image in $\widehat{L}$ *via* the diagonal embedding. We denote by $\overline{\pi}_d$ (resp. $\widehat{\pi}_d$) the canonical projection of $\widehat{L}$ onto L/dL (resp. $\widehat{L}/d\widehat{L}$). Obviously π_d is the restriction of $\overline{\pi}_d$ to L and the kernel of $\overline{\pi}_d$ is the topological closure $\overline{dL}$ of dL in $\widehat{L}$.

$$\begin{tikzcd} L \arrow[rr] \arrow[d, "\pi_d"'] & & \widehat{L} \arrow[d, "\widehat{\pi}_d"] \arrow[dll, "\overline{\pi}_d"'] \\ L/dL & & \widehat{L}/d\widehat{L} \end{tikzcd}$$

Theorem 6.1 *Let L be a Hausdorff precompact co-Heyting algebra. Then for every positive integer d:*

1. *dL and $d\widehat{L}$ are principal*[8] *and $\varepsilon_d(L) = \varepsilon_d(\widehat{L})$;*
2. *$\overline{dL} = d\widehat{L}$ and $\widehat{L}/d\widehat{L}$ identifies with L/dL.*

As a consequence the projective topology on $\widehat{L}$ coincides with its codimetric topology and $(\widehat{L}, \delta_{\widehat{L}})$ is the completion of the metric space (L, δ_L).

We first need a lemma. Recall that $\widehat{L}$ can be represented as:

$$\widehat{L} = \left\{ (x_k)_{k<\omega} \,\big/\, \forall k,\ x_k \in L/kL \text{ and } \pi_{k,k+1}(x_{k+1}) = x_k \right\}$$

Note that if L is precompact then for every positive integer d and every k, $d(L/kL)$ is obviously principal because L/kL is finite. Moreover $\big(\varepsilon_d(L/kL)\big)_{k<\omega}$ belongs to $\widehat{L}$ by corollary 3.9(3), using the above representation of $\widehat{L}$. Let us denote by $\widehat{\varepsilon}_d$ this element of $\widehat{L}$. Note that $\widehat{\varepsilon}_{d+1} \ll \widehat{\varepsilon}_d$ because $\varepsilon_{d+1}(L/kL) \ll \varepsilon_d(L/kL)$ for every k by corollary 3.11. A basis of neighborhood of any element $x \in \widehat{L}$ is given as d ranges over the positive integers, by[9]:

$$B(x,d) = \{ y \in \widehat{L} \,/\, x \triangle y \leq \widehat{\varepsilon}_d \}$$

In particular $\{\widehat{\varepsilon}_d\!\downarrow\}_{d<\omega}$ is a basis of neighborhood of $\mathbf{0}$ in $\widehat{L}$.

Lemma 6.2 *Let L be a Hausdorff precompact co-Heyting algebra. Then an element $x \in \widehat{L}$ is isolated (with respect to the projective topology) if and only if $\widehat{\varepsilon}_d \leq x$ for some d. In this case $x \in L$. In particular $\widehat{\varepsilon}_d \in L$ for every d.*

Proof: Let $x = (x_k)_{k<\omega}$ be any element of $\widehat{L}$. If x is isolated in $\widehat{L}$ then for some integer d we have $B(x,d) = \{x\}$. On the other hand $x \triangle (x \vee \widehat{\varepsilon}_d) = \widehat{\varepsilon}_d - x \leq \widehat{\varepsilon}_d$, that is $x \vee \widehat{\varepsilon}_d \in B(x,d)$ so we are done.

[8] So dL is principal for every finitely presented co-Heyting algebra L, by corollary 5.5. This is actually true for finitely generated co-Heyting algebras, as we will show in section 8.

[9] We simply use here that in each L/kL, a basis of neighborhood of x_k with respect to the discrete/codimetric topology is given by $\{U(x_k,d)\}_{d<\omega}$ (see (1) in section 4).

Conversely assume that $x = x \vee \widehat{\varepsilon}_d$ for some d. Then $\widehat{\varepsilon}_{d+1} \ll \widehat{\varepsilon}_d \leq x$ hence $x - \widehat{\varepsilon}_{d+1} = x$. For every $y \in B(x, d+1)$ we get:

$$x = x - \widehat{\varepsilon}_{d+1} = \big[(x \wedge y) \vee (x - y)\big] - \widehat{\varepsilon}_{d+1} = (x \wedge y) - \widehat{\varepsilon}_{d+1} \leq y$$

And:

$$y = (x \wedge y) \vee (y - x) \leq x \vee \widehat{\varepsilon}_{d+1} = x$$

This proves that $B(x, d+1) = \{x\}$.

The last assertion follows because L is dense in $\widehat{L}$ for the projective topology, and an isolated point in a topological space obviously belongs to every dense subspace.

■

We can now achieve the proof of theorem 6.1.

Proof: For every positive integer d we have $dL \subseteq d\widehat{L}$ by corollary 3.9(1) because the inclusion is an $\mathcal{L}_{\mathrm{HA}^*}$-morphism. Moreover $d\widehat{L} \subseteq \overline{dL}$ by corollary 3.9(2) because $\dim L/dL < d$ and $\overline{dL} = \operatorname{Ker} \overline{\pi}_d$.

By construction the ideal generated in $\widehat{L}$ by $\widehat{\varepsilon}_d$ is precisely $\overline{dL}$. By lemma 6.2 $\widehat{\varepsilon}_d$ actually belongs to L. Moreover it belongs to dL because:

$$\widehat{\varepsilon}_d \ll \cdots \ll \widehat{\varepsilon}_0 = \mathbf{1}$$

Since $dL \subseteq d\widehat{L} \subseteq \overline{dL}$ it immediately follows that $d\widehat{L} = \overline{dL}$ hence $\varepsilon_d(\widehat{L}) = \widehat{\varepsilon}_d$. Moreover $dL = \overline{dL} \cap L$ hence $dL = \widehat{L} \cap L$. We conclude that $\varepsilon_d(L) = \widehat{\varepsilon}_d$.

The identification of $\widehat{L}/\widehat{dL}$ with L/dL follows since $\widehat{\pi}_d$ and $\overline{\pi}_d$ have the same kernel.

We have proved that $d\widehat{L} = \widehat{\varepsilon}_d{\downarrow}$ hence $B(x, d) = U(x, d)$ (see (1) in section 4) for every positive integer d and every $x \in \widehat{L}$. As a consequence the projective topology on $\widehat{L}$ coincide with its codimetric topology. Since $\widehat{L}$ is compact, it is complete, and since L is dense in $\widehat{L}$ the last statement follows.

