Model-completion of scaled lattices*

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Abstract

It is known from Grzegorczyk's paper [Grz51] that the lattice of real semi-algebraic closed subsets of $\mathbb{R}^n$ is undecidable for every integer $n \geq 2$. More generally, if X is any definable set over a real or algebraically closed field K, then the lattice $L(X)$ of all definable subsets of X closed in X is undecidable whenever $\dim X \geq 2$. Nevertheless, we investigate in this paper the model theory of the class $\mathrm{SC}_{\mathrm{def}}(K,d)$ of all such lattices $L(X)$ with $\dim X \leq d$ and K as above or a henselian valued field of characteristic zero.

We show that the universal theory of $\mathrm{SC}_{\mathrm{def}}(K,d)$, in a natural expansion by definition of the lattice language, is the same for every such field K. We give a finite axiomatization of it and prove that it is locally finite and admits a model-completion, which turns to be decidable as well as all its completions. We expect $L(\mathbb{Q}_p^d)$ to be a model of (a little variant of) this model-completion. This leads us to a new conjecture in p-adic semi-algebraic geometry which, combined with the results of this paper, would give decidability (via a natural recursive axiomatization) and elimination of quantifiers for the complete theory of $L(\mathbb{R}_p^d)$, uniformly in p.

1 Introduction

In this paper we study the model-theory of a class of lattices coming from the following examples.

Example 1.1 Let K be a henselian valued field of characteristic zero, a real closed field or an algebraically closed field. There exists a good notion of dimension for definables sets A over K (see [vdD89] for the henselian case, and any book of real or complex algebraic geometry for the other cases). For any positive integer i let:

$$\mathrm{C}^i(A) = \overline{\{a \in A \,/\, \dim(A,a) = i\}}$$

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where $\dim(A, a)$ is the maximal dimension of definable neighborhood of a in A, and the overline stands for the topological closure in A. This is a definable subset of A which we call the i-pure[1]component of A. Given a definable set X over K of dimension d, let $\mathrm{L}_{\mathrm{def}}(X)$ be the lattice of all definable subsets of X closed in X, enriched[2] with the unary functions $(\mathrm{C}^i)_{i \leq d}$ and the binary function:

$$A - B = \overline{A \setminus B}$$

Eventually let $\mathrm{SC}_{\mathrm{def}}(K, d)$ denote the class of lattices $\mathrm{L}_{\mathrm{def}}(X)$ for all definable sets over K of dimension at most d.

Example 1.2 Let K be any infinite field, X a constructible subset over K (that is a boolean combination of Zariski closed subsets of K^n for some positive integer n) and $\mathrm{L}_{\mathrm{Zar}}(X)$ be the lattice of all constructible subsets of X which are Zariski closed in X enriched with the following structure. For any A, B in $\mathrm{L}_{\mathrm{Zar}}(X)$ let $A - B$ be as in the above example and $C^i(A)$ be the union of the irreductible components of A of Krull dimension i (in the usual sense for topological spaces).

Eventually let $\mathrm{SC}_{\mathrm{Zar}}(K, d)$ denote the class of lattices $\mathrm{L}_{\mathrm{Zar}}(X)$ for all definable sets over K of dimension at most d. Of course $\mathrm{SC}_{\mathrm{Zar}}(K, d) = \mathrm{SC}_{\mathrm{def}}(K, d)$ when K is algebraically closed.

It is known from an argument of [Grz51] that the complete theory of $\mathrm{L}_{\mathrm{def}}(K^n)$ is undecidable for every real closed field K and every integer $n \geq 2$, and the argument can easily be adapted to algebraically closed fields K. This paper gives some reason to believe that the complete theory of $\mathrm{L}_{\mathrm{def}}(K^n)$ is decidable for every p-adically closed field K and every n. It is organized as follows.

We first give in Section 2 a finite list of universal axioms of a theory T_d in a language $\mathcal{L}_{\mathrm{SC}_d}$ extending the language of lattices, the model of which we call d-subscaled lattices. The examples given above are all d-subscalled lattices. After some preliminar techinal results in Section 3 we prove in Section 4 that every finiteley generated d-subscaled lattice is finite. Combining this result with a linear representation for finite d-subscaled lattices and with the model-theoretic compactness theorem, we then prove in Section 5 that T_d is precisely the universal theory of $\mathrm{SC}_{\mathrm{Zar}}(K, d)$. In particular this theory is finitely axiomatizable and, remarkably enough, does not depend on K. Eventually a detailed study of finitely generated extensions of finite d-subscaled lattices, achieved in Sections 6 and 7 allows us to exhibit in Section 8 a model-completion $\bar{T}_d$ for T_d, having a finite axiomatisation. Moreover we show that $\bar{T}_d$ has finitely many completions, each of which is finitely axiomatizable and $\aleph_0$-categorical. It follows that $\bar{T}_d$ is decidable.

It is difficult to find a model of $\bar{T}_d$ coming from geometry because such models are atomless. We present in the last section a similar model completion

[1]If $\mathrm{C}^i(A)$ is non-empty it has pure dimension i, that is the local dimension $\dim(A, a) = i$ at every point of A.

[2]The additionnal functions are definable in the lattice structure of $\mathrm{L}_{\mathrm{def}}(X)$.

and decidability result for a theory $\bar{T}_d^*$ authorizing atoms. These results lead us to the following conjecture (or question):

Conjecture 1.3 *Let K be a p-adically closed field and A be an infinite definable subset of K^n which is open in its closure. Let $(B_k)_{k\leq q}$ be a finite collection of closed definable subsets of $\bar{A}\setminus A$. Then there exists a collection $(A_k)_{k\leq q}$ of non-empty definable subsets of A clopen in A such that:*

$$\forall k \leq q, \quad \overline{A_k} = A_k \cup B_k$$

If this conjecture is true then it follows immediatly that $\mathrm{L}_{\mathrm{def}}(K^n)$ is a model of T_n^* hence has a decidable complete theory (not depending on p).

Remark 1.4 Since 0-subscaled lattices are exactly non-trivial boolean algebras (with the $\mathcal{L}_{\mathrm{SC}_0}$-structure and the boolean structure being quantifier-free bi-definable) our model-completion result for subscaled lattices is a generalisation to arbitrary finite dimension d of the well known theorem on the model-completion of boolean algebras.

Remark 1.5 The duals of d-subscaled lattices form an elementary class of Heyting algebras so this paper may also be considered as a contribution to the model-theory of Heyting algebras. However the usual geometric objects whose study motivated this paper are closed sets (points, curves, surfaces, and so on). From this point of view the lattice $\mathrm{L}_{\mathrm{Zar}}(K^d)$ is a more natural object to consider than its dual, the Heyting algebra of *open* algebraic sets. This is the reason why we had to present our results in this settings and not in terms of Heyting algebra.

2 Notation and definitions

The set of all positive integers is denoted by $\mathbb{N}$, and $\mathbb{N}^* = \mathbb{N}\setminus\{0\}$. If $\mathcal{N}$ is an unbounded subset of $\mathbb{N}$ (resp. the empty subset) we set $\max\mathcal{N} = \infty$ (resp. $\max\mathcal{N} = -1$). The symbols $\subseteq$ and $\subset$ denote respectively the inclusion and the strict inclusion.

2.1 Lattices

Let $\mathcal{L}_{\mathrm{lat}} = \{\mathbf{0}, \mathbf{1}, \vee, \wedge\}$ be the language of lattices. An upper semi-lattice is an $\mathcal{L}_{\mathrm{up}}$-substructure of a lattice, with $\mathcal{L}_{\mathrm{up}} = \{\mathbf{0}, \vee\}$. As usually $b \leq a$ is an abreviation for $a \vee b = a$ and similarly for $b < a$, $b \geq a$ and $b > a$. Iterated $\vee$ and $\wedge$ operations are denoted by $\bigvee_{i\in I} a_i$ and $\bigwedge_{i\in I} a_i$ respectively. If the index set I is empty then of course $\bigvee_{i\in I} a_i = \mathbf{0}$ and $\bigwedge_{i\in I} a_i = \mathbf{1}$. The logical connectives 'or', 'and' and their iterated forms will be denoted by $\bigwedge$, $\bigvee$, $\bigwedge\!\!\bigwedge$ and $\bigvee\!\!\bigvee$ respectively. We consider the following relation, definable in any lattice:

$$\begin{aligned} b \ll a \quad &\iff \quad b < a \text{ and } \forall c\ (c < a \Rightarrow b \vee c < a) \\ &\iff \quad b \leq a \neq \mathbf{0} \text{ and } \forall c\ (c < a \Rightarrow b \vee c < a) \end{aligned}$$

The **spectrum** of a distributive lattice L is the set $\mathrm{Spec}(L)$ of all prime filters of L, endowed with the so-called Zarisky topology, defined by taking as a basis of closed sets the family:

$$P(a) = \{\mathbf{p} \in \mathrm{Spec}(L) \bigm/ a \in \mathbf{p}\}$$

for a ranging over L. Stone-Priestley's duality asserts that $a \mapsto P(a)$ is an isomorphism between L and the lattice of closed subsets of $\mathrm{Spec}(L)$ whose complement in $\mathrm{Spec}(L)$ is compact. We call a lattice **noetherian** if it is isomorphic to the lattice of closed sets of a noetherian topological space. By Stone-Priestley's duality a lattice L is noetherian if and only if its spectrum is a noetherian topological space. In such a lattice every filter is principal and every element a writes uniquely as the join of its **$\vee$-irreducible components**, which are the (finitely many) maximal elements in the set of non-zero $\vee$-irreducible elements of L smaller than a. We denote by $\mathcal{I}(L)$ the set of all non-zero $\vee$-irreducible elements of L.

We define the **lattice dimension of an element a in a lattice L** as the least upper bound (in $\mathbb{N} \cup \{-1, \infty\}$) of the set of positive integers n such that:

$$\exists \mathbf{p}_0 \subset \cdots \subset \mathbf{p}_n \in P(a)$$

This is nothing but the ordinary topological dimension (defined by chains of irreducible closed subsets) of the spectral space $P(a)$. We denote this dimension by $\dim_L a$. By construction $\dim_L a = -1$ if and only if $a = \mathbf{0}_L$. The index L is necessary since $\dim_L a$ is not preserved by $\mathcal{L}_{\mathrm{lat}}$-embeddings, nevertheless we omit it whenever the ambiant lattice is clear from the context and we often call the lattice dimension simply the dimension. We let the **lattice dimension of** L be the lattice dimension of $\mathbf{1}_L$ in L.

Fact 2.2 *If $L = L(X)$ is any of the lattices of the introduction, then for any $A \in L(X)$, $\dim_{L(X)} A$ is exactly the usual dimension of A as a definable (or constructible) set over K.*

Fact 2.4 below is a key argument in the proof of this result, which is non-obvious because the definition of $\dim_{L(X)} A$ lies on prime filters of closed definable subsets of A, an object which do not have a natural geometric meaning.

2.3 TC-lattices

Let $\mathcal{L}_{\mathrm{TC}} = \mathcal{L}_{\mathrm{lat}} \cup \{-\}$ with '$-$' a binary function symbol. A **topologically complemented lattice**, or **TC-lattice** for short, is an $\mathcal{L}_{\mathrm{TC}}$-structure which is a lattice and in which the **relative topological complement** $a - b$ is defined as the least element c such that $a \leq b \vee c$ (or equivalently $P(a-b)$ is the topolological closure of the relative complement $P(a) \setminus P(b)$, so the name). This is clearly the dual of a Heyting algebra with $a - b$ in the TC-lattice becoming $b \to a$ in its dual. So we know from the theory of Heyting algebra (see for example[Joh82]) that every TC-lattice is distributive, and that the class of all

TC-lattices is a variety (in the sense of universal algebra). Observe that in TC-lattices the $\ll$ relation is quantifier-free definable since:

$$a - b = a \iff b \wedge a \ll a \text{ or } a = \mathbf{0}$$

So it will be preserved by $\mathcal{L}_{\mathrm{TC}}$-embeddings. On the other hand the lattice dimension will not be preserved in general by $\mathcal{L}_{\mathrm{TC}}$-embeddings of TC-lattices.

We will use the following rules, the proof of which are elementary exercises (using either dual properties, if known, of Heyting algebras or, more directly, Stone-Priestley's duality).

