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Luck Darnière

Pseudo algebraically closed rings

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Abstract. We use a variant of the nonsingular Hasse Principle to define a class of rings analogeous to PAC fields, and prove that this class has nice model theoretical properties. Since there are a lot of examples of such rings which are of interest for arithmetical purpose we apply our study to those examples, obtaining several decidability results and one purely algebraic application.

1. Introduction

In this paper we continue the presentation started in [2] of the results of [3] by introducing the notion of pseudo algebraically closed (PAC) domains. The reader is expected to be familiar with the definitions and notations of [2], specially the Local-Gobal Principle with respect to Henselisations (LGPH) which is a variant of the nonsingular Hasse principle. Indeed PAC domains can be defined as those domains satisfying LGPH which have a PAC fraction field, hence they form an elementary class. Other equivalent characterisations, in model theoretic or in algebraic terms, are given by the following (W_A denotes the set of all valuation rings centered at the maximal ideals of a domain A with the same fraction field as A, and R^a is any extension of such a valuation ring to the algebraic closure K):

Theorem 1. *For any domain A with fraction field K such that the set $\{\mathbf{m} \in \mathrm{Specmax}(A) \,/\, a \notin \mathbf{m}\}$ is compact for every $a \in A$, the following are equivalent.*

1. *A is PAC.*
2. *For any variety V defined over K we have:*

$$V(A) \neq \emptyset \iff \forall R \in W_A,\ V(R^a) \neq \emptyset.$$

3. *$A \leq_\exists B$ for any domain B containing A such that:* $\mathrm{Frac}(B)$ *is a regular extension of K and for every $\mathbf{m} \in \mathrm{Specmax}(A)$ there is $\mathbf{n} \in \mathrm{Specmax}(B)$ such that $\mathbf{n} \cap A = \mathbf{m}$.*

Luck Darnière: Université d'Angers, 2 Bd Lavoisier, 49045 Angers Cedex 01 (France)
e-mail: Luck.Darniere@univ-angers.fr
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4. *A variety V defined over K has no point in A if and only if there exists $f \in A[X_1, \dots, X_n]$ and $\alpha \in A \setminus A^*$ such that $\alpha f - 1$ belongs to the ideal of V.*

It appears that in order to deal with non necessarily PAC fraction fields it is convenient to consider a larger class of domains that we call PAC* domains. We can define them by using a variant of the above theorem: a PAC* domain is a domain A such that $\{\mathbf{m} \in \mathrm{Specmax}(A) \,/\, a \notin \mathbf{m}\}$ is compact for every non zero $a \in A$, and $A \leq_\exists B$ for every domain B containing A such that $\mathrm{Frac}(A) \leq_\exists \mathrm{Frac}(B)$ and every maximal ideal of A can be lifted to a maximal ideal of B.

The notion of the residual (semi-)lattice $L(A)$ attached to a ring A and basic facts about radical relations and r-rings $(A, \leq)$ (see [18]) are reminded in section 2. They play a key role in the model theory of PAC* domains. The following result of model completeness (section 4) is representative of the very flexible criterions which can be obtained for theories of r-domains.

Theorem 2. *Let T_l be a model-complete theory of lattices, T_f a model-complete theory of fields. Then the theory of r-domains $(A, \leq)$ such that A is PAC*, the radical relation of $(A, \leq)$ is defined by the maximal spectrum of A, the Jacobson radical $J(A) = \bigcap \mathrm{Specmax}(A)$ is (resp. is not) the zero ideal and:*

$$\mathrm{Frac}(A) \models T_f \quad \text{and} \quad L(A) \models T_l$$

is model-complete.