■

Join irreducible elements

We denote as follows the sets of **join irreducible**, **completely join irreducible**, **meet irreducible** and **completely meet irreducible** elements respectively:

$$\begin{aligned}
\mathcal{I}^{\vee}(L) &= \{x \in L \setminus \{\mathbf{0}\} \,/\, \forall a, b \in L,\ x \leq a \vee b \Rightarrow x \leq a \text{ or } x \leq b\} \\
\mathcal{I}^{!\vee}(L) &= \{x \in L \setminus \{\mathbf{0}\} \,/\, \forall A \subseteq L,\ x \leq \textstyle\bigvee A \Rightarrow \exists a \in A,\ x \leq a\} \\
\mathcal{I}^{\wedge}(L) &= \{x \in L \setminus \{\mathbf{1}\} \,/\, \forall a, b \in L,\ a \wedge b \leq x \Rightarrow a \leq x \text{ or } b \leq x\} \\
\mathcal{I}^{!\wedge}(L) &= \{x \in L \setminus \{\mathbf{1}\} \,/\, \forall A \subseteq L,\ \textstyle\bigwedge A \leq x \Rightarrow \exists a \in A,\ a \leq x\}
\end{aligned}$$

Remark 6.3 If x is join irreducible and $x \not\leq y$ then $x - y = x$. Indeed $x \wedge y < x$ and $x = (x - y) \vee (x \wedge y)$, then use the join irreducibility of x. In particular $y \ll x$ whenever $y < x$.

The following lemma is folklore.

Lemma 6.4 *Let ε be any element of a co-Heyting algebra L, let L' be the quotient of L by the ideal $\varepsilon\downarrow$ and let $\pi : L \to L'$ be the canonical projection.*

1. *$\forall a \in L$, $a - \varepsilon = \min \pi^{-1}(\{\pi(a)\})$ and $a \vee \varepsilon = \max \pi^{-1}(\{\pi(a)\})$.*
   So the restrictions of π to $\{a - \varepsilon\}_{a\in L}$ and $\{a \vee \varepsilon\}_{a\in L}$ are one-to-one.
2. *If in addition L' is finite then every prime filter (resp. ideal) of L disjoint from $\varepsilon\downarrow$ (resp. containing $\varepsilon\downarrow$) is generated by a completely join (resp. meet) irreducible element. So π induces a one-to-one order preserving correspondence between the following sets:*

$$\begin{array}{ccc} \mathcal{I}^{\wedge}(L') & \longleftrightarrow & \{x \in \mathcal{I}^{!\wedge}(L) \,/\, \varepsilon \leq x\} \\ \mathcal{I}^{\vee}(L') & \longleftrightarrow & \{x \in \mathcal{I}^{!\vee}(L) \,/\, x \not\leq \varepsilon\} \end{array}$$

Proof: For every x, y in L, $\varphi(y) \leq \varphi(x) \iff y - x \leq \varepsilon$. The first point then follows from straightforward calculations:

$$y - \varepsilon \leq a \iff y \leq x \vee \varepsilon \iff y - x \leq \varepsilon$$

Now assume that L' is finite. Then every prime ideal of L' is generated by a completely meet irreducible element. As a surjective $\mathcal{L}_{\mathrm{HA}^*}$-morphism, π induces a one-to-one order preserving correspondence between the prime ideals of L containing $\varepsilon\downarrow$ (its kernel) and the prime ideals of L' (its image) which preserves inclusions. So it is sufficient to prove that, given an element x' of L' having a unique successor x'^{+}, the ideal $\varphi^{-1}(x'\downarrow)$ is generated by an element having a unique successor. In order to do this let x (resp. a) be any element of L such that $\varphi(x) = x'$ (resp. $\varphi(a) = x'^{+}$). For every $b \in L$ we have:

$$\varphi(b) \in x'\downarrow \iff \varphi(b) \leq \varphi(x) \iff b \leq x \vee \varepsilon$$

So $x \vee \varepsilon$ is the generator of $\varphi^{-1}(x'\downarrow)$. Moreover:

$$\begin{array}{rcl} x \vee \varepsilon < b \vee \varepsilon & \iff & x' = \varphi(x) < \varphi(b) \\ & \iff & x'^{+} = \varphi(a) \leq \varphi(b) \\ & \iff & a \vee \varepsilon \leq b \vee \varepsilon \end{array}$$

So $a \vee \varepsilon$ is the unique successor of $x \vee \varepsilon$ in L.

The case of join irreducible elements is similar: π induces a one-to-one order preserving correspondence between the prime filters disjoint from $\varepsilon\downarrow$ and the prime filters of L'. Given an element $x' \in L' \setminus \{\mathbf{0}\}$ having a unique predecessor x'^{-}, the inverse image by π of $x'\uparrow$ is generated by an element x having a unique predecessor. We take any two elements $x, a \in L$ such that $\pi(x) = x'$ and $\pi(a) = x'^{-}$. The reader may easily check that $x - \varepsilon$ is a generator of $\pi^{-1}(x'\uparrow)$ and $a - \varepsilon$ is its unique predecessor.

■

Remark 6.5 If L is a precompact Hausdorff co-Heyting algebra and d a positive integer then $\operatorname{Ker}\pi_d = \varepsilon_d(L)\downarrow$ by theorem 6.1. Then lemma 6.4 applied to $\varepsilon_d(L)$ tells us that every join (resp. meet) irreducible element of $L \setminus dL$ (resp. of $\varepsilon_d(L)\uparrow$) is completely join (resp. meet) irreducible, and that π_d induces a one-to-one correspondence between the following sets:

$$\begin{array}{rcl} \mathcal{I}^{\vee}(L/dL) & \longleftrightarrow & \mathcal{I}^{!\vee}(L) \setminus dL \\ \mathcal{I}^{\wedge}(L/dL) & \longleftrightarrow & \mathcal{I}^{!\wedge}(L) \cap \varepsilon_d(L)\uparrow \end{array}$$

These sets are finite, in particular there are finitely many completely join irreducible elements in L of any given finite codimension.