$\mathbf{TC_1}$: $a = (a \wedge b) \vee (a - b)$.
In particular if a is $\vee$-irreducible then $b < a \implies b \ll a$.

$\mathbf{TC_2}$: $(a_1 \vee a_2) - b = (a_1 - b) \vee (a_2 - b)$.

$\mathbf{TC_3}$: $(a - b) - b = a - b$.
So either $a - b = \mathbf{0}$ or $(a - b) \wedge b \ll a - b \leq a$.

$\mathbf{TC_4}$: More generally $a - (b_1 \vee b_2) = (a - b_1) - b_2$.
So if $a - b_1 = a$ then $a - (b_1 \vee b_2) = a - b_2$.

Fact 2.4 *For any TC-lattice L and any $a \in L$, $\dim_L a$ is exactly the least upper bound of the set of positive integers n such that there exists $a_0, \dots, a_n \in L$ such that:*

$$\mathbf{0} \neq a_0 \ll a_1 \ll \cdots \ll a_n \leq a$$

The proof is a good exercise that we leave to the reader. The fact that $\dim_L a$ is at least equal to the above least upper bound is true in any lattice. Equality holds in TC-lattices because they are min-compact (that is for any $a \in L$ the set of elements of $P(a)$ which are minimal with respect to the inclusion, is compact).

2.5 (Sub)scaled lattices.

For a given positive integer d let $\mathcal{L}_{\mathrm{SC}_d} = \mathcal{L}_{\mathrm{TC}} \cup \{\mathrm{C}^i\}_{0 \leq i \leq d}$ where the C^i's are unary function symbols. With the examples of the introduction in mind, define the **sc-dimension of an element** a **of an** $\mathcal{L}_{\mathrm{SC}_d}$**-structure** L as:

$$\text{sc-dim}\, a = \min\big\{ l \leq d \,/\, a = \bigvee_{0 \leq i \leq l} \mathrm{C}^i(a) \big\} \in \mathbb{N} \cup \{-1, \infty\}$$

The **sc-dimension of** L, denoted sc-dim(L), is by definition the sc-dimension of $\mathbf{1}_L$. In general the lattice dimension of an element is not preserved by $\mathcal{L}_{\mathrm{lat}}$-embeddings neither by $\mathcal{L}_{\mathrm{TC}}$-lattices. On the other hand the sc-dimension of an element is obviously preserved by $\mathcal{L}_{\mathrm{SC}_d}$-embeddings.

A d**-subscaled lattice** is then an $\mathcal{L}_{\mathrm{SC}_d}$-structure which is a topologically complemented lattice and which satisfies the following list of axioms:

$\mathbf{SC_1}$: $\bigvee_{0 \leq i \leq d} \mathrm{C}^i(a) = a$

SC$_2$: $\forall I \subseteq \{0, \dots, d\}$, $\forall k$:

$$\mathrm{C}^k\Big(\bigvee_{i\in I} \mathrm{C}^i(a)\Big) = \begin{cases} \mathbf{0} & \text{if } k \notin I \\ \mathrm{C}^k(a) & \text{if } k \in I \end{cases}$$

SC$_3$: $\forall k \geq \max(\text{sc-dim}(a), \text{sc-dim}(b)), \quad \mathrm{C}^k(a \vee b) = \mathrm{C}^k(a) \vee \mathrm{C}^k(b)$

SC$_4$: $\forall i \neq j, \quad \text{sc-dim}\left(\mathrm{C}^i(a) \wedge \mathrm{C}^j(b)\right) < \min(i, j)$

SC$_5$: $\forall k \geq \text{sc-dim}(b), \quad \mathrm{C}^k(a) - b = \mathrm{C}^k(a) - \mathrm{C}^k(b)$
In particular, by SC$_3$: $\text{sc-dim}\, b < a \implies \mathrm{C}^k(a) - b = \mathrm{C}^k(a)$.

SC$_6$: If $b \ll a$ then $\text{sc-dim}\, b < \text{sc-dim}\, a$.

Axioms SC$_1$ to SC$_5$ are easily seen to be equivalent to a finite set of equations in $\mathcal{L}_{\mathrm{SC}_d}$. On the other hand SC$_6$ is expressible by a universal formula in $\mathcal{L}_{\mathrm{SC}_d}$ but not by an equation (indeed the class of d-subscaled lattices is not preserved by $\mathcal{L}_{\mathrm{SC}_d}$-projections, hence is not a variety). We call ***d*-scaled lattices** the d-subscaled lattices satisfying the following additional property:

SC$_0$: $\text{sc-dim}\, a = \dim a$

This is an elementary class, which is not preserved by $\mathcal{L}_{\mathrm{SC}_d}$-substructures. By Fact 2.2 all the examples of $\mathcal{L}_{\mathrm{SC}_d}$-structures given in the introduction are d-scaled lattices in which the lattice dimension and the sc-dimension coincide with the usual geometric dimension. As the terminology suggests, we will see that d-subscaled lattices are precisely the $\mathcal{L}_{\mathrm{SC}_d}$-substructures of d-scaled lattices.

The finite language $\mathcal{L}_{\mathrm{SC}_d}$ allows to write *finite* axiomatisations. When this is not essential we consider the language $\mathcal{L}_{\mathrm{SC}} = \mathcal{L}_{\mathrm{TC}} \cup \{\mathrm{C}^i\}_{i\in\mathbb{N}}$ and define **subscaled lattices** (resp. **scaled lattices**) as those $\mathcal{L}_{\mathrm{SC}}$-structures whose $\mathcal{L}_{\mathrm{SC}_d}$-reduct is a d-subscaled lattice (resp. a d-scaled lattice) for every d large enough. This is not an elementary class, but for any fixed integer d the class of (sub)scaled lattices of sc-dimension at most d (resp. exactly d) is elementary. Any d-(sub)scaled lattice expands uniquely to a (sub)scaled lattice of dimension at most d by realizing C^i as the constant map $x \mapsto \mathbf{0}$ for every $i > d$. Conversely every (sub)scaled lattice L is of that kind for every $d \geq \text{sc-dim}(L)$.

2.6 Basic properties

The next additionnal properties follow easily from the axioms of subscaled lattices.

SC$_7$: $\text{sc-dim}\, a = \max\{k \,/\, \mathrm{C}^k(a) \neq \mathbf{0}\}$
In particular $\forall k$, $\text{sc-dim}\, \mathrm{C}^k(a) = k \iff \mathrm{C}^k(a) \neq \mathbf{0}$

SC$_8$: $\text{sc-dim}\, a \vee b = \max(\text{sc-dim}\, a, \text{sc-dim}\, b)$
In particular $b \leq a \Rightarrow \text{sc-dim}\, b \leq \text{sc-dim}\, a$.

$\mathbf{SC_9}$: $\forall k, \quad \mathrm{C}^k(a) = \bigvee \Big\{ b \,\big/\, b \le \bigvee_{0\le i\le k} \mathrm{C}^i(a) \text{ and } \mathrm{C}^k(b) = b \Big\}$

$\mathbf{SC_{10}}$: sc-dim $a \le \dim a$

$\mathbf{SC_{11}}$: $\forall I \subseteq \{0,\dots,d\}, \quad a - \bigvee_{i\in I} \mathrm{C}^i(a) = \bigvee_{i\notin I} \mathrm{C}^i(a)$

In particular sc-dim$(a - \mathrm{C}^d(a)) < d$.

$\mathbf{SC_{12}}$: $\forall k, \quad \mathrm{C}^k(a) - b = \mathrm{C}^k(\mathrm{C}^k(a) - b)$

$\mathbf{SC_{13}}$: $\forall k, \quad \mathrm{C}^k(a) = a \iff \forall b \, \big(a - b \ne \mathbf{0} \Rightarrow \text{sc-dim}\, a - b = k\big)$

Proof: (Sketch) SC_7 follows from SC_1 and SC_2; SC_8 from SC_2, SC_3 and SC_7; SC_9 from SC_2 and SC_3; eventually SC_{10} is equivalent to SC_6 modulo the other axioms. Only the three last properties require a little effort.

SC_{11}**:** For every $l \in I$, $\mathrm{C}^l(a) \le \bigvee_{i\in I} \mathrm{C}^i(a)$ hence $\mathrm{C}^l(a) - \bigvee_{i\in I} \mathrm{C}^i(a) = \mathbf{0}$. On the other hand for every $l \notin I$ and every $i \in I$, $\mathrm{C}^l(a) - \mathrm{C}^i(a) = \mathrm{C}^l(a)$ by SC_4 and SC_5. So $\mathrm{C}^l(a) - \bigvee_{i\in I} \mathrm{C}^i(a) = \mathrm{C}^l(a)$ by TC_4. Eventually by SC_1 and TC_2:

$$a - \bigvee_{i\in I} \mathrm{C}^i(a) = \bigvee_{l\le d} \Big(\mathrm{C}^l(a) - \bigvee_{i>k} \mathrm{C}^i(a) \Big) = \bigvee_{l\notin I} \mathrm{C}^l(a)$$

In particular if $I = \{d\}$ then $a - \mathrm{C}^d(a) = \bigvee_{k<d} \mathrm{C}^k(a)$ hence sc-dim$(a - \mathrm{C}^d(a)) < d$.

SC_{12}**:** By TC_1, $\mathrm{C}^k(a) = (\mathrm{C}^k(a) - b) \vee (\mathrm{C}^k(a) \wedge b)$, and by SC_8 both $\mathrm{C}^k(a) - b$ and $\mathrm{C}^k(a) \wedge b$ have sc-dimension at most k, so by SC_3:

$$\mathrm{C}^k(\mathrm{C}^k(a)) = \mathrm{C}^k(\mathrm{C}^k(a) - b) \vee \mathrm{C}^k(\mathrm{C}^k(a) \wedge b)$$

By SC_1, $\mathrm{C}^k(\mathrm{C}^k(a) \wedge b) \le \mathrm{C}^k(a) \wedge b \le b$, and by SC_2, $\mathrm{C}^k(a) = \mathrm{C}^k(\mathrm{C}^k(a))$ so:

$$\mathrm{C}^k(a) \le \mathrm{C}^k(\mathrm{C}^k(a) - b) \vee b$$

It follows that $\mathrm{C}^k(a) - b \le \mathrm{C}^k(\mathrm{C}^k(a) - b)$, and equality holds by SC_1.

SC_{13}**:** If $a = \mathrm{C}^k(a)$ then $a - b = \mathrm{C}^k(a - b)$ for every b by SC_{12}. If moreover $a - b \ne \mathbf{0}$ then sc-dim$(a - b) = k$ by SC_7. Conversely assume that $a \ne \mathrm{C}^k(a)$ (hence $a \ne \mathbf{0}$). If sc-dim $a \ne k$ then $a - \mathbf{0} = a \ne \mathbf{0}$ hence sc-dim $a - \mathbf{0} \ne k$. If sc-dim $a = k$, let $b = \mathrm{C}^k(a)$. Then by assumption $a - b \ne \mathbf{0}$, and by SC_{11}, $a - b = \bigvee_{i<k} \mathrm{C}^i(a)$ hence sc-dim $a - b < k$.
■

2.7 Miscellanies

Given an integer k we say that an element a of a distributive lattice L is k**-pure in** L if and only if[3]:

$$\forall b \in L \ (a - b \ne \mathbf{0} \Rightarrow \dim_L a - b = k)$$

[3] Remember Example 1.1 and Footnote 1.

Then either $a = \mathbf{0}$ or $\dim_L a = k$. In the latter case we say that a **has pure dimension k in L**.

Similarly in any d-subscaled lattice we say that an element a is **k-sc-pure** if and only if:

$$\forall b \in L\ (a - b \neq \mathbf{0} \Rightarrow \text{sc-dim}\, a - b = k)$$

By SC_{13}, a is k-sc-pure if and only if $a = C^k(a)$. Then by SC_7, either $a = \mathbf{0}$ or sc-dim $a = k$. In the latter case we say that a **has pure sc-dimension k**. For any a, the element $C^k(a)$ is called the **k-sc-pure component of a**, and simply its **k-pure component** if L is a scaled lattice.