However this preliminary theorem does not answer the question of knowing wether or not an embedding between two particular PAC or PAC* domains is elementary, because one can have $\mathrm{Frac}(A) \preceq \mathrm{Frac}(A')$ whithout $\mathrm{Frac}(A)$ having a model-complete theory. This is a much more difficult question in general. We answer it in section 6 for all PAC domains (regardless of their characteristic) and for PAC* domains having a fraction field in a certain class we call restricted RC-local fields. This class of fields contains the usual examples of fields satisfying a local-global principle, namely PAC fields, pseudo real closed fields (PRC) and pseudo p-adically closed fields (PpC). We then obtain the following:

Theorem 3. *Let A, A' be PAC* domains with restricted RC-local fraction fields (this contains the case of PAC domains) such that the Jacobson radical $J(A)$ is zero if and only if $J(A')$ is so.*

1. *If $A \subseteq A'$ then $A \preceq A'$ if and only if $\mathrm{Frac}(A) \preceq \mathrm{Frac}(A')$ and $L(A) \preceq L(A')$.*

2. *A similar criterion is given for elementary equivalence, which boils down in positive characteristic to the very simple following one:* $A \equiv A'$ *if and only if* $\mathrm{Frac}(A) \equiv \mathrm{Frac}(A')$ *and* $L(A) \equiv L(A')$.

All the results of that kind given in section 6 are based on a deep and somewhat intricate embedding theorem whose proof is the subject of the whole section 5.

In the last section of this paper we apply this to various natural rings. A ring is decidable if there exists an algorithm which can prove every first order theorem which holds in this ring. If a domain is decidable then so is its fraction field but the converse is false in general. Our paper can be considered, from the point of view of the decidability results, as a continuation of [4] and [18] where the integral closure of the ring of integers (resp. of $\mathbf{F}_p[t]$) inside any algebraically closed field was proved among other things to be decidable.

If $\mathcal{F}_0$ is a finite set of p-adically closed fields (different values for p as well as ramification are allowed) and if we call absolutely $\mathcal{F}_0$-adic integers those algebraic integers which belong *as well as all their conjugates over* $\mathbf{Q}$ to every F in $\mathcal{F}_0$, then our first application is:

Theorem 4. *The ring of all absolutely $\mathcal{F}_0$-adic integers is decidable if and only if its fraction field is decidable.*

Since the field of all absolutely $\mathcal{F}_0$-adic numbers, that is the fraction field of the ring of all absolutely $\mathcal{F}_0$-adic integers, has been proved to be decidable in several cases, in all those cases we obtain that the ring of all absolutely $\mathcal{F}_0$-adic integers is decidable.

We also consider the case of various overrings of the ring $\mathbf{Z}_{r-abs}$ of all absolutely real integers, that is the ring of algebraic integers which belong to $\mathbf{R}$ as well as all their conjugates over $\mathbf{Q}$. Although this ring has been proved by J. Robinson to be undecidable, its fraction field is decidable (see [9]). This is a rather suprising property, and even more surprising is the following:

Theorem 5. *For any $n \in \mathrm{N} \setminus \{0, 1\}$ the ring $\mathbf{Z}_{r-abs}[1/n]$ is decidable.*

We then turn to some cases of positive characteristic. We prove for example that the integral closure of $\mathbf{F}_p[t]$ in any separably closed field is decidable. More generally:

Theorem 6. *Any integrally closed ring whose fraction field is separably closed and algebraic over $\mathbf{F}_p(t)$ is decidable.*

Eventually let us mention a natural application to algebra of our model theoretic methods:

Theorem 7. *Let A be a PAC domain with algebraically closed fraction field and a dense residual lattice. Then for every $f_1, \dots, f_m \in A[X_1, \dots, X_n, Y]$ the following assertions are equivalent:*

1. *For every $\alpha \in A^n$ the $f_i(\alpha, Y)$'s have a common root in A.*
2. *There exists a non-constant $f \in A[X_1, \dots, X_n, Y]$ monic in Y which divides simultaneously $f_1, \dots, f_m$.*

Remark 1. All the results published in this paper are part of the doctoral thesis of the author, made under the direction of F. Delon (University Paris 7, France) and the co-direction of L. Moret-Bailly (University Rennes 1, France). While this work was achieved, other people were studying very closed topics.