Given an element $a \in L$ the maximal elements of $\mathcal{I}^{\vee}(L) \cap a\downarrow$, if they exist, are called the **join irreducible components** of a in L. The set of join irreducible components of a is denoted $\operatorname{Comp}^{\vee}_L(a)$. As usually the index L is often omitted. The **meet irreducible components** of a in L and the set $\operatorname{Comp}^{\wedge}_L(a)$ are defined dually.

Proposition 6.6 *Let L be a precompact Hausdorff co-Heyting algebra.*

1. *L and $\widehat{L}$ have the same completely join irreducible elements.*
2. *Every join irreducible element of L is completely join irreducible.*
3. *For every $x \in \mathcal{I}^{!\vee}(L)$, the cofoundation rank of x in $\mathcal{I}^{!\vee}(L)$ is finite. It is the codimension of x.*
4. *$\mathcal{I}^{!\vee}(L)$ satisfies the ascending chain condition.*
5. *For every $a \in L$, $a = \bigvee\!\!\!\!\bigvee \operatorname{Comp}^{\vee} a$.*

Proof: Since $L/dL = \widehat{L}/d\widehat{L}$ for every d and $\bigcap_{d<\omega} dL = \{\mathbf{0}\}$, the two first points follow immediately from lemma 6.4 applied to $\varepsilon_d(L)$ (see remark 6.5). For the third point, note simply that it is true in every finite lattice, because every prime filter is generated by a completely join irreducible element, and apply lemma 6.4 with $\varepsilon = \varepsilon_d(L)$ for any d such that $x \not\leq \varepsilon_d(L)$. The ascending chain condition follows: every element in $\mathcal{I}^{!\vee}(L)$ has finite corank because it has finite codimension.

For the last point, fix an element $a \in L \setminus \{\mathbf{0}\}$. For every positive integer d, let:

$$a_d = \bigvee\!\!\!\!\bigvee \{x \in \mathcal{I}^{\vee}(L) \,/\, x \leq a \text{ and } x \not\leq \varepsilon_d(L)\}$$

By lemma 6.4, $a_d = a - \varepsilon_d(L)$ hence by continuity of $x \mapsto a - x$ the sequence $(a_d)_{d<\omega}$ is convergent to a. So a is the complete join of all the join irreducible elements of $L \cap a\downarrow$. These elements are completely join irreducible, hence by the ascending chain condition each of them is smaller than a maximal one, which proves the last point.

■

Meet irreducible elements

The case of meet irreducible elements in a precompact Hausdorff co-Heyting algebra is slightly more complicated. For example they are not always completely irreducible, contrary to the join irreducible elements (see proposition 6.11 below).

In finite distributive lattices there is a correspondence between (completely) join and meet irreducible elements which is defined as follows. For every $x \in L$ let:

$$x^{\vee} = \bigwedge\!\!\!\bigwedge\{y \in L \,/\, y \not\leq x\} \qquad x^{\wedge} = \bigvee\!\!\!\bigvee\{y \in L \,/\, x \not\leq y\}$$

Then $x \in \mathcal{I}^{!\wedge}(L) \Rightarrow x^{\vee} \in \mathcal{I}^{!\vee}(L)$ and symmetrically $x \in \mathcal{I}^{!\vee}(L) \Rightarrow x^{\wedge} \in \mathcal{I}^{!\wedge}(L)$. These two operations are easily seen to define reciprocal, order preserving bijections between $\mathcal{I}^{!\wedge}(L)$ and $\mathcal{I}^{!\vee}(L)$.

This correspondence generalizes to join complete and meet complete lattices which satisfy the infinite distributive laws:

$$x \wedge \bigvee\!\!\!\bigvee_{y \in Y} y = \bigvee\!\!\!\bigvee_{y \in Y} (x \wedge y) \qquad x \vee \bigwedge\!\!\!\bigwedge_{y \in Y} y = \bigwedge\!\!\!\bigwedge_{y \in Y} (x \vee y)$$

In particular it holds for profinite lattices, and we take advantage of this in the following proposition.

Proposition 6.7 *Let L be a precompact Hausdorff co-Heyting algebra.*

1. *L and $\widehat{L}$ have the same completely meet irreducible elements.*
2. *$x \mapsto x^{\vee}$ and $x \mapsto x^{\wedge}$ are well-defined, reciprocal, order preserving bijections between $\mathcal{I}^{!\vee}(L)$ and $\mathcal{I}^{!\wedge}(L)$.*
3. *For every $x \in \mathcal{I}^{!\wedge}(L)$, the cofoundation rank of x in $\mathcal{I}^{!\wedge}(L)$ is finite.*
4. *$\mathcal{I}^{!\wedge}(L)$ satisfies the ascending chain condition.*
5. *Every element $a \in L$ is the complete meet of $\mathcal{I}^{!\wedge}(L) \cap a\!\uparrow$.*

Proof: For every element $\hat{a}$ in $\widehat{L}$ and every positive integer d, lemma 6.4 applied to $\varepsilon_d(\widehat{L})$ shows that $\mathcal{I}^{\wedge}(\widehat{L}) \cap (\hat{a} \vee \varepsilon_d(\widehat{L}))\!\uparrow$ is finite, contained in $\mathcal{I}^{!\wedge}(\widehat{L})$, and its complete meet is equal to $\hat{a} \vee \varepsilon_d(\widehat{L})$. The sequence $\hat{a} \vee \varepsilon_d(\widehat{L})$ is convergent to $\hat{a}$ hence:

$$\hat{a} = \bigwedge\!\!\!\bigwedge\{x \in \mathcal{I}^{\wedge}(\widehat{L}) \,/\, \exists d < \omega,\ \varepsilon_d(\widehat{L}) \leq x\}$$

It follows that if $\hat{a}$ is completely meet irreducible, it must be greater than $\varepsilon_d(\widehat{L})$ for some d, hence it belongs to L by lemma 6.2. Conversely if $a \in L$ is completely meet irreducible in L then by the above equality and lemma 6.2 it must be greater than $\varepsilon_d(L)$ for some d. The filter generated by $\varepsilon_d(L)$ in $\widehat{L}$ is finite by lemma 6.4 and contained in L by lemma 6.2 hence a remains completely meet irreducible in $\widehat{L}$. This proves the first and the last point.

Since $\mathcal{I}^{!\vee}(L) = \mathcal{I}^{!\vee}(\widehat{L})$ and $\mathcal{I}^{!\wedge}(L) = \mathcal{I}^{!\wedge}(\widehat{L})$, L inherits from the profinite lattice $\widehat{L}$ the correspondence between $\mathcal{I}^{!\vee}(L)$ and $\mathcal{I}^{!\wedge}(L)$. This proves the second point, and the remaining points then follow from proposition 6.6.

