Axiomatizing intersection theory over local fields

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1 Motivations

Given an algebraically closed field k and a positive integer n let us define:

$$\mathrm{L}_{\mathrm{Zar}}(k^n) = \{\text{Zariski closed subsets of } k^n\}$$

This is a distributive, bounded lattice. Studying the model-theory of the lattice $\mathrm{L}_{\mathrm{Zar}}(k^n)$ might give a model-theoretic approach of intersection theory (that is the study of intersection of algebraic varieties over an algebraically closed field) so our title.

Question 1.1 *Is* $\mathrm{L}_{\mathrm{Zar}}(k^n)$ *decidable?*

Question 1.2 $k \equiv k' \implies \mathrm{L}_{\mathrm{Zar}}(k^n) \equiv \mathrm{L}_{\mathrm{Zar}}(k'^n)$*?*

Question 1.3 $k \preceq k' \implies \mathrm{L}_{\mathrm{Zar}}(k^n) \preceq \mathrm{L}_{\mathrm{Zar}}(k'^n)$*?*

If $\mathrm{L}_{\mathrm{Zar}}(k^n)$ would be uniformly interpretable in k then these questions would have immediate answers. One can show that this is not the case, more precisely:

Proposition 1.4 $\mathrm{L}_{\mathrm{Zar}}(k^n)$ *is interpretable in the ring* $k[X_1, \dots, X_n]$ *but not in the field* k*.*

One can show that question 1.1 has a negative answer, more precisely[1]:

Proposition 1.5 *The ring of integers is uniformly interpretable in* $\mathrm{L}_{\mathrm{Zar}}(k^n)$ *for every algebraically closed field* k *and every integer* $n \geq 2$*.*

Nevertheless questions 1.2 and 1.3 are still widely open. Let us consider now the following natural variants. By a **local field** k we mean an algebraically closed, real closed or p-adicaly closed field. Endow k^n with its natural topology and for any definable subset X of k^n let:

$$\mathrm{L}_{\mathrm{def}}(X) = \{\text{definable subsets of } k^n \text{ closed in } X\}$$

If k is algebraically closed then $\mathrm{L}_{\mathrm{def}}(k^n)$ is nothing but $\mathrm{L}_{\mathrm{Zar}}(k^n)$ by the quantifier elimination of the theory of algebraically closed fields. In any case $\mathrm{L}_{\mathrm{def}}(k^n)$ is not interpretable in k, so all the questions above still arise for $\mathrm{L}_{\mathrm{def}}(k^n)$. Question 1.1 has again a negative answer for real closed fields but we expect the the p-adic case to be different, due to the lack of connectedness for p-adic definable sets not reduced to a single point. The results presented here include:

[1] I thank Luc Bélair for giving me the reference [Grz51] where an argument is given from which proposition 1.5 follows immediately.

1. A finite axiomatisation of the universal theory of the class $\Sigma(k,d)$ of all lattices $\mathrm{L}_{\mathrm{Zar}}(X)$ for X ranging over the Zariski closed sets of dimension at most d over an arbitrary infinite field k.

2. A model-completion for this theory in an appropriate finite expansion by definition of the language of lattices.

In the last section we discuss how this is related to the above questions, specially for p-adically closed fields.

2 Axiomatization and local finiteness

$\mathcal{L}_{\mathrm{lat}} = \{\mathbf{0}, \mathbf{1}, \vee, \wedge\}$ is the language of lattices. In any lattice the order is quantifier-free definable in $\mathcal{L}_{\mathrm{lat}}$ by $b \leq a \iff a \vee b = a$ so we can consider $\leq$ as well as the strict order $<$ as part of the lattice language.
$\mathcal{L}_{\mathrm{SC}_d} = \mathcal{L}_{\mathrm{lat}} \cup \{-\} \cup \{\mathrm{C}^k\}_{0 \leq k \leq d}$ where '$-$' is a binary function symbol and the C^k's are unary function symbols.

Example 2.1 Let X be any topological space of dimension at most d, and $L(X)$ the lattice of closed subsets of X. For any subset Y of X let $\overline{Y}$ denote the topological closure of Y in X. We endow $L(X)$ with an $\mathcal{L}_{\mathrm{SC}_d}$-structure as follows. For any $A, B \in L(X)$ and any positive integer $k \leq d$ let:

- $A - B =$ the topological closure of $A \setminus B$ in X.
- $\mathrm{C}^k(A) = \overline{\{a \in A \,/\, \dim(A,a) = i\}}$ where $\dim(A,a)$, the **local dimension** of A at a, is defined as the least possible dimension of the closure of of a neighborhood of a in A.

In case X is an algebraic variety, the lattice $L(X)$ is $\mathrm{L}_{\mathrm{Zar}}(X)$, and the above $\mathrm{C}^k(A)$ is simply the union of the irreducible components of A of dimension k.