The author has to mention first Y.-L. Ershov, whose work on boolean families of valuation rings in [7] has been a source of inspiration for him. It is not surprising then that the methods of [3] and those developed by Y.-L. Ershov after [7] lead to similar, although not identical, results. In particular Y.-L. Ershov has informed us in february 1998 that the decidability of the ring of all absolutely $\{\mathbf{Q}_p\}$-adic integers was part of his own recent results.

Another notion of PAC domains has also been introduced independently by A. Razon in [21], which eventually appears to coincide with a special case of our PAC domains, namely those which have a dense residual lattice.

Y.-L. Ershov, A. Razon and the author shared a common inspiration taken from the fundamental papers [18] and [4] as well as their ancestor [22] and the literature on local-global principle for fields which has grown after the doctoral thesis of L. van den Dries. However they used different methods and achieved different goals. The results concerning overrings of the absolutely real integers, for example, are specific to the author as well as the above application to algebra (for which no algebraic proof has been found until now).

2. Notations and preliminaries

For any ring A and $a \in A$ we denote by:

$$M_A(a) = \{\mathbf{m} \in \mathrm{Specmax}(A) \;/\; a \in \mathbf{m}\}$$

$$M_A(a)^c = \mathrm{Specmax}(A) \setminus M(a).$$

The subscript A is omitted whenever it is not confusing. The set of valuation rings centered at the maximal ideals of a domain A having the same fraction field as A is denoted by W_A. The Jacobson radical $J(A)$ is the intersection of all the maximal ideals of A.

If K is a field then K^s (resp. K^a) is the separable (resp. algebraic) closure of K. The **absolute Galois group** $G(K)$ of K is the Galois group of the extension $K^s|K$. Given a local ring R we denote by $m(R)$ its unique maximal ideal. If R is a valuation ring then R^h denotes its henselisation and R^s (resp. R^a) the unique extension of R^h to the separable (resp. algebraic) closure of the fraction field of R.

Our varieties are always affine. A variety V is **defined over** K if $K(V)|K$ is a regular extension, or equivalently if the ideal of V in $K^a[X_1, \ldots, X_n]$ is a prime ideal which admits a system of generators with coefficients in K. The set of smooth points of V is denoted by V°.

The reader is supposed to be familiar with the notion of ring with a radical relation or r-ring $(A, \leq)$ introduced in [18]. For convenience we recall here the main properties of r-rings that we will use. An **r-ring** is a ring A together with a **radical relation** on A, that is a binary relation $\leq$ such that for every a, b, c in A:

1. $a \leq a$;
2. $a \leq b,\ b \leq c \Rightarrow a \leq c$;
3. $a \leq b \Rightarrow ac \leq bc$;
4. $1 \leq a$;
5. $a \leq b^2 \Rightarrow a \leq b$.

Prestel and Schmid also required that $0 \not\leq 1$ but this restriction is not necessary and it may be convenient to avoid it. An **r-domain** is an r-ring whose underlying ring is a domain. It is easy to check that if $\mathcal{P}$ is any set of prime ideals of A, the relation defined by:

$$a \leq_{\mathcal{P}} b \iff (\forall \mathbf{p} \in \mathcal{P},\ a \in \mathbf{p} \Rightarrow b \in \mathbf{p})$$

is a radical relation. Prestel and Schmid ([18], theorem 2.5) proved that every radical relation is of that kind. More precisely, they denoted by $\mathrm{Spec}(A, \leq)$ the set of prime ideals $\mathbf{p}$ of A such that:

$$(a \in \mathbf{p} \text{ and } a \leq b) \Rightarrow\ b \in \mathbf{p}.$$

They proved that $a \leq b$ if and only if:

$$\forall \mathbf{p} \in \mathrm{Spec}(A, \leq),\ a \in \mathbf{p} \Rightarrow b \in \mathbf{p}.$$

Prestel and Schmid actually proved in [18] that if $0 \not\leq 1$ then $\mathrm{Spec}(A, \leq)$ is not empty and $a \leq b$ is defined by $\mathrm{Spec}(A, \leq)$ as above. However it is easy to check that $0 \leq 1$ if and only if $\mathrm{Spec}(A, \leq)$ is empty, so the above result is true in this case too.