■

Proposition 6.6 shows that the cofoundation rank of any completely join irreducible inside $\mathcal{I}^{!\vee}(L)$ is equal to its codimension in L. There is a symmetric interpretation for the cofoundation rank in $\mathcal{I}^{!\wedge}(L)$.

Proposition 6.8 *Let L be a precompact Hausdorff co-Heyting algebra, and $x \in \mathcal{I}^{!\wedge}(L)$. Let r be its cofoundation rank in $\mathcal{I}^{!\vee}(L)$. Then:*

$$\dim_{L^*} x^* = r = \operatorname{codim}_L x^\vee$$

Proof: The cofoundation rank of x in $\mathcal{I}^{!\vee}(L)$ is the cofoundation rank of $x^\vee$ in $\mathcal{I}^{!\wedge}(L)$, because the map $y \mapsto y^\vee$ from $\mathcal{I}^{!\wedge}(L)$ to $\mathcal{I}^{!\vee}(L)$ is one-to-one and order preserving. We have seen in proposition 6.6 that the latter is the codimension of $x^\vee$ in L, so the second equality is proved.

Note that the prime filters of L^* are exactly the sets $\mathfrak{i}^*$ where $\mathfrak{i}$ is a prime ideal of L. Since x belongs to $\mathcal{I}^{!\wedge}(L)$, we get that $x^* \in \mathcal{I}^{!\vee}(L^*)$, hence the dimension of x^* in L^* is exactly the height of the prime filter generated by x^* in L^*. Now a prime filter $\mathfrak{i}^*$ of L^* contains x^* if and only if the corresponding prime ideal $\mathfrak{i}$ of L contains x. By proposition 6.7, x is greater than $\varepsilon_d(L)$ for some d, hence $L/(x\downarrow)$ is finite. Then by lemma 6.4 every prime ideal of L containing x is generated by a completely meet irreducible element. Since $x \leq y$ if and only if $x\downarrow \subseteq y\downarrow$, it follows that the height of $(x^*)\uparrow$ in $\operatorname{Spec} L^*$ is exactly the cofoundation rank of x in $\mathcal{I}^{!\wedge}(L)$.

■

Remark 6.9 One may wonder what are $\dim x$ for $x \in \mathcal{I}^{!\vee}(L)$, and $\operatorname{codim}_{L^*} y^*$ for $y \in \mathcal{I}^{!\wedge}(L)$. They do have a good behaviour when L and L^* are finite dimensional. However the special case of $\mathcal{F}_n$, the free co-Heyting algebra with n generators, shows that although $\mathcal{F}_n$ is bi-Heyting, these notions do not provide any significant information, contrary to the codimension. Indeed one can prove that the foundation rank of x in $\mathcal{I}^{!\vee}(\mathcal{F}_n)$ is $+\infty$. The cofoundation rank of y^* in $\mathcal{I}^{!\vee}(\mathcal{F}_n^*)$ is also the foundation rank of y in $\mathcal{I}^{!\wedge}(\mathcal{F}_n)$, which is $+\infty$ as well (see [DJ08], comments after lemma 4.1). It follows that:

$$\dim_{\mathcal{F}_n} x = \operatorname{codim}_{\mathcal{F}_n^*} y^* = +\infty$$

As a consequence of this and propositions 6.6 and 6.7, $\dim_{\mathcal{F}_n} a$ and $\operatorname{codim}_{\mathcal{F}_n^*} a^*$ are $+\infty$ for every element $a \in \mathcal{F}_n \setminus \{\mathbf{0}\}$, and $\dim_{\mathcal{F}_n^*} a^*$ is finite only if a is a finite meet of completely meet irreducible elements, or equivalently if $a \geq \varepsilon_d(\mathcal{F}_n)$ for some d.

In every distributive lattice, if an element x is the complete meet of a set Y of meet irreducible elements such that Y is downward filtering[10] then x itself is meet irreducible. Indeed if $x_1 \wedge x_2 \leq x$, $x_1 \not\leq x$ and $x_2 \not\leq x$, let $y_1, y_2 \in Y$ such that $x_1 \not\leq y_1$ and $x_2 \not\leq y_2$. The assumption on Y gives $y \in Y$ smaller than $y_1 \wedge y_2$. Then $x_1 \wedge x_2 \leq x \leq y$ hence $x_1 \leq y$ or $x_2 \leq y$ (because y is meet irreducible) so $x_1 \leq y_1$ or $x_2 \leq y$, a contradiction.

[10] An ordered set Y is downward filtering if for every $y_1, y_2 \in Y$ there exists $y \in Y$ smaller than y_1 and y_2.

In particular, if L is a precompact Hausdorff co-Heyting algebra, then the complete meet in $\widehat{L}$ of any chain of completely meet irreducible elements is meet irreducible. By Zorn's lemma it follows that for every $a \in \widehat{L}$, every element in $\mathcal{I}^{!\wedge}(\widehat{L}) \cap a\uparrow$ is greater than a minimal one. So the last point of proposition 6.7 leads to:

Corollary 6.10 *Let L be a precompact Hausdorff co-Heyting algebra. For every $a \in \widehat{L}$, $a = \bigwedge\!\!\!\bigwedge \operatorname{Comp}^\wedge a$.*

We turn now to a characterisation of the meet irreducible elements of L.

Proposition 6.11 *Let L be a precompact Hausdorff co-Heyting algebra.*

1. *An element $a \in L$ is meet irreducible if and only if $\mathcal{I}^{!\wedge}(L) \cap a\uparrow$ is downward filtering.*
2. *A meet irreducible element is completely meet irreducible if and only if its cofoundation rank in L (with respect to the strict order $<$ of L) is finite.*

In particular if L is not finite then **0** *is meet irreducible, but not completely meet irreducible.*

Proof: Since a is the complete meet of $\mathcal{I}^{!\wedge}(L) \cap a\uparrow$, if this set is downward filtering then a is meet irreducible by the above general argument. Conversely assume that a is meet irreducible. Let $y_1, y_2 \in \mathcal{I}^{!\wedge}(L) \cap a\uparrow$ and $b_i = a \vee y_i^\vee$. By definition $y_i^\vee \nleq a$ since $a \leq y_i$, hence $y_1^\vee \wedge y_2^\vee \nleq a$ (because a is meet irreducible). So by proposition 6.7 there is a join irreducible component x of $y_1^\vee \wedge y_2^\vee$ which is not smaller than a. By construction $x^\wedge \leq y_i$ because $x \leq y_i^\vee$ (and $y_i = y_i^{\vee\wedge}$). Moreover $a \leq x^\wedge$ because $x \nleq a$ (by definition of $x^\wedge$). So $x \in \mathcal{I}^{!\wedge}(L) \cap a\uparrow$ and the first point is proved.