Let $\Sigma(d)$ denote the class of all $\mathcal{L}_{\mathrm{SC}_d}$-structures $L(X)$ as in the above example, where X ranges over the class of all noetherian topological spaces of dimension at most d.

Definition 2.2 *A* **d-subscaled lattice** *is a model of the universal theory (in $\mathcal{L}_{\mathrm{SC}_d}$) of the class $\Sigma(d)$.*

Given an $\mathcal{L}_{\mathrm{SC}_d}$-structure L we define for any $a, b \in L$ (in the two last definitions we use the convention that $\max(\emptyset) = -1$):

- $b \ll a \iff b < a$ and $a - b = a$.
- $\dim_L a = \max\{k \,/\, \exists a_0, \dots, a_k \in L,\ \mathbf{0} \neq a_0 \ll a_1 \ll \cdots \ll a_k \leq a\}$
- $\mathrm{scdim}_L\, a = \max\{k \,/\, \mathrm{C}^k(a) \neq \mathbf{0}\}$

For any $A, B \in L(X)$ in the above example, $A \ll B$ if and only A is non-empty and B has empty interior in A. So $\dim_{L(X)} A$ is exactly the usual (topological) dimension of A. Moreover one can show that:

$$\forall A \in L(X), \quad \mathrm{scdim}_{L(X)}(A) = \dim_{L(X)}(A)$$

This is true in particular for $\mathrm{L}_{\mathrm{Zar}}(k^d)$ (k an infinite field). More generaly it remains true for every $\mathrm{L}_{\mathrm{def}}(X)$ (X definable over a local field).

Definition 2.3 *A d-subscaled lattice L such that* $\dim_L(a) = \mathrm{scdim}_L(a)$ *for every $a \in L$ is called a d-***scaled lattice**[2]**.**

A distributive bounded lattice admits at most one structure of d-scaled lattice extending its lattice structure. This $\mathcal{L}_{\mathrm{SC}_d}$-structure is then an extension by definition of its lattice structure.

Theorem 2.4 *The theory of d-subscaled lattices (resp. of d-scaled lattices) is finitely axiomatizable.*

An explicit axiomatization has been given in [Dar04]. For seek of shortness we do not reproduce it here. A crucial tool for this axiomatisation, as well as for all the results presented here, is the following:

Theorem 2.5 (Local finiteness) *Every finitely generated d-subscaled lattice is finite. More precisely there is a bound $\mu(n, d)$ for the cardinality of any d-subscaled lattice generated by n elements.*

Since the theory of d-subscaled lattices is universal, a prime d-subscaled lattice is simply a d-subscaled lattice generated by the empty set, so:

Corollary 2.6 *There is a finite number of non-isomorphic prime d-subscaled lattice.*

Let $\Sigma(k, d)$ be as in the first section, with k an arbitrary infinite field. Endow each lattice $\mathrm{L}_{\mathrm{Zar}}(X) \in \Sigma(k, d)$ with its natural $\mathcal{L}_{\mathrm{SC}_d}$-structure (see example **??**). Using the local finiteness theorem one can prove:

Theorem 2.7 *The universal theory of $\Sigma(k, d)$ (in $\mathcal{L}_{\mathrm{SC}_d}$) is exactly the theory of d-subscaled lattices. In particular it does not depend on k.*

3 Model-completion

We call **super d-scaled lattices** the d-scaled lattices satisfying the following additionnal properties.

Catenarity: For every positive integers $r \leq q \leq p$ and every elements $c \leq a$, if $\mathrm{C}^p(a) = a$ and $\mathrm{C}^r(c) = c$ then there exists an element b such that $c \leq b \leq a$ and $\mathrm{C}^q(b) = b$.

Splitting: For every elements b_1, b_2, a, if $b_1 \vee b_2 \ll a$ then there exists non-zero elements $a_1 \geq b_1$ and $a_2 \geq b_2$ such that:

$$\begin{cases} a_1 = a - a_2 \\ a_2 = a - a_1 \\ a_1 \wedge a_2 = b_1 \wedge b_2 \end{cases}$$

Clearly the class of super d-scaled lattices is finitely axiomatizable in $\mathcal{L}_{\mathrm{SC}_d}$, using only finitely many $\forall\exists$-formulas.

Theorem 3.1 *Every d-subscaled lattice embeds in a super d-scaled lattice.*

[2]By theorem 3.1 below, the class of $\mathcal{L}_{\mathrm{SC}_d}$-substructures of d-scaled lattices is exactly the class of d-subscaled lattices.