The residual upper semi-lattice $L(A, \leq)$ associated to an r-ring $(A, \leq)$ is the set of all $P(a) := \{\mathbf{p} \in \mathrm{Spec}(A, \leq) \;/\; a \in \mathbf{p}\}$ with a ranging over A. Ordered by inclusion, $L(A, \leq)$ has a largest element $P(0)$, a least element $P(1)$ and is stable by finite union: $P(ab) = P(a) \cup P(b)$. If in addition A is Bezout (that is, any finitely generated ideal is principal) then $L(A, \leq)$

is also stable by finite intersection, hence a distributive lattice (we refer to [18] for this observation: if A is Bezout then for any a, b in A there is d in A such that $aA + bA = dA$, and for any such d, $P(d) = P(a) \cap P(b)$). Since the residual upper semi-lattice of most of our r-rings will be stable by finite intersection, we will often call $L(A, \leq)$ the residual lattice by abuse of language. The quotient ordered set canonically associated to the preorder $\leq$ is isomorphic to $L(A, \leq)$ hence $L(A, \leq)$ is interpretable in $(A, \leq)$. We define the **full residual lattice** of an r-ring $(A, \leq)$ as the lattice of all subsets of $\mathrm{Spec}(A, \leq)$ of type $P(a_1) \cap \ldots \cap P(a_n)$ for any finite set $a_1, \ldots, a_n$ of elements of A. We denote it by $\mathbf{L}(A, \leq)$.

An **embedding of r-rings**, or **r-embedding**, $\varphi : (A, \leq) \to (A', \leq)$ is an embedding of rings which preserves the radical relation, that is:

$$\forall a, b \in A, \qquad a \leq b \iff \varphi(a) \leq \varphi(b).$$

If φ is the inclusion map we write $(A, \leq) \subseteq (A', \leq)$ and say $(A', \leq)$ is an **r-extension** of $(A, \leq)$.

Proposition 1. *Let φ be an r-embedding of $(A, \leq)$ into $(B, \leq)$ then $\varphi^{-1}(\mathbf{q}) \in \mathrm{Spec}(A, \leq)$ for every $\mathbf{q} \in \mathrm{Spec}(B, \leq)$.*
Conversely if the radical relation of A (resp. B) is defined by a family $\mathcal{P}$ (resp. $\mathcal{Q}$) of prime ideals then an embedding ψ of A into B is an r-embedding if and only if it satisifies the two following conditions:

- $\forall \mathbf{q} \in \mathcal{Q},\ \psi^{-1}(\mathbf{q}) \in \mathrm{Spec}(A, \leq)$.
- $\forall \mathbf{p} \in \mathcal{P}, \exists \mathbf{q} \in \mathrm{Spec}(B, \leq),\ \mathbf{p} = \psi^{-1}(\mathbf{q})$.

Proof. This improvement of lemma 2.7 in [18] is straightforward. □

If $\varphi : (A, \leq) \to (A', \leq)$ is an r-embedding then $\bar{\varphi} : P(a) \mapsto P(\varphi(a))$ is a well-defined embedding of $L(A, \leq)$ into $L(A', \leq)$ (see [18] lemma 2.10.e). Moreover it extends uniquely to an embedding still denoted by $\bar{\varphi}$ of $\mathbf{L}(A, \leq)$ into $\mathbf{L}(A', \leq)$. Indeed for any $a_1, \ldots, a_m, b_1, \ldots, b_n \in A$ such that $\bigcap_{i \leq m} P(a_i) = \bigcap_{i \leq n} P(b_i)$ and any $\mathbf{p}' \in \mathrm{Spec}(A', \leq)$ by the above proposition $\varphi^{-1}(\mathbf{p}') \in \mathrm{Spec}(A, \leq)$ hence we have:

$$\begin{aligned} \varphi(a_1), \ldots, \varphi(a_m) \in \mathbf{p}' &\iff a_1, \ldots, a_m \in \varphi^{-1}(\mathbf{p}') \\ &\iff b_1, \ldots, b_n \in \varphi^{-1}(\mathbf{p}') \\ &\iff \varphi(b_1), \ldots, \varphi(b_n) \in \mathbf{p}' \end{aligned}$$

So $\bigcap_{i \leq m} P(\varphi(a_i)) = \bigcap_{i \leq n} P(\varphi(b_i))$, in other words the map $\bar{\varphi} : \bigcap_{i \leq n} P(a_i) \mapsto \bigcap_{i \leq n} P(\varphi(a_i))$ is well defined. We will often identify $L(A, \leq)$ and $\mathbf{L}(A, \leq)$ to their images by $\bar{\varphi}$ hence consider them as subsets of $L(A', \leq)$ and $\mathbf{L}(A', \leq)$.

The main radical relation that we will consider in a ring A is the relation defined by the maximal spectrum of A. We denote it by $\leq_A$:

$$a \leq_A b \quad \iff \quad \forall \mathbf{m} \in \mathrm{Specmax}(A),\ \ a \in \mathbf{m} \Rightarrow b \in \mathbf{m}.$$

It is important to notice that this relation is definable in the language of rings:

$$a \leq_A b \iff \forall c \in A, ((b, c) = 1 \Rightarrow (a, c) = 1)$$

where $(x, y) = 1$ means that the ideal generated by x, y in A is A hitself, in other words $A \models \exists u, \exists v(ux + vy = 1)$. All notions defined for r-rings make sense for a ring A by considering its associated r-ring $(A, \leq_A)$. For example $L(A) = L(A, \leq_A)$ and $\mathbf{L}(A) = \mathbf{L}(A, \leq_A)$ are the residual and full residual lattice of A. The first one is interpretable in A.

Remark 2. An r-domain such that $(0) \in \mathrm{Spec}(A, \leq)$ is called an $\mathbf{r}_\circ$-**domain**. We will constantly use the following trivial observation:

$$(0) \in \mathrm{Spec}(A, \leq_A) \iff J(A) = \bigcap \mathrm{Specmax}(A) = (0).$$

All the lattices or semi-lattices we consider in this paper are distributive with an upper and a least bound, hence we require all our morphisms of lattices or semi-lattices to preserve these bounds, as well as $\sup$ and $\inf$ whenever they exist.

Proposition 2. *Let A be a domain such that for every non zero a and every b in A there is c in $A \setminus \{0\}$ such that $M(a) \setminus M(b) = M(c)$ then $L(A) = \mathbf{L}(A)$ is a relatively complemented lattice, that is:*

$$\forall \alpha, \beta \in \mathbf{L}(A), \quad (\alpha \neq P(0) \ \Rightarrow\ \exists \gamma \in \mathbf{L}(A),\ \alpha \setminus \beta = \gamma)$$

and one of the following occurs:

1. *If $J(A) \neq (0)$ then $\mathrm{Spec}(A, \leq_A) = \mathrm{Specmax}(A)$ and $\mathbf{L}(A)$ is a boolean algebra.*
2. *If $J(A) = (0)$ then $\mathrm{Spec}(A, \leq_A) = \mathrm{Specmax}(A) \cup \{(0)\}$ and, provided A is not a field, $\mathbf{L}(A)$ is not a boolean algebra.*