Assume now that a meet irreducible. If its cofoundation rank in L is finite then $\mathcal{I}^{!\wedge}(L) \cap a\uparrow$ is finite. Since it is downward filtering it must have a smallest element, hence a is completely meet irreducible. Conversely if a is completely meet irreducible, then it is greater than $\varepsilon_d(L)$ for some d (see the proof of the first point of proposition 6.7). But $\varepsilon_d(L)\uparrow$ is finite by lemma 6.4 (because L/dL is finite) hence so is $a\uparrow$.
■

It was proven in [Bel86] that in free finitely generated co-Heyting algebras every join irreducible element is meet irreducible. This obviously does not hold for finite co-Heyting algebras, hence it does not generalize to precompact Hausdorff ones.

The smallest dense subalgebra

Proposition 6.12 *Let L be a precompact Hausdorff co-Heyting algebra. Then $\mathcal{I}^{!\vee}(L)$ and $\mathcal{I}^{!\wedge}(L)$ generate the same $\mathcal{L}_{\mathrm{HA}^*}$-substructure of L, which is also the smallest $\mathcal{L}_{\mathrm{HA}^*}$-substructure dense in L (with respect to the codimetric topology).*

Proof: Let $L^\vee$ (resp. $L^\wedge$) be the $\mathcal{L}_{\mathrm{HA}^*}$-substructure of L generated by $\mathcal{I}^{!\vee}(L)$ (resp. $\mathcal{I}^{!\wedge}(L)$).

Note that if an element x of L is greater than $\varepsilon_d(L)$ for some d then $\mathrm{Comp}^\vee x$ is contained in $\mathcal{I}^{!\vee}(L) \setminus (d+1)L$ hence is finite by remark 6.5. Moreover:

$$\mathcal{I}^\wedge(L) \cap x\uparrow \subseteq \mathcal{I}^\wedge(L) \cap \varepsilon_d(L)\uparrow = \mathcal{I}^{!\wedge}(L) \cap \varepsilon_d(L)\uparrow$$

so $\mathcal{I}^\wedge(L) \cap x\uparrow$ is finite also and contained in $\mathcal{I}^{!\vee}(L)$. It follows that every isolated point of L, and in particular every $\varepsilon_d(L)$, belongs both to $L^\vee$ and $L^\wedge$.

In particular $\mathcal{I}^{!\wedge}(L) \subset L^\vee$ hence $L^\wedge \subseteq L^\vee$. Conversely if x is any completely join irreducible element of L and $d > \operatorname{codim} x$ then $x = x - \varepsilon_d(L)$. Because $\varepsilon_d(L)$ and $x - \varepsilon_d(L)$ are isolated they belong to $L^\wedge$, so:

$$x = x - \varepsilon_d(L) = (x \vee \varepsilon_d(L)) - \varepsilon_d(L) \in L^\wedge$$

It follows that $L^\vee = L^\wedge$ is also the $\mathcal{L}_{\mathrm{HA}^*}$-substructure generated by the set of isolated points, hence it is contained in every dense $\mathcal{L}_{\mathrm{HA}^*}$-substructure of L. Conversely every $x \in L$ is the limit of $(x \vee \varepsilon_d(L))_{d<\omega}$ which is a sequence of isolated points, hence $L^\vee$ is dense in L.

■

Proposition 6.13 *Given a precompact Hausdorff co-Heyting algebra L, and $L^\vee$ its smallest dense subalgebra, the following conditions are equivalent:*

1. $\widehat{L} = L^\vee$.
2. $\widehat{L}$ *is countable or finite.*
3. *There is no infinite antichain in* $\mathcal{I}^{!\vee}(L)$.
4. *There is no infinite antichain in* $\mathcal{I}^{!\wedge}(L)$.

Proof: The equivalence of the two last conditions follows immediately from the one-to-one, order preserving correspondence between $\mathcal{I}^{!\vee}(L)$ and $\mathcal{I}^{!\wedge}(L)$ (see proposition 6.7). If there is an infinite antichain $(x_i)_{i<\omega}$ in $\mathcal{I}^{!\vee}(L)$ then for every subset I of $\mathbf{N}$, the complete join x_I of $(x_i)_{i\in I}$ belongs to $\widehat{L}$ since $\widehat{L}$ is join complete. These elements are two by two distinct hence $\widehat{L}$ is uncountable. Conversely, note that the join irreducible components of any element form an antichain, and $\mathcal{I}^{!\vee}(L) = \mathcal{I}^{!\vee}(\widehat{L})$. So the third condition implies that $\widehat{L} = L^\vee$ which is obviously countable our finite.

■

If L is a precompact Hausdorff co-Heyting algebra such that $L \neq L^\vee$ then obviously $L \not\equiv L^\vee$ (because the latter does not contain a proper dense subalgebra). Because of the density of $L^\vee$ in L, both of them satisfy the same identities, an argument that we will re-use and develop in section 7. Does it happen that $\widehat{L} \neq L^\vee$ but $\widehat{L} \equiv L^\vee$? Our guess is no. But the analogy with the model theory of the ring $\mathbf{Z}_p$ of p-adic numbers (which is both the completion of $\mathbf{Z}$ with respect to the p-adic ultrametric distance, and the projective limit of all the quotients $\mathbf{Z}/p^d\mathbf{Z}$) suggests the following questions.

Question 6.14 Is the existential closure of $L^{\vee}$ inside $\widehat{L}$ an elementary substructure of L ?

Question 6.15 When L is finitely presented, is L the existential closure of $L^{\vee}$ inside $\widehat{L}$?

In [DJ08] it was proven that if the free co-Heyting algebra $\mathcal{F}_n$ with n generators is elementarily equivalent to $\widehat{\mathcal{F}_n}$ then $\mathcal{F}_n \preccurlyeq \widehat{\mathcal{F}_n}$. More generally, does this hold for every precompact Hausdorff co-Heyting algebra?