Proposition 2.8 *The $\mathcal{L}_{\mathrm{SC}_d}$-structure of a d-scaled lattice L is uniformly definable in the $\mathcal{L}_{\mathrm{lat}}$-structure of L. In particular it is uniquely determined by this $\mathcal{L}_{\mathrm{lat}}$-structure.*

Proof: Clearly the TC-structure is an extension by definition of the lattice structure of L. For every positive integer k the class of k-pure elements is uniformly definable, using the definablility of $\ll$. Then so is the function C^k for every k, by decreasing induction on k. Indeed by SC_9 and SC_{11}, $C^k(a)$ is the largest k-pure element c such that $c \leq a - \bigvee_{i>k} C^i(a)$.
■

Our study of (sub)scaled lattices is motivated by the examples given in the introduction. Although they are less natural, the following examples which will be needed further in this paper.

Example 2.9 In an arbitrary noetherian lattice L an $\mathcal{L}_{\mathrm{SC}}$-structure can be defined as follows. For every $a, b \in L$, if $\mathcal{C}(a)$ denotes the set of all $\vee$-irreducible components of a, let:

$$a - b = \bigvee \{c \in \mathcal{C}(a) \,/\, c \not\leq b\}$$
$$(\forall k) \quad C^k(a) = \bigvee \{c \in \mathcal{C}(a) \,/\, \dim_L c = k\}$$

This $\mathcal{L}_{\mathrm{TC}}$-structure (resp. $\mathcal{L}_{\mathrm{SC}}$-structure) is the only one (by Proposition 2.8) which turns L into a TC-lattice (resp. a scaled lattice). On the other hand, for any strictly increasing map $D\colon \mathcal{I}(L) \to \mathbb{N}$ and any $a, b \in L$ define $a - b$ as above and:

$$(\forall k) \quad C^k_D(a) = \bigvee \{c \in \mathcal{C}(a) \,/\, D(c) = k\}$$

This $\mathcal{L}_{\mathrm{SC}}$-structure turns L into a subscaled lattice which is not a scaled lattice (except if D coincides with the map $\dim_L$). We will use without further mention the following obvious fact:

Fact 2.10 *Every noetherian (hence in particular every finite) subscaled lattice is of the above kind.*

Eventually the following notation will be convenient in induction arguments. If $\mathcal{L}$ is any of our languages $\mathcal{L}_{\mathrm{lat}}$, $\mathcal{L}_{\mathrm{TC}}$, $\mathcal{L}_{\mathrm{SC}_d}$ or $\mathcal{L}_{\mathrm{SC}}$ we let $\mathcal{L}^* = \mathcal{L} \setminus \{\mathbf{1}\}$. Given an $\mathcal{L}$-structure L whose reduct to $\mathcal{L}_{\mathrm{lat}}$ is a lattice, for any $a \in L$ we denote by:

$$L(a) = \{b \in L \,/\, b \leq a\}$$

$L(a)$ is a typical example of $\mathcal{L}^*$-substructure of L.

3 Embeddinds of subscaled lattices

We need a reasonably easy criterion for an $\mathcal{L}_{\text{lat}}$-embedding of subscaled lattices to be an $\mathcal{L}_{\text{SC}}$-embedding. The special case of a noetherian embedded lattice, presented in the next proposition, is sufficient for this paper. However combining the model-theoretic compactness theorem with the local finiteness Theorem 4.1, one can easily derive from Proposition 3.1 that an $\mathcal{L}_{\text{lat}}$-embedding $\varphi : L \to L'$ between arbitrary subscaled lattices is an $\mathcal{L}_{\text{SC}}$-embedding if and only if it preserves the sc-dimension and sc-purity, that is for every $a \in L$ and every $k \in \mathbb{N}$:

$$\mathrm{C}^k(a) = a \implies \mathrm{C}^k(\varphi(a)) = \varphi(a)$$

Proposition 3.1 *Let L_0 be a noetherian subscaled lattice and $\varphi : L_0 \to L$ an $\mathcal{L}_{\text{lat}}$-embedding such that for every $a \in \mathcal{I}(L_0)$, $\varphi(a)$ is sc-pure and has the same sc-dimension as a. Then φ is an $\mathcal{L}_{\text{SC}}$-embedding.*

Remark 3.2 Clearly the same statement remains true with $\mathcal{L}_{\text{lat}}$ and $\mathcal{L}_{\text{SC}}$ replaced respectively by $\mathcal{L}^*_{\text{lat}}$ and $\mathcal{L}^*_{\text{SC}}$ (or $\mathcal{L}^*_{\text{SC}_d}$). We will freely use these variants.

Proof: Let $d = \text{sc-dim}\, L$, and for any $a \in L_0$:

$$(a_0, \dots, a_d) = (\varphi(\mathrm{C}^0(a)), \dots, \varphi(\mathrm{C}^d(a)))$$

For every positive integer k, $\mathrm{C}^k(a)$ is k-sc-pure by SC_{13}, hence each $\vee$-irreducible component c of $\mathrm{C}^k(a)$ in L_0 has pure sc-dimension k. By assumption each such $\varphi(c)$ then has pure sc-dimension k. The join of finitely many elements of pure sc-dimension k is easily seen to be k-sc-pure by definition and by TC_2, so we have proved:

$$\forall k,\ a_k \text{ is } k\text{-sc-pure.} \tag{1}$$

Moreover for any $k \neq l$, $\text{sc-dim}(\mathrm{C}^k(a) \wedge \mathrm{C}^l(a)) < \min(k, l)$ by SC_4. It follows that each $\vee$-irreducible component c of $\mathrm{C}^k(a) \wedge \mathrm{C}^l(a)$ has sc-dimension stricly less than $\min(k, l)$, hence so does $\varphi(c)$ by assumption. By SC_8 we conclude that:

$$\forall k \neq l,\ \text{sc-dim}(a_k \wedge a_l) < \min(k, l) \tag{2}$$

For every $k > d$, $\varphi(\mathrm{C}^k(a)) = \varphi(\mathbf{0}) = \mathbf{0}$. Each $\vee$-irreducible component of a has sc-dimension at most d hence so does $\varphi(a)$ by assumption, so by SC_8, $\text{sc-dim}(\varphi(a)) \leq d$. It follows that $\mathrm{C}^k(\varphi(a)) = \mathbf{0} = \mathrm{C}^k(\varphi(a))$.

For every $k \leq d$ let:

$$b_k = \bigvee_{0 \leq l \leq k} a_l \qquad \text{and} \qquad c_k = \bigvee_{0 \leq l \leq k} \mathrm{C}^l(\varphi(a))$$

By SC_{11}, $c_k - \mathrm{C}^k(\varphi(a)) = c_{k-1}$. Moreover the proof of SC_{11} proves as well that $b_k - a_k = \bigvee_{l \leq k}(a_l - a_k) = b_{k-1}$, thanks to (1) and (2) above.

Now assume that $b_k = c_k$ for some $k \le d$. Then $\mathrm{C}^k(\varphi(a)) = \mathrm{C}^k(c_k)$ by SC_2, so $\mathrm{C}^k(\varphi(a)) = \mathrm{C}^k(b_k)$ by assumption. It follows by SC_5 that:

$$\mathrm{C}^k(\varphi(a)) - a_k = \mathrm{C}^k(b_k) - \mathrm{C}^k(a_k) = \mathrm{C}^k(b_k - a_k) = \mathrm{C}^k(b_{k-1})$$

SC_8 implies that $\text{sc-dim}(b_{k-1}) \le k-1$ so $\mathrm{C}^k(b_{k-1}) = \mathbf{0}$ by SC_2. It follows that $\mathrm{C}^k(\varphi(a)) - a_k = \mathbf{0}$ that is $\mathrm{C}^k(\varphi(a)) \le a_k$. On the other hand $a_k \le \bigvee\!\!\!\bigvee_{l \le k} \mathrm{C}^k(\varphi(a))$ and $\mathrm{C}^k(a_k) = a_k$ by (1) so $a_k \le \mathrm{C}^k(\varphi(a))$ by SC_9. We have proved:

$$b_k = c_k \implies a_k = \mathrm{C}^k(\varphi(a))$$

Then $b_{k-1} = b_k - a_k = c_k - \mathrm{C}^k(\varphi(a)) = c_{k-1}$. Since $b_d = c_d = a$ it follows by decreasing induction that $b_k = c_k$ for every k, that is $\varphi(\mathrm{C}^k(a)) = \mathrm{C}^k(\varphi(a))$. Incidentally, since φ is injective, this implies by SC_7 that for every $a \in L_0$:

$$\text{sc-dim}\, a = \text{sc-dim}\, \varphi(a) \tag{3}$$

Now let $a, b \in L_0$, and a', b' be their images by φ. We have to show that $\varphi(a - b) = a' - b'$. By TC_2, replacing if necessary a by its $\vee$-irreducible components, we may assume w.l.o.g. that a itself is $\vee$-irreducible in L_0. This implies that $a = \mathrm{C}^k(a)$ for some k. It then remains two possibilities for $a - b$:

- If $b \ge a$ then $a - b = \mathbf{0}$ and $b' \ge a'$ hence $\varphi(a - b) = \mathbf{0} = a' - b'$.
- Otherwise $c = b \wedge a < a$ hence $c \ll a$ by TC_1, so $\text{sc-dim}\, c < \text{sc-dim}\, a$ by SC_6. Let $c' = \varphi(c) = b' \wedge a'$, by assumption $a' = \mathrm{C}^k(a)$ and by (3) $\text{sc-dim}(c') < \text{sc-dim}(a') = k$, hence by SC_5 $a' - c' = a'$. We conclude that $a - b = a$ and $a' - b' = a'$ hence $\varphi(a - b) = \varphi(a) = a' = a' - b'$.

We have proved that φ preserves $-$ and the C^k's operations, so φ is an $\mathcal{L}_{\mathrm{SC}}$-embedding.
■

Corollary 3.3 *Let L_0 be a noetherian sublattice of a subscaled lattice L. If for any $b < a \in \mathcal{I}(L)$, a is sc-pure in L and* $\text{sc-dim}\, b < \text{sc-dim}\, a$ *in L, then L_0 is an $\mathcal{L}_{\mathrm{SC}}$-substructure of L_0.*

Proof: The assumptions imply that the map $D : a \mapsto \text{sc-dim}\, a$ is a stricly inreasing map from $\mathcal{I}(L_0)$ to $\mathbb{N}$. Endow L_0 with the structure of subscaled lattice determined by D as in Example 2.9. Proposition 3.1 then applies to the inclusion map φ from L_0 to L.
■

4 Local finiteness

We prove in this section that every finitely generated subscaled lattice is finite. This result is far non-obvious, due to the lack of any known normal form for terms in $\mathcal{L}_{\mathrm{SC}}$. It contrasts with the situation in TC-lattices, which can be both infinite and generated by a single element. Our main ingredient, which explains this difference, is the uniform bound given *a priori* for the sc-dimension of any element in a given subscaled lattice.

Theorem 4.1 *Any subscaled lattice L of sc-dimension d generated by n elements is finite. More precisely, the cardinality of $\mathcal{I}(L)$ is then bounded by the fonction $\mu(n,d)$ defined by:*

$$\begin{array}{rcll} \mu(n,-1) & = & 0 & (\forall n) \\ \mu(n,d) & = & 2^n + \mu(2^{n+1}, d-1) & (\forall n,\ \forall d \geq 0) \end{array}$$

Proof: If $d=-1$ the only subscaled lattice of dimension -1 is the one-element lattice $\{\mathbf{0}\}$, so the result is trivial.