Theorem 3.2 *Let L, L' be any two super d-scaled lattices, and φ an isomorphism between two finitely generated $\mathcal{L}_{\mathrm{SC}_d}$-substructures L_0 and L'_0 of L and L' respectively. Let L_1 be a finitely generated extension of L_0 in L. Then φ extends to an embedding ψ of L_1 into L'.*

The two above theorems and the local finiteness result then easily imply:

Corollary 3.3 *The theory of super d-scaled lattices is the model-completion of the theory of subscaled lattices.*

Corollary 3.4 *The theory of super d-scaled lattices containing any given prime d-subscaled lattice is complete, decidable and $\aleph_0$-categorical. Every completion of the theory of super d-scaled lattices is of that kind, hence the theory of super d-scaled lattices has finitely many completions and is decidable.*

Remark: This results generalize to higher dimensions the well known model-completion result for boolean algebras. Indeed 0-subscaled lattices are exactly non-trivial boolean algebras (up to uniform and quantifier-free interdefinability), while the splitting property for 0-scaled lattices boils down to the atomless property.

4 Back to motivations

• **The theory of** $\mathrm{L}_{\mathrm{Zar}}(k^n)$ **for an algebraically closed field** k is somewhat complicated. The following variant is enlightening: let $\mathrm{L}^{\circ}_{\mathrm{Zar}}(k^2)$ denote the $\mathcal{L}_{\mathrm{SC}_d}$-substructure of $\mathrm{L}_{\mathrm{Zar}}(k^2)$ generated by smooth curves.

Fact 4.1 *Let k be the algebraic closure of the finite field with p elements, and K the algebraic closure of $k(t)$. Then $k \preceq K$ but $\mathrm{L}^{\circ}_{\mathrm{Zar}}(k^2) \not\equiv \mathrm{L}^{\circ}_{\mathrm{Zar}}(K^2)$.*

• **The case of certain local fields** might be more promising. Let k be any local field. The points (in the affine space k^n) are the atomes of $\mathrm{L}_{\mathrm{def}}(k^n)$. Since the splitting property implies that a super n-scaled lattice is atomless, clearly $\mathrm{L}_{\mathrm{def}}(k^n)$ can not be a super n-scaled lattice. However it is not difficult to adapt the previous result in order to keep atoms in a new model-completion result.
$\mathcal{L}_{\mathrm{ASC}_d} = \mathcal{L}_{\mathrm{SC}_d} \cup \{\mathrm{At}_k\}_{k\geq 1}$ with the At_k's being new unary predicates. An **atomic d-scaled lattice** is an $\mathcal{L}_{\mathrm{ASC}_d}$-structure L such that:

1. The $\mathcal{L}_{\mathrm{SC}_d}$-reduct of L is a d-scaled lattice.
2. Each At_k is interpreted as the set of elements of L which are the join of exactly k atoms.
3. Every $a \in L$ is the least upper bound of the set of atoms of L which are smaller than a.

A **super atomic d-scaled lattice** is an atomic d-scaled lattice which satisfies the catenarity axiom and the splitting axiom *restricted to the elements a such that* $\mathrm{C}^0(a) = \mathbf{0}$.

Theorem 4.2 *The theory of super atomic d-scaled lattices is the model-completion of the theory of atomic d-scaled lattices. It admits $\aleph_0$ completions, each of which is determined by a prime atomic d-subscaled lattices, hence is decidable. As a consequence the theory of super atomic d-scaled lattices is decidable.*

If k is algebraically closed or real closed then $\mathrm{L}_{\mathrm{def}}(k^n)$, with its natural structure of atomic n-scaled lattice, is not super atomic. Indeed it admits connected elements of dimension greater than zero, which still contradicts the restricted splitting property.
On the other hand if k is p-adically closed then the only connected elements of $\mathrm{L}_{\mathrm{def}}(k^n)$ are the points of k^n, that is the atoms of $\mathrm{L}_{\mathrm{def}}(k^n)$. It might then be (and there are many other evidence) that $\mathrm{L}_{\mathrm{def}}(k^n)$ is a super atomic n-scaled lattice, which is equivalent to the following conjecture (remember that a definable subset over a local field is said to have **pure dimension** d if it has local dimension d at every point):

Conjecture 4.3 *Let k be any p-adically closed field. Let A be a closed definable subset of k^n of pure dimension $d \geq 1$. Let B_1, B_2 be two closed definable subsets of A of dimension at most $d-1$. Then there exist closed definable subsets A_1, A_2 of A of pure dimension d such that $A_i \supseteq B_i$ (for $i = 1, 2$) and:*

$$A_1 \cap A_2 = B_1 \cap B_2$$

References

[Dar04] Luck Darnière. Model-completion of scaled lattices. Prépublications mathématiques d'Angers 191, may 2004.

[Grz51] Andrzej Grzegorczyk. Undecidability of some topological theories. Fund. Math. **38** (1951) 137–152.