7 Hausdorff completion

Since the Hausdorff completion L' of a co-Heyting algebra L is the completion of $L/\omega L$ we can assume w.l.o.g. that L is Hausdorff. We identify L with its image in L' and consider it as a dense subset of L'. By proposition 4.3 the $\mathcal{L}_{\mathrm{HA}^*}$-functions $\vee, \wedge, -$ are continuous on $L \times L$. Their unique continuous extension to $L' \times L'$ defines an $\mathcal{L}_{\mathrm{HA}^*}$-structure on L'. Moreover for any two $\mathcal{L}_{\mathrm{HA}^*}$-terms t_1, t_2 with n free variables, if the corresponding functions coincide on L^n then by continuity (and density inside L'^n) they coincide on L'^n. So every equation $t_1(x) = t_2(x)$ valid on the whole of L^n remains valid on L'^n. Since the class of all co-Heyting algebras is a variety, it can be axiomatized by equations. It follows that L' with this $\mathcal{L}_{\mathrm{HA}^*}$-structure is a co-Heyting algebra. It is another story to prove that the pseudometric $\delta_{L'}$ is precisely the native metric of L', as we will do now.

Theorem 7.1 *Let L be a Hausdorff co-Heyting algebra. Let (L', δ') be the completion of the metric space (L, δ_L). Then L' is a Hausdorff co-Heyting algebra, and δ' is exactly the ultrametric $\delta_{L'}$. Moreover for every positive integer d, $L'/dL' = L/dL$ and (using remark 2.3) π_{dL} is the restriction of $\pi_{dL'}$ to L.*

It is worthwhile to notice, before starting the proof, that the "triangle inequality" for $\triangle$ (see section 2) implies that δ_L is an *ultrametric*:

$$\delta_L(a, c) \leq \max \delta_L(a, b), \delta_L(b, c)$$

It follows that a sequence $(x_n)_{n<\omega}$ is Cauchy if and only if $\delta_L(x_n, x_{n+1})$ is convergent to 0.

Proof: Note that $\delta_L(a, b) = \delta_L(a \triangle b, \mathbf{0})$ for every $a, b \in L$. By density it follows that:

$$\forall a', b' \in L', \quad \delta'(a', b') = \delta'(a' \triangle b', \mathbf{0}) \tag{2}$$

In order to show that $\delta' = \delta_{L'}$ it is then sufficient to check that they define the same balls centered at $\mathbf{0}$. Since δ' extends δ_L and L is dense in L', the ball of radius 2^{-d} and center $\mathbf{0}$ for δ' is precisely the closure $\overline{dL}$ of dL in L' with respect to δ'. So it suffices to check[11] that $dL' = \overline{dL}$ for every positive integer d.

[11] Here we use that both δ' and δ_L take their values in $\{2^{-d}\}_{d<\omega} \cup \{0\}$. Indeed by the ultrametric triangle inequality, for every $a' \neq b' \in L'$, $\delta'(a', b') = \delta'(a, b)$ for some (any) $a, b \in L$ close enough to a', b'.

The codimetric topology on L/dL is discrete by remark 4.1 so the metric of L/dL is complete. Moreover, by proposition 4.2, π_d is continuous. Hence π_d extends uniquely to a continuous map $\overline{\pi_d} : L' \to L/dL$ which is an $\mathcal{L}_{\mathrm{HA}^*}$-morphism by the same arguments as above (π_d preserves $\mathcal{L}_{\mathrm{HA}^*}$-equations hence so does $\overline{\pi_d}$ by continuity). The kernel of $\overline{\pi_d}$ is the closure of $\operatorname{Ker} \pi_d$, that is of dL, in L' with respect to δ'. This morphism is surjective and $\dim L/dL < d$ so the points (1) and (2) of corollary 3.9 give us:

$$dL \subseteq dL' \subseteq \overline{dL}$$

Conversely let $a' \in \overline{dL}$. We show by induction on d that $a' \in dL'$.

If $d = 0$ this is obvious since $0L' = L'$. So let us assume that $d \geq 1$ and $(d-1)L'$ is closed with respect to δ'. Let $(a_n)_{n<\omega}$ be a sequence of elements of dL converging to a' with respect to δ'. Then $\delta'(a_n, a_{n+1})$ is convergent to 0 hence so does $\delta_L(a_n, a_{n+1})$, as δ' and δ_L coincide on L. We may assume that $\operatorname{codim} a_n \bigtriangleup a_{n+1} \geq n+1$ for every n, by taking a subsequence of $(a_n)_{n<\omega}$ if necessary. So by theorem 3.8 we can find $x_n \in nL$ such that $a_n \bigtriangleup a_{n+1} \leq x_n$.

Since $a_d \in dL$ we can find $b_d \in (d-1)L$ such that $a_d \ll b_d$. For every $k \leq d$ let $b_k = b_d$. Assume that for some $n \geq d$ we have constructed a sequence $b_0, \dots, b_d$ of elements of $(d-1)L$ so that $a_n \ll b_n$ and $b_{n-1} \bigtriangleup b_n \in nL$. Let $b_{n+1} = b_n \vee x_{n+1}$. Since $x_{n+1} \in (n+1)L \subseteq (d-1)L$, by construction $b_{n+1} \in (d-1)L$. Moreover:

$$a_{n+1} \wedge a_n \leq a_n \ll b_n \leq b_{n+1}$$

$$a_{n+1} - a_n \leq a_n \bigtriangleup a_{n+1} \ll x_{n+1} \leq b_{n+1}$$

So $a_{n+1} = (a_{n+1} - a_n) \vee (a_{n+1} \wedge a_n) \ll b_{n+1}$. Finally $b_n \bigtriangleup b_{n+1} \leq x_{n+1}$ hence $b_n \bigtriangleup b_{n+1} \in (n+1)L$.

So we can continue this construction by induction. It gives a sequence $(b_n)_{n<\omega}$ of elements of $(d-1)L$ such that $a_n \ll b_n$ for every n. Moreover $\delta_L(b_n, b_{n+1}) \leq 2^{-n-1}$ hence this is a Cauchy sequence. Let b' be its limit in L' with respect to δ'. By the induction hypothesis $\overline{(d-1)L} = (d-1)L'$ hence $b' \in (d-1)L'$, that is $\operatorname{codim}_{L'} b' \geq d-1$. Since $a_n \vee b_n = b_n$ and $b_n - a_n = b_n$ for every n, the same holds for a', b' by continuity. So $a' \leq b'$ and $b' - a' = b'$ that is $a' \ll b'$. By theorem 3.8 we conclude that $\operatorname{codim}_{L'} a' \geq d$ that is $a' \in dL'$.