Assume the $d \geq 0$ and that the result is proved for every $d' < d$ and every positive integer n. Let L be a subscaled lattice of sc-dimension d generated by elements $x_1, \dots, x_n$. Let Ω_n be the family of all subsets of $\{1,\dots,n\}$ (so $\Omega_0 = \{\emptyset\}$). For every $I \in \Omega_n$ let $I^c = \Omega_n \setminus I$ and:

$$y_I = \Big(\bigwedge_{i\in I} x_i\Big) - \Big(\bigvee_{i\in I^c} x_i\Big), \qquad z_I = \mathrm{C}^d(y_I)$$

The family of all $\mathcal{Y}_I = \bigcap_{i\in I} P(x_i) \cap \bigcap_{i\in I^c} P(x_i)^c$ is a partition of $\mathrm{Spec}(L)$. Indeed the $\mathcal{Y}_i$'s are the atoms of the boolean algebra generated in the power set $\mathcal{P}(\mathrm{Spec}(L))$ by the $P(x_i)$'s. Moreover each $P(y_I)$ is the topological closure $\overline{\mathcal{Y}}_I$ of $\mathcal{Y}_I$ in $\mathrm{Spec}(L)$ hence for every $x \in L$:

$$P(x) = \bigcup_{I\in\Omega_n} P(x) \cap \mathcal{Y}_I \subseteq \bigcup_{I\in\Omega_n} P(x) \cap \overline{\mathcal{Y}}_I = P\Big(\bigvee_{I\in\Omega_n} x \wedge y_I\Big)$$

So $x \leq \bigvee_{I\in\Omega_n} (x \wedge y_I)$ by Stone-Priestley's duality. The reverse inequality being obvious we have proved:

$$\forall x \in L, \quad x \leq \bigvee_{I\in\Omega_n} (x \wedge y_I) \tag{4}$$

In particular SC_3 also gives:

$$\mathrm{C}^d(\mathbf{1}) = \mathrm{C}^d\Big(\bigvee_{I\in\Omega_n} y_I\Big) = \bigvee_{I\in\Omega_n} z_I \tag{5}$$

For every $I \neq J \in \Omega_n$, if for example $I \not\subseteq J$ choose any $i \in I \setminus J$ and observe that $y_I \leq x_i$ and $y_J \leq \mathbf{1} - x_i$ so $y_I \wedge y_J \ll \mathbf{1} - x_i$ by TC_3. By SC_6 and the d-sc-purity of the z_I's it follows that:

$$\text{sc-dim}\, z_I \wedge z_J < d \quad \text{hence} \quad z_I - z_J = z_I \tag{6}$$

It follows from SC_8, SC_{11} and (6) above, that the element:

$$a = \big(\mathbf{1} - \mathrm{C}^d(\mathbf{1})\big) \vee \Big(\bigvee_{I\in\Omega_n} (y_I - z_I)\Big) \vee \Big(\bigvee_{I\neq J\in\Omega_n} (z_I \wedge z_J)\Big)$$

has sc-dimension strictly smaller than d. So the induction hypothesis applies to the $\mathcal{L}_{\mathrm{SC}}$-substructure L_0^- of $L(a)$ generated by the $(y_I - z_I)$'s and the $(z_I \wedge a)$'s: L_0^- is finite, with at most $\mu(2|\Omega_n|, d-1)$ non-zero $\vee$-irreducible elements. Eventually let L_1 be the upper semi-lattice generated in L by $L_0^- \cup \{z_I\}_{I\in\Omega_n}$. By construction L_1 is finite and $\mathcal{I}(L_1) \subseteq \mathcal{I}(L_0) \cup \{z_I\}_{I\in\Omega_n}$, so $|\mathcal{I}(L_1)| \leq 2^n + \mu(2^{n+1}, d-1) = \mu(n,d)$. It is then sufficient to show that $L_1 = L$.

By (5), $\mathbf{1} = \mathrm{C}^d(\mathbf{1}) \vee a = \bigvee_{I\in\Omega_n} z_I \vee a \in L_1$. For every $I \in \Omega_n$ and every $b \in L_0^-$, $z_I \wedge b = (z_I \wedge a) \wedge b \in L_0^-$. Eventually for every $I \neq J \in \Omega_n$, $z_I \wedge z_j = (z_I \wedge a) \wedge (z_J \wedge a) \in L_0^-$. So by the distributivity law, L_1 is a sublattice of L.

Since $\mathcal{I}(L_1) \subseteq \mathcal{I}(L_0^-) \cup \{z_I\}_{I\in\Omega_n}$ it is immediate that for any $b' < b$ in $\mathcal{I}(L_1)$, sc-dim b' < sc-dim b. So L_1 is an $\mathcal{L}_{\mathrm{SC}}$-substructure of L by Corollary 3.3. Moreover each $y_I = (y_I - z_I) \vee z_I \in L_1$ and for every $i \leq n$, (4) gives:

$$x_i = \bigvee_{I\in\Omega_n} x_i \wedge y_I \leq \bigvee_{\substack{I\in\Omega_n \\ i\in I}} y_I \leq x_i$$

So equality holds, hence each $x_i \in L_1$ and eventually $L = L_1$.
∎

Corollary 4.2 *For every n, d there are finitely many non-isomorphic subscaled lattices of sc-dimension d generated by n elements.*

Proof: Any such subscaled lattice L is finite, with $|\mathcal{I}(L)| \leq \mu(n,d)$ by Theorem 4.1. Clearly there are finitely many non-isomorphic lattices such that $|\mathcal{I}(L)| \leq \mu(n,d)$ and each of them admits finitely many non-isomorphic $\mathcal{L}_{\mathrm{SC}_d}$-structures. The conclusion follows.
∎

5 Linear representation

In this section we prove that the theory of d-subscaled lattices is the universal theory of various natural classes of $\mathcal{L}_{\mathrm{SC}_d}$-structures, including $\mathrm{SC}_{\mathrm{Zar}}(K,d)$. The argument is based on an elementary representation theorem for d-subscaled lattices, combined with the local finiteness result of Section 4.

Given an arbitrary field K, a non-empty linear variety $X \subseteq K^m$ is determined by the data of an arbitrary point $P \in X$ and the vector subspace $\overrightarrow{X}$ of K^m, *via* the relation $X = P + \overrightarrow{X}$ (the orbit of P under the action of $\overrightarrow{X}$ by translation). We call X a **special linear variety** (resp. a **special linear set**) if X is a linear variety such that $\overrightarrow{X}$ is generated by a subset of the canonical basis of K^m (resp. if X is a finite union of special linear varieties). For example the empty set is a special linear set, as the union of an empty family of special linear varieties. The family $\mathrm{L}_{\mathrm{lin}}(X)$ of all special linear subsets of X is a noetherian lattice (because it is the family of closed sets of a noetherian topology on X). So it has a natural structure of scaled lattice defined as in Example 2.9.

Remark 5.1 For every $A \in \mathrm{L_{lin}}(X)$, sc-dim $A = \dim_{\mathrm{L_{lin}}(X)} A$ = the dimension of A as defined in linear algebra. If K is infinite then this dimension coincides with the Krull dimension of A. Moreover if A is $\vee$-irreducible in $\mathrm{L_{lin}}(X)$ then it is pure dimensionnal, hence it is sc-pure both in $\mathrm{L_{lin}}(X)$ and $\mathrm{L_{Zar}}(X)$. By Proposition 3.1 it follows that if K is infinite then $\mathrm{L_{lin}}(X)$ is an $\mathcal{L}_{\mathrm{SC}}$-substructure of $\mathrm{L_{Zar}}(X)$. Similarly if K is a henselian valued field of characteristic zero, a real closed field or an algebraically closed field then $\mathrm{L_{lin}}(X)$ is an $\mathcal{L}_{\mathrm{SC}}$-substructure of $\mathrm{L_{def}}(X)$.

In the following proposition K^m is identified to the subset $K^m \times \{0\}^r$ of K^{m+r}.

Proposition 5.2 *Given any two special linear sets $C \subseteq B \subseteq K^m$ and any $N \geq \dim C$ there exists a positive integer r and a special linear set $A \subseteq K^{m+r}$ of pure dimension n such that $A \cap B = C$.*

Proof: For any integer n let $(e_1, \ldots, e_n)$ be the canonical basis of K^n. If I is a subset of $\{1, \ldots, n\}$ we denote $\overrightarrow{E}(I)$ the vector subspace of K^n generated by $(e_i)_{i \in I}$.

Decompose C as a union of special linear varieties: $C_1, \ldots, C_p$ with each $C_i = P_i + \overrightarrow{E}(J_i)$ and $|J_i| = \dim C_i \leq n$. Let r be larger than 0, $N - m$ and every $N - |J_i|$. For each i take an arbitrary subset J_i' of cardinality $N - |J_i|$ inside $\{m+1, \ldots, m+r\}$ and let $I_i = J_i \cup J_i'$. Choose an arbitrary point $P_0 \in K^{m+r} \setminus K^m$ and let $I_0 = \{1, \ldots, N\}$. Eventually let $A_i = P_i + \overrightarrow{E}(I_i)$ for every $i \leq p$, and $A = A_0 \cup \ldots \cup A_p$. By construction each A_i has dimension $|I_i| = N$, hence A is N-pure. At least A_0 is non empty (it is the only one in case $p = 0$ that is if C is empty) hence A has pure dimension N. Clearly $A \cap K^m = C$ hence *a fortiori* $A \cap B = C$.
■

Proposition 5.3 (Linear representation) *Let K be an infinite field and let L be any finite subscaled lattice. Then there exists a linear set X over K and an $\mathcal{L}_{\mathrm{SC}}$-embedding $\varphi\colon L \to \mathrm{L_{lin}}(X)$.*

Remark 5.4 Since φ, in the above proposition, is an $\mathcal{L}_{\mathrm{SC}}$-embedding it preserves the sc-dimension hence $\dim X =$ sc-dim L. So if sc-dim $L \leq d$ we can indentify L and X with their respective $\mathcal{L}_{\mathrm{SC}_d}$-reduct, and φ is then also an $\mathcal{L}_{\mathrm{SC}_d}$-embedding.

Proof: Remember that $\mathcal{L}^*_{\mathrm{SC}} = \mathcal{L}_{\mathrm{SC}} \setminus \{\mathbf{1}\}$. We prove by induction on the number n of non-zero $\vee$-irreducible elements of a subscaled lattice L that there exists an $\mathcal{L}^*_{\mathrm{SC}}$-embedding φ of L into $\mathrm{L_{lin}}(K^m)$ for some m depending on L.

For $n = 0$, L is the one-element lattice $\{\mathbf{0}\}$ hence it is an $\mathcal{L}^*_{\mathrm{SC}}$-substructure of $\mathrm{L_{lin}}(K)$.

Let $n \geq 1$, assume the result proved for $n-1$ and take a subscaled lattice L with non-zero $\vee$-irreducible elements $a_1, \ldots, a_n$. Reordering if necessary we may assume that a_n is maximal among the a_i's. Let $a = a_n$, $b = \bigvee_{1 \le i < n} a_i$, $c = a \wedge b$ and φ an $\mathcal{L}^*_{\rm SC}$-embedding of $L(b)$ into some $\mathrm{L}_{\rm lin}(K^m)$ given by induction hypothesis. Since a is $\vee$-irreducible in L it is sc-pure. Moreover $c \ll a$ by TC_1, hence a has pure sc-dimension N for some $N > \text{sc-dim}(c)$ by SC_6. Let B, C be the respective images of b, c by φ. Proposition 5.2 gives a positive integer r and a special linear set $A \subseteq K^{m+r}$ of pure dimension N such that $A \cap B = C$. Since we indentified K^m with $K^m \times \{0\}^r$ we consider $\mathrm{L}_{\rm lin}(K^m)$ as an $\mathcal{L}^*_{\rm SC}$-substructure of $\mathrm{L}_{\rm lin}(K^{m+r})$. So φ actually embeds $L(b)$ into $\mathrm{L}_{\rm lin}(K^{m+r})$.

Every element x of L writes uniquely $x_a \vee x_b$ with $x_a \in \{\mathbf{0}, a\}$ and $x_b \in L(b)$ (group appropriately the $\vee$-irreducible components of x, using the maximality of a_n) hence:

$$\bar\varphi(x) = \begin{cases} \varphi(x_b) & \text{if } x_a = \mathbf{0} \\ A \cup \varphi(x_b) & \text{if } x_a = a \end{cases}$$

is a well-defined $\mathcal{L}^*_{\rm lat}$-embedding of L into $\mathrm{L}_{\rm lin}(K^{m+r})$. Moreover $\bar\varphi$ is an $\mathcal{L}^*_{\rm SC}$-embedding by Proposition 3.1. This finishes the induction.

We have constructed an $\mathcal{L}^*_{\rm SC}$-embedding φ of L into $\mathrm{L}_{\rm lin}(K^m)$ for some m. Then $X = \varphi(\mathbf{1}_L)$ is a special linear set, so φ induces an $\mathcal{L}_{\rm SC}$-embedding of L into $\mathrm{L}_{\rm lin}(X)$.