Proof. We first check that every non zero $\mathbf{p} \in \mathrm{Spec}(A, \leq_A)$ is a maximal ideal. Suppose this is not so, let $\mathbf{m}$ be a maximal ideal containing $\mathbf{p}$, choose $a \in \mathbf{p} \setminus \{0\}$ and $b \in \mathbf{m} \setminus \mathbf{p}$. Since $a \neq 0$ there is $c \in A$ such that $M(c) = M(a) \setminus M(b)$. Then $a \leq_A bc$ but since neither b (by construction) nor c (because $c \notin \mathbf{m}$) belongs to $\mathbf{p}$ it follows that $bc \notin \mathbf{p}$ and this contradicts $\mathbf{p} \in \mathrm{Spec}(A, \leq_A)$ because $a \in \mathbf{p}$ and $a \leq_A bc$. Consequently $P(a) = M(a)$ for any non zero element of A.

Since $M(a) \cap M(b) = M(a) \setminus (M(a) \setminus M(b))$ the hypothesis implies that $L(A) = \mathbf{L}(A)$ and it is relatively complemented (the case of the zero element is left to the reader). The point 1 follows, using remark 2. In order to finish the proof of point 2 suppose that $L(A)$ is a boolean algebra and A is not a field. Take $a \in A \setminus (A^* \cup \{0\})$ and $b \in A$ such that $P(b)^c = P(a)$. Then $b \neq 0$ because $P(b)^c$ is nonempty, and $0 \leq_A ab$ because $P(a) \cup P(b) = \mathrm{Spec}(A, \leq_A)$, hence $ab \neq 0$ (A is a domain) and $ab \in J(A)$. □

3. Pseudo algebraically closed domains and variants

Let us say that a ring A is **relatively pseudo algebraically closed (PAC*)** if and only if A is a domain satisfying the two following properties:

1. $\forall a \in A \setminus \{0\}$, $M(a)^c$ is compact (with respect to the Zariski topology).
2. Every variety V defined over the fraction field K of A satisfies:

$$V^\circ(A) \neq \emptyset \quad \iff \quad \begin{cases} V^\circ(K) \neq \emptyset \ \text{ which we call condition } (*), \\ \forall R \in W_A,\ V^\circ(R^a) \neq \emptyset. \end{cases}$$

The definition of **pseudo algebraically closed (PAC)** domains is obtained from the above one by removing the $(*)$ condition. Observe that if A is a field, property 1 is trivial and property 2 without $(*)$ is exactly the usual definition of pseudo algebraically closed fields, which substantiates our terminology.

In practice PAC* domains are of some interest only if condition $(*)$ is not too hard to deal with, for example if the fraction field is pseudo real closed or pseudo p-adically closed. We will prove further in this section that PAC domains are exactly PAC* domains with a PAC fraction field (one implication is inevident, the other requires to check that property 2 remains true in a PAC* domain if V is replaced by an affine open subset defined over K).

The LGPH introduced in [2] can also give rise to a $(*)$ version. A ring A satisfies LGPH* if and only if it is a domain such that:

1. $\forall a \in A \setminus \{0\}$, $M(a)^c$ is compact (with respect to the Zariski topology).
2. Every variety V defined over the fraction field K of A satisfies:

$$V^\circ(A) \neq \emptyset \quad \iff \quad \begin{cases} V^\circ(K) \neq \emptyset, \\ \forall R \in W_A,\ V^\circ(R^h) \neq \emptyset. \end{cases}$$

Although we have not been able to eliminate the first technical condition on $M(a)^c$ (which is satisfied for example by any ring of dimension ≤ 1) it is obviously the second one which is the most important. Since this last condition is analogeous to the nonsingular Hasse principle of algebraic geometry, we call it the **geometric condition of LGPH***.

Since for every valuation ring R one can embed R^h in R^a over R (usually we will simply consider that R^a contains R^h) it is immediate that PAC* domains satisfy LGPH*. As a consequence all the properties derived from LGPH in [2] which actually only used LGPH* still hold for PAC domains (we refer the reader to [2] for the definitions).