This ends the proof that $dL' = \overline{dL}$ for every positive integer d. It follows that $\delta_{L'} = \delta'$. In particular $\delta_{L'}$ is a metric on L'. Moreover $L'/dL' = L/dL$ since $\operatorname{Ker} \pi_{dL'} = \operatorname{Ker} \overline{\pi_{dL}}$.

■

As in section 6, for every co-Heyting algebra L, let $\widehat{L}$ denote the limit of the projective system:

$$\cdots \to L/(d+1)L \to L/dL \to \cdots \to L/0L = \{\mathbf{0}\}$$

with projections $\pi_{d,d+1} : L/(d+1)L \to L/dL$. Recall that $\widehat{L}$ can be represented as:

$$\widehat{L} = \left\{ (x_d)_{d<\omega} \;\middle/\; \forall d,\ x_d \in L/dL \text{ and } \pi_{d,d+1}(x_{d+1}) = x_d \right\}$$

Theorem 7.2 *Let L be a co-Heyting algebra. Then $\widehat{L}$ is the Hausdorff completion of L, and the projective topology on $\widehat{L}$ coincides with its codimetric topology.*

Proof: Let $L_0 = L/\omega L$ be the largest Hausdorff quotient of L. Then L is also the completion of L_0, and $L_0/dL_0 = L/dL$ for every positive integer d. So we may assume that $L = L_0$, that is L is Hausdorff.

Let L' be the completion of L. We know that $L'/dL' = L/dL$ for every positive integer d by theorem 7.1. So $x \mapsto (\pi_{dL'}(x))_{d<\omega}$ defines an $\mathcal{L}_{\mathrm{HA}^*}$-morphism $\varphi : L' \to \widehat{L}$ whose restriction to L is the canonical embedding of L in $\widehat{L}$.

$\operatorname{Ker}\varphi = \bigcap_{d<\omega} \operatorname{Ker}\pi_{dL'} = \{\mathbf{0}\}$ hence φ is injective. In order to show that it is surjective let us take any element $y = (y_d)_{d<\omega}$ in the projective limit. Then each $y_d = \pi_{dL}(x_d)$ for some $x_d \in L$. Since $\pi_{d,d+1}(y_{d+1}) = y_d$ and $\pi_{d,d+1} \circ \pi_{(d+1)L} = \pi_{dL}$ we have $\pi_{dL}(x_{d+1}) = \pi_{dL}(x_d)$ so $x_d \bigtriangleup x_{d+1} \in dL$. It follows that $(x_d)_{d<\omega}$ is a Cauchy sequence in L hence it converges to some $x \in L'$.

$$\pi_{dL'}(x) = \lim \pi_d L(x_n) = \pi_{dL'}(x_d)$$

So $\varphi(x) = (\pi_{dL'}(x))_{d<\omega} = (\pi_{dL'}(x_d))_{d<\omega} = (y_d)_{d<\omega} = y$. This ends the proof that φ is an $\mathcal{L}_{\mathrm{HA}^*}$-isomorphism. By theorem 7.1 it follows that $\widehat{L}/d\widehat{L} = L/dL$ (see remark 2.3) for every positive integer d. So the projective topology of $\widehat{L}$ coincides with the codimetric topology.

■

Remark 7.3 A quotient of a co-Heyting algebra L by an ideal I is finite dimensional if and only if $dL \subseteq I$ for some d (see corollary 3.9). So $\widehat{L}$ is also the limit of the projective system of *all* finite dimensional quotients of L.

Corollary 7.4 *A subset X of a complete Hausdorff co-Heyting algebra L is compact if and only if is closed and $\pi_d(X)$ is finite for every positive integer d.*

Proof: If X is compact it is obviously closed. Moreover for any positive integer d the sets $U(x,d) = \{y \in L \,/\, x \bigtriangleup y \in dL\}$ form an open cover of X as x ranges over X. By compactness there is a finite subset X_d of X such that $\{U(x,d)\}_{x\in X_d}$ covers X. Then $\pi_d(X) = \pi_d(X_d)$ is finite.

Conversely since $L = \widehat{L}$ by corollary 7.2, the topological closure of X is known to be:

$$\overline{X} = \{(x_k)_{k<\omega} \in \widehat{L} \,/\, \forall k,\ x_k \in \pi_k(X)\}$$

So if X is closed and every $\pi_k(X)$ is finite then $X = \overline{X}$ is compact as the limit of a projective system of finite discrete spaces.

■

A pseudometric space is called **precompact** if and only if its Hausdorff completion is compact. The following corollary, which immediately follows from corollaries 7.2 and 7.4 justifies our terminology for precompact co-Heyting algebras.

Corollary 7.5 *The Hausdorff completion of a co-Heyting algebra L is compact if and only if L/dL is finite for every positive integer d.*

We conclude with two delightful results which show that some metric properties of complete co-Heyting algebra have a familiar flavour. Recall that a sequence $(x_n)_{n<\omega}$ in a pseudometric space (X, δ) is convergent to y if and only if $\delta_L(x, y)$ is convergent to 0. The uniqueness of the limit holds only in the Hausdorff case.

Theorem 7.6 *Consider three sequences in a co-Heyting algebra L such that $c_n \leq b_n \leq a_n$ for every $n < \omega$. If a_n and c_n converge to the same limit l then b_n is convergent to l.*

Proof: Let $u_n = (a_n - l) \vee (c_n - l)$, this sequence is convergent to $\mathbf{0}$ (by continuity of the terms). By assumption $b_n \bigtriangleup l \leq u_n$ hence $\operatorname{codim} b_n \bigtriangleup l \geq \operatorname{codim} u_n$ for every positive integer n. So $\delta_L(b_n, l) \leq \delta_L(u_n, \mathbf{0})$ is convergent to 0.
■

Corollary 7.7 *Every monotonic sequence in a compact subset X of a co-Heyting algebra L is convergent.*

Proof: Let $(a_n)_{n<\omega}$ be a monotonic sequence in X. Let $(a_{\sigma(n)})_{n<\omega}$ a subsequence convergent in X. If $(a_n)_{n<\omega}$ is increasing, for every integer k let n_k be the smallest integer n such that $a_k \leq a_{\sigma(n)}$.

$$a_{\sigma(n_k - 1)} \leq a_k \leq a_{k+1} \leq a_{\sigma(n_{k+1})} \tag{3}$$

Conversely if $(a_n)_{n<\omega}$ is decreasing let n_k be the smallest integer n such that $a_k \geq a_{\sigma(n)}$. We have the same inequalities as in (3) with reverse order. In both cases $a_{\sigma(n_k - 1)}$ and $a_{\sigma(n_{k+1})}$ converge to the same limit hence so does a_k by theorem 7.6.
■

8 Appendix

Proposition 5.4 allows a slight improvement of the finite model property (to be compared with fact 5.2).