■

For any infinite field K and positive integer d let $\mathrm{SC}_{\rm lin}(K, d)$ be the class of d-scaled lattices $\mathrm{L}_{\rm lin}(X)$ for every special linear variety X over K of dimension at most d.

Theorem 5.5 *For any infinite field K and positive integer d, the universal theories of $\mathrm{SC}_{\rm lin}(K, d)$ and $\mathrm{SC}_{\rm Zar}(K, d)$ are exactly the theory of d-subscaled lattices.*

If moreover K is a henselian valued field of characteristic zero, a real closed field or an algebraically closed field then the same holds for $\mathrm{SC}_{\rm def}(K, d)$.

Proof: By Remark 5.1 it suffices to prove the theorem for $\mathrm{SC}_{\rm lin}(K, d)$. Let $T(K, d)$ be its universal theory.

The linear representation Proposition 5.3 shows that every finite d-subscaled lattice embeds into some $L \in \mathrm{SC}_{\rm lin}(K, d)$ hence is a model of $T(K, d)$. Since every finitely generated d-subscaled lattice is finite by Theorem 4.1, the model-theoretic compactness argument then implies that any d-subscaled lattice is a model of $T(K, d)$.

Conversely every $L \in \mathrm{SC}_{\rm lin}(K, d)$ is a d-scaled lattice hence obviously its universal theory contains the theory of d-subscaled lattices.

■

6 Primitive extensions

This section is devoted to the study of minimal proper extensions of finite subscaled lattices. Let L_0 be a finite subscaled lattice, L an $\mathcal{L}_{\rm SC}$-extension of L_0 and $x \in L$. We introduce the following notation.

- For every $a \in L_0$, $a^- = \bigvee\{b \in L_0 \,/\, b < a\}$.
- $L_0\langle x\rangle$ denotes the $\mathcal{L}_{\rm SC}$-substructure of L generated by $L_0 \cup \{x\}$.
- $g(x, L_0) = \bigwedge\{a \in L_0 \,/\, x \leq a\}$.

Clearly $a \in \mathcal{I}(L_0)$ if and only if a^- is the unique predecessor of a in L_0 (otherwise $a^- = a$). We say that a tuple (x_1, x_2) of elements of L is **primitive over** L_0 if there exists $g \in \mathcal{I}(L_0)$ such that:

1. x_1, x_2 are sc-pure of the same sc-dimension.
2. Each $g^- \wedge x_i \in L_0$.
3. One of the following happens:
   - $x_1 = x_2$ and $g^- \wedge x_1 \ll x_1 \ll g$.
   - $x_1 \neq x_2$, $x_1 \wedge x_2 \in L_0$ and $g - x_1 = x_2$, $g - x_2 = x_1$.

The above conditions imply that each $x_i \notin L_0$ and:

$$g = g(x_1, L_0) = g(x_2, L_0)$$

We say that L is **primitively generated over** L_0, or simply that it is a **primitive extension** of L_0, if there exists (x_1, x_2) primitive over L_0 such that $L = L_0\langle x_1, x_2\rangle$ (then clearly $L = L_0\langle x\rangle_1 = L_0\langle x\rangle_2$). By the following proposition such a tuple is necessarily unique.

Proposition 6.1 *Let L_0 be a finite subscaled lattice, L an extension generated over L_0 by a primitive tuple (x_1, x_2), and let $g = g(x_1, L_0)$.*

Then L is exactly the upper semi-lattice generated over L_0 by x_1, x_2. It is a finite subscaled lattice and one of the following happens:

1. *$x_1 = x_2$, $\text{sc-dim}\, x_1 < \text{sc-dim}\, g$ and $\mathcal{I}(L) = \mathcal{I}(L_0) \cup \{x_1\}$.*
2. *$x_1 \neq x_2$, $\text{sc-dim}\, x_1 = \text{sc-dim}\, g$ and $\mathcal{I}(L) = (\mathcal{I}(L_0) \setminus \{g\}) \cup \{x_1, x_2\}$.*

Proof: Let L_1 be the upper semi-lattice generated over L_0 by x_1, x_2. In order to show that $L_1 = L$ it is sufficient to check that L_1 is an $\mathcal{L}_{\rm SC}$-substructure of L.

Let $p = \text{sc-dim}\, g$ and $q = \text{sc-dim}\, x_1 = \text{sc-dim}\, x_2$. By TC_1, $g^- \ll g$ hence $g^- \wedge x_i \ll g$ since $g^- \wedge x_i \in L_0$ by assumption. So $\text{sc-dim}\, g^- < p$. We will need the following facts, for every $a \in L_0$:

$$g \not\leq a \quad\Longrightarrow\quad a \wedge x_i \in L_0 \;\text{ and }\; a \wedge x_i \ll x_i \tag{7}$$

Indeed $g \wedge a = g^- \wedge a$ hence $a \wedge x_i = (a \wedge g) \wedge x_i = a \wedge (g^- \wedge x_i)$. By assumption $x_i \wedge g^- \in L_0$ hence $a \wedge x_i \in L_0$ is proved. Moreover $x_i \wedge (g^- \wedge a) \leq x_i \wedge g^-$ hence it suffices to check that $g^- \wedge x_i \ll x_i$. If $x_1 = x_2$ this is an assumption. Otherwise $x_1 = g - x_2$ and $x_2 = g - x_1$ are p-sc-pure by SC_{13}. Then $x_i - g^- = x_i$ by SC_5, since $\text{sc-dim}\, g^- < p$, so $g^- \wedge x_i \ll x_i$.

L_1 is a sublattice of L then follows easily from (7), the distributivity law, and the fact that by assumption $x_1 \wedge x_2 \in L_0 \cup \{x_1\}$.

Since L_0 is finite and L_1 is generated by $L_0 \cup \{x_1, x_2\}$ as an upper semilattice, it follows immediatly that L_1 is finite and:

$$\mathcal{I}(L_1) \subseteq \mathcal{I}(L_0) \cup \{x_1, x_2\} \tag{8}$$

So for any $b' < b \in cI(L_1)$ it is easily seen that sc-dim$(b') <$ sc-dim b in L. Corollary 3.3 then shows that L_1 is an $\mathcal{L}_{\mathrm{SC}}$-substructure of L, hence $L_1 = L$.

We turn now to the description of $\mathcal{I}(L)$. If $x_1 \neq x_2$ then of course $g = x_1 \vee x_2 \notin \mathcal{I}(L)$. Conversely if $x_1 = x_2$ then (8) implies that g is $\vee$-irreducible in L, so:

$$g \in \mathcal{I}(L) \iff x_1 \neq x_2 \tag{9}$$

Assume that $\mathcal{I}(L_0) \not\subseteq \mathcal{I}(L)$. Let $b \in \mathcal{I}(L_0) \setminus \mathcal{I}(L)$ and let $y_1, \ldots, y_r$ $(r \geq 2)$ be its $\vee$-irreducible components in L_1. By (8), each y_i either belongs to L_0 or to $\{x_1, x_2\}$, and at least one of them does not belong to L_0. We may assume without loss of generality that $y_1 = x_1$. Then $x_1 \leq b$ hence $g \leq b$. If $g < b$ then $g \ll b$ since $b \in \mathcal{I}(L_0)$ so $b - g = b$, but then we have a contradiction:

$$y_1 \leq b - g \leq b - x_1 = \bigvee_{i=2}^{r} y_i$$

So $b = g$. We have proved that:

$$\mathcal{I}(L_0) \setminus \{g\} \subseteq \mathcal{I}(L) \tag{10}$$

The conclusion follows by combining (8), (9), (10) and an obvious argument of cardinality.
■

It is not difficult to deduce from Proposition 6.1 that any primitively generated extension L of a finite subscaled lattice L_0 is minimal, in the sense that there is no intermediate proper extension $L_0 \subset L_1 \subset L$. Only the converse, which we prove now, is actually needed in the remaining of this paper.

Proposition 6.2 *Any finitely generated proper extension L of a finite d-subscaled lattice L_0 is the union of a finite chain of primitively generated extensions of L_0.*

Proof: Since L is finite by the local finiteness Theorem 4.1, it suffices to show that L contains a primitive extension of L_0. Take any element x minimal in $\mathcal{I}(L) \setminus L_0$. Observe that if y is any element of L stricly smaller than x then all the $\vee$-irreducible components of y in L actually belong to L_0, so $y \in L_0$.

Let $g = g(x, L_0)$. For every $a \in L_0$, if $a < g$ then $a \not\leq x$ hence $a \wedge x < x$, so $a \wedge x \in L_0$. It follows that $g^- \wedge x \in L_0$, hence $g^- < g$. In particular $g \in \mathcal{I}(L_0)$.

Since $x \in \mathcal{I}(L)$ it is sc-pure, and moreover $g^- \wedge x < x$ implies that $g^- \wedge x \ll x$ by TC_1. So if moreover $x \ll g$ then we have proved that (x, x) is primitive over L_0.

On the other hand if $x \not\ll g$ let $x_1 = x$ and $x_2 = g - x$. Since $g \in \mathcal{I}(L_0)$ it is sc-pure, and so are x_1 and x_2, with the same sc-dimension $p = \text{sc-dim}\, g$ (indeed $g - x \neq \mathbf{0}$ since $x < g$). Moreover $x_1 \wedge x_2 \ll x_2$ by TC_3 so $\text{sc-dim}\, x_1 \wedge x_2 < p$ hence $g - x_1 = x_2$ and $g - x_2 = x_1$ by TC_1, TC_2 and the p-sc-purity of x_1 and x_2. Eventually $x_1 \wedge x_2 < x$ imply that $x_1 \wedge x_2 \in L_0$. We conclude that (x_1, x_2) is primitive over L_0.
■

7 Signatures

A triple (g, q, H) will be called a **signature** in a finite subscaled lattice L_0 if and only if $g \in \mathcal{I}(L_0)$, $q \leq \text{sc-dim}\, g$ is a positive integer, and H is a subset of two (not necessarily distinct) elements $h_1, h_2 \in L_0$ such that:

- $h_1 \vee h_2 < g$ (hence $h_1 \vee h_2 \ll g$ by TC_1)
- $\text{sc-dim}(h_1 \vee h_2) < q$
- If $q < \text{sc-dim}\, g$ then $h_1 = h_2$.

By Proposition 6.1 and the definition, any extension L primitively generated over L_0 by x_1, x_2 determines a unique signature:

$$\sigma(L) = (g(x_1, L_0), \text{sc-dim}\, x_1, \{x_1 \wedge g^-, x_2 \wedge g^-\})$$

which we call the **signature of** L **in** L_0. It determines the extension $L|L_0$ as follows.

Proposition 7.1 *Two primitively generated extensions of a finite subscaled lattice L_0 are isomorphic over L_0 if and only if they have the same signature in L_0.*

Proof: Let L (resp L') be an extension of L_0 generated over L_0 by a primitive tuple (x_1, x_2) (resp. x'_1, x'_2). If they are isomorphic over L_0 then obviously $\sigma(L) = \sigma(L')$. Conversely assume that:

$$\sigma(L) = \sigma(L') = (g, q, \{h_1, h_2\})$$

Reordering if necessary we may assume that each $h_i = x_i \wedge g^- = x'_i \wedge g^-$. Let $\varphi(x_i) = x'_i$ for each $i \in \{1, 2\}$, and $\varphi(a) = a$ for every $a \in \mathcal{I}(L_0)$. By Proposition 6.1 this is a well defined bijection from $\mathcal{I}(L)$ to $\mathcal{I}(L')$ which preserves the order, hence it extends uniquely to an isomorphism of upper semi-lattice $\varphi\colon L' \to L$. The $\mathcal{L}_{\text{lat}}$-structure of a lattice being an extension by definition of its upper semi-lattice structure, this is an $\mathcal{L}_{\text{lat}}$-isomorphism. Moreover by construction φ preserves the sc-dimensions of the $\vee$-irreducible elements of L and L', hence by Proposition 3.1 it is an $\mathcal{L}_{\text{SC}}$-isomorphism, whose restriction to L_0 is the identity.
■

Proposition 7.2 *Let $\sigma = (g, q, \{h_1, h_2\})$ be a signature in a finite subscaled lattice L_0. There exists a primitive extension L of L_0 whose signature in L_0 is precisely σ.*

Proof: We only treat the case when $q = \text{sc-dim}\, g$. The case when $q < \text{sc-dim}\, g$ (hence $h_1 = h_2$) is similar, and left to the reader.

Let x_1, x_2 be any two distinct elements in a set disjoint from $\mathcal{I}(L_0)$ and let:

$$\mathcal{I} = (\mathcal{I}(L_0) \setminus \{g\}) \cup \{x_1, x_2\}$$

The order on $\mathcal{I}(L_0) \setminus \{g\}$ inherited from L_0 can be extended to $\mathcal{I}$ by stating that $x_1 \not\leq x_2$, $x_2 \not\leq x_1$, and for every $b \in \mathcal{I}(L_0) \setminus \{g\}$ and every $j \in \{1, 2\}$:

- $b < x_j \iff b \leq h_j$.
- $x_j < b \iff g \leq b$

For every $z \in \mathcal{I}$ let $z\!\uparrow = \{y \in \mathcal{I} \,/\, z \leq y\}$. Let L be the upper-semilattice generated in the power set $\mathcal{P}(\mathcal{I})$ of $\mathcal{I}$ by all the $z\!\uparrow$'s with z ranging over $\mathcal{I}$. This is a sublattice of $\mathcal{P}(\mathcal{I})$ such that $\mathcal{I}(L) = \{z\!\uparrow \,/\, z \in \mathcal{I}\}$, that is:

$$\mathcal{I}(L) = \big(\{b\!\uparrow \,/\, b \in \mathcal{I}(L_0)\} \setminus \{g\!\uparrow\}\big) \cup \{x_1\!\uparrow, x_2\!\uparrow\}$$

Let $\varphi(g) = x_1\!\uparrow \cup x_2\!\uparrow$, and $\varphi(b) = b\!\uparrow$ for every $b \in \mathcal{I}(L_0) \setminus \{g\}$. This application uniquely extends to an embedding of upper semi-lattice that we still denote φ from L_0 to L, which is easily seen to be an $\mathcal{L}_{\text{lat}}$-embedding. For every $z \in \mathcal{I}$ let:

$$D(z\!\uparrow) = \begin{cases} \text{sc-dim}(z) & \text{if } z \in \mathcal{I}(L_0) \setminus \{g\}, \\ \text{sc-dim}(g) & \text{otherwise.} \end{cases}$$

This is a strictly increasing map from $\mathcal{I}(L)$ to $\mathbb{N}$ hence it determines an $\mathcal{L}_{\text{SC}_d}$-structure on L as in Example 2.9. Proposition 3.1 asserts that φ is an $\mathcal{L}_{\text{SC}_d}$-embedding. By construction L is primitively generated over L_0 by $x_1\!\uparrow, x_2\!\uparrow$, with signature σ in L_0.

The case when $q < \text{sc-dim}\, g$ (hence $h_1 = h_2$) is similar, and left to the reader.
■

8 Model-completion

We call **super d-scaled lattice** (resp. **super scaled lattice**) any d-subscaled lattice (resp. subscaled lattice) L which satisfy the following additionnal properties.

Scaling: L is a scaled lattice.

Catenarity: For every positive integers $r \leq q \leq p$ and every elements $c \leq a$, if a has pure dimension p and c has pure dimension r then there exists a q-pure element b such that $c \leq b \leq a$.

Splitting: For every elements b_1, b_2, a, if $b_1 \vee b_2 \ll a$ then there exists non-zero elements $a_1 \geq b_1$ and $a_2 \geq b_2$ such that:

$$\begin{cases} a_1 = a - a_2 \\ a_2 = a - a_1 \\ a_1 \wedge a_2 = b_1 \wedge b_2 \end{cases}$$

All these properties are clearly axiomatizable in $\mathcal{L}_{\mathrm{SC}_d}$, using only finitely many $\forall\exists$-formulas. The name of the second one comes form the fact that in a subscaled lattice whose spectrum is a noetherian topological space, this property is equivalent to the usual notion of catenarity, namely that any two maximal chains in $\mathrm{Spec}(L)$ having the same first and last element have the same length.

Proposition 8.1 *Every subscaled lattice $\mathcal{L}_{\mathrm{SC}}$-embeds in a superscaled lattice.*

Proof: Let L_0 be a finitely generated subscaled lattice and $a, b_1, b_2 \in L_0$ such that $b_1 \vee b_2 \ll a$. By Theorem 4.1, L_0 is finite. Let $g_1, \dots, g_r$ be the $\vee$-irreducible components of a in L_0, and for every $i \leq r$ let g_i^- be the unique predecessor of g_i in L_0, $h_{i,1} = b_1 \wedge g_i^-$ and $h_{i,2} = b_2 \wedge g_i^-$.

$\sigma_1 = (g_1, \text{sc-dim}\, g_1, \{h_{1,1}, h_{1,2}\})$ is a signature in L_0. Proposition 7.2 gives an extension L_1 primitively generated over L_0 by $x_{1,1}, x_{1,2}$, with signature σ_1 in L_0. By construction $x_{1,1} \wedge x_{1,2}$ belongs to L_0 and is stricly smaller than g_1 hence $x_{1,1} \wedge x_{1,2} \leq g_1^-$ For every $i \geq 2$, $g_i \in \mathcal{I}(L_0) \setminus \{g_1\} \subseteq \mathcal{I}(L_1)$, and moreover g_i^- is still the unique predecessor of g_i in L_1. Indeed let $g_i^\dagger$ be the union of every $c \in \mathcal{I}(L_1)$ strictly smaller than g_i. Neither $x_{1,1}$ nor $x_{1,2}$ are smaller than g_i because $g_1 \not\leq g_i$, so every such c must belong to $\mathcal{I}(L_0)$ by Proposition 6.1. It follows that $g_i^\dagger = g_i^-$.

So we can repeat in L_1 the same construction applied to $g_2, h_{2,1}, h_{2,2}$, and after r steps we obtain a chain of extensions $(L_i)_{i \leq r}$ and non-zero elements $x_{i,j} \in L = L_r$ such that for every i ($1 \leq i \leq r$):

$$g_i - x_{i,1} = x_{i,2} \quad \text{and} \quad g_i - x_{i,2} = x_{i,1} \tag{11}$$

$$x_{i,1} \wedge g_i^- = h_{i,1} \quad \text{and} \quad x_{i,2} \wedge g_i^- = h_{i,2} \tag{12}$$

$$x_{i,1} \wedge x_{i,2} \leq g_i^- \tag{13}$$

Let $x_1 = \bigvee_{i \leq r} x_{i,1}$ and $x_2 = \bigvee_{i \leq r} x_{i,2}$. For every $i \leq r$ and every $k \neq i$, $g_i \wedge x_{k,1} \leq g_i \wedge g_k \ll g_i$, so by TC_4 and (11):

$$g_i - x_1 = g_i - \bigvee_{1 \leq k \leq r} x_k = g_i - x_{i,1} = x_{i,2}$$

So by TC_2, $a - x_1 = \bigvee_{i \leq r} (g_i - x_1) = x_2$. Symmetrically $a - x_2 = x_1$.

(12) and (13) imply that $x_{i,1} \wedge x_{i,2} = x_{i,1} \wedge g_i^- \wedge x_{i,2} = h_{i,1} \wedge h_{i,2}$. Similarly for every $k \neq i$:

$$x_{i,1} \wedge x_{k,2} = x_{i,1} \wedge g_i \wedge g_k \wedge x_{k,2} = x_{i,1} \wedge g_i^- \wedge g_k^- \wedge x_{k,2} = h_{i,1} \wedge h_{k,2}$$

So eventually:

$$x_1 \wedge x_2 = \bigvee_{1\le i\le r} \bigwedge_{1\le k\le r} x_{i,1} \wedge x_{k,2} = \bigvee_{1\le i\le r} \bigwedge_{1\le k\le r} h_{i,1} \wedge h_{k,2} = b_1 \wedge b_2$$

We have proved that for every finitely generated scaled lattice L_0 and every $a, b_1, b_2 \in L_0$ such that $b_1 \vee b_2 \ll a$, there exists an $\mathcal{L}_{\rm SC}$-embeding φ from L_0 to a subscaled lattice L in which there exists non-zero elements x_1, x_2 such that, after identifying L_0 to its image:

$$\begin{cases} x_1 = a - a_2 \\ x_2 = a - a_1 \\ x_1 \wedge x_2 = \varphi(b_1) \wedge \varphi(b_2) \end{cases}$$

On the other hand, Theorem 4.1 and Proposition 5.3, show that every finitely generated subscaled lattice also $\mathcal{L}_{\rm SC}$-embeds into some $\mathrm{L}_{\rm lin}(X)$, which is a catenary scaled lattice. The model-theoretic compactness argument then implies that every subscaled lattice $\mathcal{L}_{\rm SC}$-embeds in a superscaled lattice.
■

Proposition 8.2 *Let L_0 be a finite $\mathcal{L}_{\rm SC}$-substructure of a super scaled lattice $\hat{L}$. Then for every signature σ in L_0 there exists a primitive tuple $(x_1, x_2) \in \hat{L}$ such that σ is the signature of $L_0\langle x_1, x_2\rangle$ in L_0.*

Proof: Let $\sigma = (g, q, \{h_1, h_2\})$ be a signature in L_0, let $p = \text{sc-dim}\, g$ and $r = \text{sc-dim}(h_1 \vee h_2)$. Observe that since g is $\vee$-irreducible in L_0 it is sc-pure, and admits a unique predecessor g^- in L_0. Let $y_1, y_2 \in \hat{L}$ given by the splitting property applied to g, h_1, h_2. By construction $y_1 \vee y_2 = g$, and since g has pure sc-dimension p so does each y_i. Moreover:

$$y_1 \wedge h_2 \le y_1 \wedge y_2 = h_1 \wedge h_2$$

hence $y_1 \wedge (h_1 \vee h_2) = h_1 \vee (y_1 \wedge h_2) = h_1$. Since $h_1 \vee h_2 < g$ it follows that $y_1 \wedge g^- = h_1 \in L_0$, and symetrically $y_2 \wedge g^- = h_2 \in L_0$. So (y_1, y_2) is primitive over L_0, and the signature of $L_1 = L_0\langle y_1, y_2\rangle$ in L_0 is $(g, p, \{h_1, h_2\})$.

If $p = q$ then we are done. Assume now that $q < p$, hence $h_1 = h_2$. For every $i \le r$, the catenarity gives an element $x_i \in \hat{L}$ of pure sc-dimension p such that $\mathrm{C}^i(h_1) \le x_i \le y_1$. Even in case $r = -1$ the catenarity gives $x_{-1} \in \hat{L}$ of pure sc-dimension p such that $x_{-1} \le y_1$. Let $x = \bigvee_{-1\le i\le r} x_i$, by construction x has pure sc-dimension p and $h_1 \le x \le y_1$. Then $x \le g$, and in view of their sc-dimension $x \ll g$. Moreover:

$$h_1 \le x \wedge g^- \le y_1 \wedge g^- = h_1$$

hence $h_1 \wedge x = h_1 \in L_0$ and in view of the sc-dimensions, $h_1 \ll x$. So (x, x) is a primitive tuple over L_0, of signature σ in L_0.
■

Theorem 8.3 *The theory of super d-scaled lattices is the model-completion of the theory of d-subscaled lattices*

Remark 8.4 Obviously the theorem remains true by replacing everywhere d-(sub)scaled lattices by (sub)scaled lattices of sc-dimension at most d (resp. exactly d).

Proof: Since the asioms of super d-scaled lattices are $\forall\exists$, it follows from Proposition 8.1 that every existentially closed d-subscaled lattice is super d-scaled. By classical model-theoretic arguments, it is then sufficient to show that: given a d-superscaled lattice $\hat{L}$, a finitely generated d-subscaled lattice L, and a common $\mathcal{L}_{\mathrm{SC}_d}$-substructure L_0 of L and $\hat{L}$, there exists an $\mathcal{L}_{\mathrm{SC}_d}$-embedding of L into $\hat{L}$ whose restriction to L_0 is the identity.