Proposition 8.1 *Let $\mathcal{V}$ be a variety of co-Heyting algebras having the finite model property and $\theta(x)$ be a quantifier free $\mathcal{L}_{\mathrm{HA}^*}$-formula. If there exists a $\mathcal{V}$-algebra L such that $L \models \exists x\ \theta(x)$ then there exists a finite $\mathcal{V}$-algebra having this property.*

Proof: We may assume that $\theta(x)$ is a conjunction of atomic and negatomic formulas with n variables. Since $t(x) \leq t'(x)$ is equivalent, modulo the theory of co-Heyting algebras, to $t(x) - t'(x) = \mathbf{0}$, we can suppose that every atomic formula is of type $t(x) = \mathbf{0}$. Finally $t(x) = \mathbf{0}$ and $t'(x) = \mathbf{0}$ is equivalent to $t(x) \vee t'(x) = \mathbf{0}$ so we can assume:

$$\theta(x) \equiv \bigwedge_{i \leq r} t_i(x) \neq \mathbf{0} \bigwedge t(x) = \mathbf{0}$$

Let a be a tuple of elements of L such that $L \models \theta(a)$. We may assume that L is generated by a. Let $\mathcal{F}_n$ be the free $\mathcal{V}$-algebra with n generators and $\pi : \mathcal{F}_n \to L$ the projection which maps the free generators X of $\mathcal{F}_n$ onto a. Let $(g_k)_{k<\omega}$ be an enumeration of the kernel of π. By construction $t(X) = g_k(X)$ for some k, but $t_i(X) \not\leq g_l(X)$ for every positive integer l and every $i \leq r$. By proposition 5.4, $\mathcal{F}_n$ is Hausdorff so:

$$\max_{i \leq r} \operatorname{codim} t_i(X) - g_k(X) < \omega$$

Let d denote this integer. Let I be the ideal of $\mathcal{F}_n$ generated by $g_k(X)$ and $\varepsilon_{d+1}(\mathcal{F}_n)$, and let b be the image of X in $\mathcal{F}_n/I$ *via* the canonical projection. By construction $\mathcal{F}_n/I$ is a quotient of $\mathcal{F}_n/(d+1)\mathcal{F}_n$. By proposition 5.4 and the assumption on $\mathcal{V}$, $\mathcal{F}_n/(d+1)\mathcal{F}_n$ is finite hence so is $\mathcal{F}_n/I$. Moreover $t(X)$ belongs to I and none of the $t_i(X)$'s belongs to I so $\mathcal{F}_n/I \models \theta(b)$.

■

We have seen that if a co-Heyting algebra L is finitely presented, then dL is a principal ideal for every positive integer d (corollary 5.5 and lemma 6.2). This is actually true for finitely generated co-Heyting algebras, and even more is true:

Proposition 8.2 *For every positive integers n, d there exists an $\mathcal{L}_{\mathrm{HA}^*}$-term $t_{n,d}$ in n variables such that for every co-Heyting algebra L generated by some $a \in L^n$, $t_{n,d}(a) = \varepsilon_d(L)$.*

Proof: Let $t_{n,d}$ be an $\mathcal{L}_{\mathrm{HA}^*}$-term such that in the free co-Heyting algebra $\mathcal{F}_n$ generated by an n-tuple X, $t_{n,d}(X) = \varepsilon_d(\mathcal{F}_n)$. Let L be any co-Heyting algebra generated by some n-tuple a and φ the projection of $\mathcal{F}_n$ onto L which maps X onto a. By corollary 3.9(3) $\varphi(\varepsilon_d(\mathcal{F}_n)) = \varepsilon_d(L)$ so:

$$t_{n,d}(a) = \varphi(t_{n,d}(X)) = \varphi(\varepsilon_d(\mathcal{F}_n)) = \varepsilon_d(L)$$

■

Remark 8.3 Our approach does not give any explicit form for $t_{n,d}$. Such an expression can be derived from Bellissima's construction. Indeed an explicit formula for all the join irreducible elements of fixed dimension d in the free co-Heyting algebra $\mathcal{F}_n$ with n generators is provided by this construction (see [Bel86], or theorem 3.3 in [DJ08] for a slightly better formula). Their join gives an expression for $t_{n,d}$, but its complexity seems to be discouraging for practical computations.

Let $\mathcal{V}_{n,d}$ be the variety of co-Heyting algebras axiomatized by the equation $t_{n,d+1} = \mathbf{0}$. This is the variety of co-Heyting algebras L such that every subalgebra of L generated by n elements has dimension at most d. So a variety $\mathcal{V}$ is contained in $\mathcal{V}_{n,d}$ if and only if the algebra freely generated in $\mathcal{V}$ by n elements has dimension at most d. Of course a variety $\mathcal{V}$ of co-Heyting algebras is locally finite (that is every finitely generated algebra in $\mathcal{V}$ is finite) if and only if for every positive integer n there is an integer $d(n)$ such that $\mathcal{V} \subseteq \mathcal{V}_{n,d(n)}$. For every $n \geq 1$, $\mathcal{V}_{n,0}$ is nothing but the variety of boolean algebras, hence it is locally finite. On the other hand one can easily show by adapting an example of Mardaev [Mar84] that the varieties $\mathcal{V}_{1,d}$ for $d > 1$ are distinct and not locally finite.

Question 8.4 For which integers n, d is $\mathcal{V}_{n,d}$ locally finite?

It is asked in [BG05] if $\mathcal{V}$ is a locally finite variety whenever the algebra freely generated in $\mathcal{V}$ by 2 elements is finite. This is equivalent to the local finiteness of $\mathcal{V}_{2,d}$ for every d, and it would imply that $\mathcal{V}_{n,d}$ is locally finite for every $n \geq 2$ and every d because $\mathcal{V}_{n,d}$ is obviously contained in $\mathcal{V}_{2,d}$.

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