By the local finiteness Theorem 4.1, L_0 is finite. By Proposition 6.2 an immediate induction allows us to assume w.l.o.g. that L is primitively generated over L_0. Let σ be the signature of L over L_0. By Proposition 8.2 there exists a primitively generated extension L_1 of L_0 in $\hat{L}$ whose signature is σ. Eventually L is isomorphic to L_1 over L_0 by Proposition 7.1.
■

The completions of the theory of super d-scaled lattices are easy to classify. Let us say that a d-subscaled lattice is **prime** if it does not contain any proper d-subscaled lattice, or equivalently if it is generated by the empty set. Every prime d-subscaled lattice is finite. By Corollary 4.2 their exists finitely many prime d-subscaled lattices up to isomorphism.

Corollary 8.5 *The theory of super d-scaled lattices containing (a copy of) a given prime d-subscaled lattice is $\aleph_0$-categorical, hence complete. It is also finitely axiomatisable, hence decidable. Since every completion of the theory of super d-scaled lattices is of that kind, the theory of super d-scaled lattices is decidable.*

Proof: Let L, L' be any two countable super d-scaled lattices containing isomorphic prime d-subscaled lattice L_0 and L'_0. Any partial ismorphism between L and L' (extending the given isomorphism between L_0 and L'_0) can be extended by va-et-vient, using exactly the same argument as in the proof of Theorem 8.3. So the first statement is proved. The other ones are immediate consequences.
■

9 Atomic scaled lattices

A natural example of a super scaled lattice is not easy to find. Indeed if X is any topological space in which points are closed, the points of X are the atoms of $L(X)$, and the splitting axiom imply that a super scaled lattice has no atom. In this section we explore a possible solution to this problem, which lead us to a new conjecture in p-adic semi-algebraic geometry.

Let $\mathcal{L}_{\text{ASC}} = \mathcal{L}_{\text{SC}} \cup \{\text{At}_k\}_{k\in\mathbb{N}^*}$ with each At_k a new unary predicate symbol. We call **sub-ASC-lattices** the $\mathcal{L}_{\text{ASC}}$-structures whose $\mathcal{L}_{\text{SC}}$-reduct is a subscaled lattice and which satisfy the following lists of universal axioms:

$\mathbf{ASC_1}$: $(\forall k \neq l), \quad \forall a,\ \text{At}_k(a) \to \neg\text{At}_l(a)$

$\mathbf{ASC_2}$: $(\forall k), \quad \forall a, a_0, \ldots, a_{2^k}, \quad \text{At}_k(a) \longrightarrow$

$$\text{sc-dim}\, a = 0 \bigwedge \left[\left(\bigvee_{0\le i\le 2^k} a_i = a\right) \to \left(\bigvee_{0\le i<j\le 2^k} a_i = a_j\right)\right]$$

$\mathbf{ASC_3}$: $(\forall k, n), \quad \forall a, a_1, \ldots, a_n,$

$$\left[\left(a = \bigvee_{1\le i\le n} a_i\right) \bigwedge \left(\bigwedge_{1\le j<i\le n} a_i \wedge a_j = \mathbf{0}\right)\right] \longrightarrow$$

$$\left[\text{At}_k(a) \longleftrightarrow \bigvee_{l_1+\cdots+l_n=k}\ \bigwedge_{1\le i\le n} \text{At}_{l_i}(a_i)\right]$$

For any $\mathcal{L}_{\text{ASC}}$-structure L we denote by $\text{At}_k(L)$ the set of elements a in L such that $L \models \text{At}_k(a)$, and we let $\text{At}_0(L) = L \setminus \bigcup_{k>0} \text{At}_k(L)$. If L is a sub-ASC-lattice then ASC_1 asserts that $(\text{At}_k(L))_{k\in\mathbb{N}}$ is a partition of L. For any $a \in L$ we then define $\text{asc}(a)$ as the unique $k \in \mathbb{N}$ such that $a \in \text{At}_k(L)$. The other axioms of sub-ASC-lattices have the following meaning:

- If $\text{asc}(a) = k > 0$ then ASC_2 asserts that $L(a)$ is a boolean algebra generated by n atoms $a_1, \ldots, a_n$ with $n \le k$, and ASC_3 then implies that each $\text{asc}(a_i) > 0$ and for every $b \in L(a)$:

  $$\text{asc}(b) = \sum_{c\in\mathcal{C}(b)} \text{asc}(c)$$

  where $\mathcal{C}(b)$ is the set of atoms in $L(a)$ (as well as in L) smaller than b.

- Conversely ASC_3 implies that if $L(a)$ is finite and $\text{asc}(c) > 0$ for each atom c in $L(a)$:

  $$\text{asc}(a) = \sum_{c\in\mathcal{I}(L(a))} \text{asc}(c)$$

  In particular if $a \neq \mathbf{0}$ then $\text{asc}(a) > 0$.

Remark 9.1 It follows immediatly that an $\mathcal{L}_{\text{SC}}$-embedding of sub-ASC-lattices $\varphi\colon L \to L'$ is an $\mathcal{L}_{\text{ASC}}$-embedding if and only if for every $k > 0$ and every *atom* $a \in L$:

$$L \models \text{At}_k(a) \iff L' \models \text{At}_k(\varphi(a))$$

Every natural example of sub-ASC-lattice L satisfy the following additionnal property, which imply ASC_1 to ASC_3:

$\mathbf{ASC_0}$: L is a scaled lattices, and for every $k > 0$, $\text{At}_k(L)$ is the set of elements of L which are the join of exactly k atoms in L.

We call **ASC-lattices**[4] the sub-ASC-lattices which satisfy ASC_0. Every subscaled lattice L admits a unique structure of ASC-lattice which is an extension by definition of its lattice structure. We denote by L^{At} this expansion of L.

Proposition 9.2 (Linear representation) *Let K be an infinite field and let L_0 be a finite sub-*ASC*-lattice. For any positive integer N there exists a positive integer m, not depending on N, and an $\mathcal{L}_{\mathrm{SC}}^*$-embedding $\varphi_N : L_0 \to \mathrm{L}_{\mathrm{lin}}^{\mathrm{At}}(K^m)$ such that for every element a of L_0 of sc-dimension zero:*

$$\begin{aligned} \mathrm{asc}(a) > 0 &\implies \mathrm{asc}(\varphi_N(a)) = \mathrm{asc}(a) \\ \mathrm{asc}(a) = 0 &\implies \mathrm{asc}(\varphi_N(a)) \geq N \end{aligned}$$

Proof: We prove by induction on tuples (r, s) ordered lexicographically that the proposition holds for every finite sub-ASC-lattice L_0 having r non-zero $\vee$-irreducible elements, s of which have the same sc-dimension as L_0.

If $r = 0$ then $s = 0$ and the unique embedding of $L_0 = \{\mathbf{0}\}$ into $\mathrm{L}_{\mathrm{lin}}^{\mathrm{At}}(K^0)$ has the required property. So let us assume that $r \geq 1$ and that the result is proved for every $(r', s') < (r, s)$. Let $a_1 \dots, a_r$ be the elements of $\mathcal{I}(L_0)$ ordered by increasing sc-dimension, so $d = \text{sc-dim}\, L = \text{sc-dim}\, a_r \geq 0$.

Case $d = 0$**.** Then L is a boolean algebra, and $a_1, \dots, a_r$ are its atoms. For every $i \leq r$ we choose a finite subset A_i of $K \setminus \bigcup_{j<i} A_j$ such that:

- If $\mathrm{asc}(a_i) > 0$, A_i has $\mathrm{asc}(a_i)$ elements, so $\mathrm{asc}(A_i) = \mathrm{asc}(a_i)$.
- If $\mathrm{asc}(a_i) = 0$, A_i has N elements, so $\mathrm{asc}(A_i) = N$.

Clearly $m = 1$ does not depend on N, and the map φ which maps each a_i to A_i extends uniquely to an $\mathcal{L}_{\mathrm{SC}}^*$-embedding of L_0 into $\mathrm{L}_{\mathrm{lin}}^{\mathrm{At}}(K)$ which has the required properties.

Case $d > 0$**.** The upper semi-lattice L_0^- generated by $a_1, \dots, a_{r-1}$ is an $\mathcal{L}_{\mathrm{ASC}}^*$-substructure of L_0 to which the induction hypothesis applies. This gives a positive integer m not depending on N and an $\mathcal{L}_{\mathrm{SC}}^*$-embedding $\varphi : L_0^- \to \mathrm{L}_{\mathrm{lin}}^{\mathrm{At}}(K^m)$ having the required properties. One can extend φ to an embedding $\bar{\varphi}$ of L_0 into some $\mathrm{L}_{\mathrm{lin}}^{\mathrm{At}}(K^{m+p})$ exactly like in the proof of Proposition 5.3. The integer $m+p$ does not depend on N and $\bar{\varphi}$ inherits from φ the required properties because all the elements of L with sc-dimension zero belong to L_0^-.
∎

Let $\mathrm{ASC}_{\mathrm{Zar}}(K, d)$ (resp. $\mathrm{ASC}_{\mathrm{lin}}(K, d)$, resp. $\mathrm{ASC}_{\mathrm{def}}(K, d)$) the class of all ASC-lattices L^{At} with L ranging over $\mathrm{SC}_{\mathrm{Zar}}(K, d)$, (resp. $\mathrm{SC}_{\mathrm{lin}}(K, d)$, resp. $\mathrm{SC}_{\mathrm{def}}(K, d)$).

Corollary 9.3 *For any infinite field K and positive integer d, the universal theories of $\mathrm{ASC}_{\mathrm{Zar}}(K, d)$ and $\mathrm{ASC}_{\mathrm{lin}}(K, d)$) are exactly the theory of sub-*ASC*-lattices.*

The same holds for $\mathrm{ASC}_{\mathrm{def}}(K, d)$ if moreover K is a henselian valued field of characteristic zero, a real closed field or an algebraically closed field.

[4]Of course we will show that the sub-ASC-lattices are precisely the $\mathcal{L}_{\mathrm{ASC}}$-substructures of ASC-lattices.

Proof: Since $\mathrm{ASC_{lin}}(K,d)$ is contained in the other classes, all of which are contained in the class of ASC-lattices, it suffices to prove that every sub-ASC-lattice $\mathcal{L}_{\mathrm{ASC}}$-embeds into an ultraproduct of elements of $\mathrm{ASC_{lin}}(K,d)$.

For any positive integer N let $\varphi_N : L_0 \to \mathrm{L}^{\mathrm{At}}_{\mathrm{lin}}(K^m)$ be the $\mathcal{L}^*_{\mathrm{SC}}$-embedding given by Proposition 9.2. We still denote by φ_N the induced $\mathcal{L}_{\mathrm{SC}}$-embedding of L_0 into $\mathrm{L}^{\mathrm{At}}_{\mathrm{lin}}(X_N)$ where $X_N = \varphi_N(\mathbf{1}_{L_0})$. Let $\mathcal{U}$ be a non principal ultrafilter in the boolean algebra of subsets of $\mathbb{N}$, and $L = \prod_{N\in\mathbb{N}} \mathrm{L}^{\mathrm{At}}_{\mathrm{lin}}(X_N)/\mathcal{U}$. Then $\varphi = \prod_{N\in\mathbb{N}} \varphi_N/\mathcal{U}$ is an $\mathcal{L}_{\mathrm{SC}}$-embedding of L_0 into the ultraproduct L. Let a be an atom of L_0 and $k = \mathrm{asc}(a)$.

If $k > 0$ then for every $N \geq k$, $\mathrm{L}^{\mathrm{At}}_{\mathrm{lin}}(X_N) \models \mathrm{At}_k(\varphi_N(a))$ by construction. So $L \models \mathrm{At}_k(\varphi(a))$, that is $\mathrm{asc}(\varphi(a)) = k$.

If $k = 0$, let l be any stricly positive integer. For every $N \geq l$, $\mathrm{L}^{\mathrm{At}}_{\mathrm{lin}}(X_N) \models \mathrm{At}_N(\varphi_N(a))$ by construction, hence $lalin(X_N) \not\models \mathrm{At}_l(\varphi_N(a))$. So $L \not\models \mathrm{At}_l(\varphi(a))$, and this being true for every $l > 0$ it follows that $\mathrm{asc}(\varphi(a)) = 0$.

By Remark 9.1, φ is then an $\mathcal{L}_{\mathrm{ASC}}$-embedding.

■

Let us call **super ASC-lattices** those ASC-lattices which satisfy the following axioms:

Atomicity: Every element x is the least upper bound in L of the set of atoms of L smaller that x.

Catenarity: For every positive integers $r \leq q \leq p$ and every $c \leq a$, if a has pure dimension p and c has pure dimension r then there exists a q-pure element b such that $c \leq b \leq a$.

ASC-Splitting: For every b_1, b_2, a, if $b_1 \vee b_2 \ll a$ and $\mathrm{C}^0(a) = \mathbf{0}$ there exists non-zero elements $a_1 \geq b_1$ and $a_2 \geq b_2$ such that:

$$\begin{cases} a_1 = a - a_2 \\ a_2 = a - a_1 \\ a_1 \wedge a_2 = b_1 \wedge b_2 \end{cases}$$

The class of super-ASC-lattices of sc-dimension at most d (resp. exactly d) is clearly axiomatisable by $\forall\exists$-formulas in $\mathcal{L}_{\mathrm{ASC}}$. We are going to show that its theory is the model-completion of the theory of sub-ASC-lattices of dimension at most d (resp. exactly d).

Remark 9.4 An immediate consequence of the atomicity axiom is that for every elements x, y in a super ASC-lattice L such that $y \ll x$, there are infinitely many atoms $a \in L$ such that $a \leq x$ and $a \wedge y = \mathbf{0}$. Indeed $y < x$ hence by the atomicity axiom there is an atom $a_1 \in L$ such that $a_1 \leq x$ and $a_1 \not\leq y$. Then $a_1 \leq x$ and $a_1 \wedge y < a_1$ hence $a_1 \wedge y = \mathbf{0}$ (because a_1 is an atom). Moreover $\text{sc-dim}\, y \vee a_1 < \text{sc-dim}\, x$ because $y \ll x$, hence $y \vee a_1 < x$ so the same argument applies to x and $y \vee a_1$. It gives another atom $a_2 \leq x$ such that $a_2 \wedge y = \mathbf{0}$, and so on.

Primitive tuples and primitive extensions are defined for sub-ASC-lattices exactly like for subscaled lattices. If L is an extension of a sub-ASC-lattice L_0 and $x \in L$ we denote now $L_0\langle x\rangle$ the $\mathcal{L}_{\mathrm{ASC}}$-substructure of L generated by $L_0 \vee \{x\}$ (that is the subscaled lattice generated by $L_0 \cup \{x\}$ endowed with the $\mathcal{L}_{\mathrm{ASC}}$-structure induced by L).

We define **ASC-signatures** in a finite sub-ASC-lattice L_0 as triples (g, p, H) with H a set of non-necessarily distinct ordered pairs (h_1, k_1), (h_2, k_2) such that $(g, q, \{h_1, h_2\})$ is a signature in the $\mathcal{L}_{\mathrm{SC}}$-reduct of L_0, k_1, k_2 are positive integers and:

1. If $q < \text{sc-dim}\, g$ then $(h_1, k_1) = (h_2, k_2)$.
2. If $q \neq 0$ then $k_1 = k_2 = 0$.
3. If $k_1 \neq 0$, $k_2 \neq 0$ and $\text{sc-dim}\, g = 0$ then $\mathrm{asc}(g) = k_1 + k_2$
4. If $k_1 = 0$ or $k_2 = 0$ then $\mathrm{asc}(g) = 0$.

Example 9.5 Let L_0 be a finite sub-ASC-lattice, and L an extension of L_0 generated by a primitive tuple (x_1, x_2). Let $(g, p, \{h_1, h_2\})$ be the signature of L in L_0 (in the sense of subscaled lattices) and $k_i = \mathrm{asc}(x_i)$. Then $(g, p, \{(h_1, k_1), (h_2, k_2)\}$ is easily seen to be an ASC-signature in L_0, uniquely determined by L. We call it the **ASC-signature of L in L_0**.

The same argument as in Proposition 7.1 shows (using Remark 9.1 in addition to Proposition 3.1) that two primitively generated extensions of a finite sub-ASC-lattice L_0 are isomorphic over L_0 if and only if they have the same signature in L_0.

Proposition 9.6 *Let L_0 be a finite $\mathcal{L}_{\mathrm{ASC}}$-substructure of an $\aleph_0$-saturated super* ASC-*lattice $\hat{L}$. Then for every* ASC-*signature σ_{At} in L_0 there exists a primitive tuple $(x_1, x_2) \in \hat{L}$ such that σ_{At} is the signature of $L_0\langle x_1, x_2\rangle$ in L_0.*

Proof: Let $\sigma_{\mathrm{At}} = (g, q, \{(h_1, k_1), (h_2, k_2)\})$ be an ASC-signature in L_0, let $\sigma = (g, q, \{h_1, h_2\})$ and $p = \text{sc-dim}\, g$. In each of the following cases, the fact that the constructed tuple (x_1, x_2) is primitive over L_0 and generates an extension whose signature in L_0 is precisely σ_{At} is straighforward.

Case $p \geq 1$**.** Then g is not an atom, so the ASC-splitting property applies to g, h_1, h_2. Moreover by Remark 9.4, there are infinitely many atoms a in L smaller than g and not in L_0. By $\aleph_0$-saturation it follows that there exists $x_0 \in L \setminus L_0$ such that $\text{sc-dim}\, x_0 = 0$ and $L \not\models \mathrm{At}_k(a)$ for every $k > 0$, that is $\mathrm{asc}(x_0) = 0$.

If $q \geq 1$ then the construction in the proof of Proposition 8.2 applies here. It gives a primitive tuple (x_1, x_2) in L such that the $\mathcal{L}_{\mathrm{SC}}$-substructure L_1 of L generated by (x_1, x_2) over L_0 has signature σ in L_0. Moreover each $k_i = 0$ since $p \neq 0$, and on the other hand each $\mathrm{asc}\, x_i = 0$ since $\text{sc-dim}\, x_i = p \geq 1$.

Otherwise $q = 0 < p$ hence $(h_1, k_1) = (h_2, k_2)$. If $k_1 \neq 0$ let $x_1 = x_2 = a_1 \vee \ldots \vee a_{k_1}$ with the a_i's being any distinct atoms in L smaller than g and not belonging to L_0. If $k_1 = 0$ let $x_1 = x_2 = x_0$.

Case $p = 0$**.** Then $q = 0$ and $h_1 = h_2 = \mathbf{0}$.

If k_1 and k_2 are non-zero then $\mathrm{asc}(g) = k_1 + k_2$ hence $L(g)$ contains $k_1 + k_2$ atoms. Let x_1 be the join of k_1 of them, and x_2 be the join of the others.

Eventually if $k_1 = 0$ or $k_2 = 0$ then $\mathrm{asc}(g) = 0$ so $L(g)$ contains infinitely many atoms, hence by saturation there exists an element x in L smaller than g such that both x and $g - x$ are non-zero and $\mathrm{asc}(x) = \mathrm{asc}(g - x) = 0$. If $k_1 = k_2 = 0$ let $(x_1, x_2) = (x, g - x)$. If $k_1 \neq 0$ let $x_1 = a_1 \vee \ldots \vee a_{k_1}$ with the a_i's atoms in L smaller than g and $x_2 = g - x_1$. If $k_2 \neq 0$ exchange k_1 and k_2.
■

Proposition 9.7 *Every sub-*ASC*-lattice embeds in a super-*ASC*-lattice.*

Proof: Obviously every finitely generated substructure of a sub-ASC-lattices is finite by the local finiteness Theorem 4.1 because $\mathcal{L}_{\mathrm{SC}}$ and $\mathcal{L}_{\mathrm{ASC}}$ have the same function symbols.

Let L_0 be a finitely generated sub-ASC-lattice, and $a, b_1, b_2 \in L_0$ such that $b_1 \vee b_2 \ll a$ and $\mathrm{C}^0(a) = \mathbf{0}$. So L_0 is finite, let $g_1, \ldots, g_r$ be the $\vee$-irreducible components of a in L_0. As in the proof of Proposition 8.1 we construct a tower of $\mathcal{L}_{\mathrm{SC}}$-extensions $(L_i)_{i \leq r}$ so that each L_r contains elements x_1, x_2 satisfying

$$\begin{cases} x_1 = a - a_2 \\ x_2 = a - a_1 \\ x_1 \wedge x_2 = \varphi(b_1) \wedge \varphi(b_2) \end{cases}$$

and each L_i is generated over L_{i-1} by a primitive tuple of elements $x_{i,1} \neq x_{i,2}$ such that $g_i = g(x_{i,1}, L_{i-1}) = g(x_{i,2}, L_{i-1})$. Proposition 6.1 then implies that:

$$\mathcal{I}(L_r) = \big(\mathcal{I}(L_0) \setminus \{g_1, \ldots, g_r\}\big) \cup \{x_{i,1}, x_{i,2}\}_{1 \leq i \leq r}$$

By assumption all the g_i's have non-zero sc-dimension, hence the $x_{i,1}$'s and $x_{i,2}$'s do the same. In particular L_0 and L_r have the same elements of sc-dimension zero. Defining $\mathrm{At}_k(L_r) = \mathrm{At}_k(L_0)$ for every k then endows L_r with an $\mathcal{L}_{\mathrm{ASC}}$-structure which makes it a sub-ASC-lattice and an $\mathcal{L}_{\mathrm{ASC}}$-extension of L_0.

On the other hand every $\mathrm{L}^{\mathrm{At}}_{\mathrm{lin}}(X)$ is an ASC-lattice satisfying the atomicity and catenarity properties, hence Corollary 9.3 implies that every finitely generated sub-ASC-lattice embeds in an ASC-lattice having these two properties.

The conclusion follows by the model-theoretic compactness argument.
■

Theorem 9.8 *The theory of super* ASC*-lattices of sc-dimension at most d (resp. exactly d) is the model-completion of the theory of* ASC*-lattices of dimension at most d (resp. exactly d). It admits $\aleph_0$ completions, each of which is decidable, and it is decidable.*

Proof: The proof of the first statement is very similar to Theorem 8.3 and Corollary 8.5, with the only difference that in the embedding argument we have to assume the super ASC-lattice $\hat{L}$ to be $\aleph_0$-saturated, in order to apply Proposition 9.6 in place of Proposition 8.2.

The last statement then follows from the remark that there are $\aleph_0$ prime sub-ASC-lattices of dimension at most d (resp. exactly d). Indeed there are finitely many subscaled lattices of dimension at most d (resp. exactly d) and on each of them $\aleph_0$ different structures of sub-ASC-lattices.
■

Let L be any super ASC-lattice and L' the quotient of L by the equivalence relation $x \sim y$ if and only if $x - y$ and $y - x$ are the join of finitely many atoms. Then it is easily seen that L' is a super scaled lattice. So the problem of finding a natural example of super scaled lattice boyls down to finding a natural example of super ASC-lattice. This and related ideas lead us to the following conjecture:

Conjecture 9.9 *Let K be a p-adically closed field and A be an infinite definable subset of K^n which is open in its closure. Let $(B_k)_{k\leq q}$ be a finite collection of closed definable subsets of $\bar{A} \setminus A$. Then there exists a collection $(A_k)_{k\leq q}$ of non-empty definable subsets of A clopen in A such that:*

$$\forall k \leq q, \quad \overline{A_k} = A_k \cup B_k$$

It is an elementary exercise to deduce from this conjecture that $\mathrm{L}^{\mathrm{At}}_{\mathrm{def}}(X)$ models the ASC-splitting property for every definable set X over K. Moreover the $\mathcal{L}_{\mathrm{SC}}$-substructure of $\mathrm{L}^{\mathrm{At}}_{\mathrm{def}}(K^n)$ generated by the empty set is simply the two element lattice with the obvious $\mathcal{L}_{\mathrm{SC}}$-structure (because K^d is d-pure). This gives an explicit recursive axiomatisation of $\mathrm{L}^{\mathrm{At}}_{\mathrm{def}}(K^n)$.

Corollary 9.10 (Modulo Conjecture 9.9) *Let K be a p-adically closed field, then $\mathrm{L}^{\mathrm{At}}_{\mathrm{def}}(K^n)$ is a super-ASC-lattice. In particular its complete theory is decidable and eliminates the quantifier in $\mathcal{L}_{\mathrm{ASC}}$.*

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