Proposition 3. *Let A be a PAC* domain, or more generally a domain satisfying LGPH*, then:*

1. *A is a Bezout domain.*
2. *A has the block factorisation property. In particular for any $a, b \in A$ if $a \neq 0$ then there exists $c \in A$ such that $M(c) = M(a) \setminus M(b)$.*
3. *A has the block approximation property.*
4. *Any two $R \neq R'$ in W_A are independent valuation rings.*
5. *A is dense in $A^h_{\mathbf{m}}$, for every maximal ideal $\mathbf{m}$ of A.*

Proof. See propositions 3.1 to 3.7, and proposition 4.7 of [2]. They were proved with LGPH in place of LGPH*, but the adaptations of the proofs are straightforward. $\square$

Remark 3. It follows from proposition 3 that proposition 2 applies to A hence $L(A) = \mathbf{L}(A)$ is a relatively complemented lattice.

The relation between PAC* (resp. PAC) and LGPH* (resp. LGPH) can be made more precise. We need two lemmas.

Lemma 1. *If a domain A which is not a field is PAC*, then $R^h = R^s$ for every R in W_A.*

Proof. Let A be a PAC* domain which is not a field, let K denote its fraction field and choose an element R_0 of W_A. We have to prove that R_0^h has a separably closed fraction field, hence that any monic separable non constant polynomial $f \in R_0[X]$ has a zero in R_0^h. By density of A inside R_0 it is sufficient to prove it for $f \in A[X]$.

Let $\varepsilon \in A \setminus \{0\}$, and V be the variety defined over K by $f(X) - \varepsilon Y$. This polynomial generates a prime ideal of $K^a[X, Y]$ and V is smooth. The point $(1, f(1)/\varepsilon)$ belongs to $V(K)$, and for every R in W_A since f is monic it has a zero x_R in R^a hence $(x_R, 0) \in V^\circ(R^a)$. By PAC* we get $(a, b) \in V^\circ(A)$, which satisfies $|f(a)|_{R_0} = |\varepsilon b|_{R_0} \leq |\varepsilon|_{R_0}$. It follows that 0 belongs to the topological closure of $f(A)$ inside $\mathrm{Frac}(R_0^h)$. Since f is separable, lemma 7.5 of [20] shows that $f(R_0^h)$ is closed inside $\mathrm{Frac}(R_0^h)$, and since it contains $f(A)$ we conclude that $0 \in f(R_0^h)$. $\square$

Lemma 2. *Let V be a variety defined over the fraction field F of a valuation ring R. Then $V^\circ(R^s) \neq \emptyset$ if and only if $V^\circ(R^a) \neq \emptyset$.*

Only the implication $V^\circ(R^a) \neq \emptyset \Rightarrow V^\circ(R^s) \neq \emptyset$ has to be proved, the other one being trivial. We can actually prove a stronger result:

Lemma 3. *With the above notations, suppose that are given a smooth point x^a of $V(R^a)$ and arbitrary polynomials $h_1, \dots, h_r$ with coefficients in R^s. Then there exists $x \in V^\circ(R^s)$ such that for every $i \leq r$:*

$$h_i(x) \in m(R^s) \iff h_i(x^a) \in m(R^a).$$

Remark 4. Observe that in lemma 3, R^a can be replaced by any valuation ring R' containing R^s such that $R' \cap \mathrm{Frac}(R^s) = R^s$. Indeed, if a smooth point y of V is initially given in any such valuation ring R' then $R^a \preceq R'^a$ because the theory of non trivial valuation rings with an algebraically closed fraction field is model complete (the case of trivial valuation rings is simpler: separably closed fields are PAC hence the conclusion of lemma 3 is immediate in that case). It follows that y can be replaced by a point of R^a to which lemma 3 applies.

Proof. Replacing R by R^s we can suppose F to be separably closed, hence $R = R^s$ is henselian. Let $\bar{R}$ be an elementary extension of R with infinitesimal elements, that is such that:

$$\mathbf{p} := \{\bar{x} \in \bar{R} \,/\, \forall x \in R \setminus \{0\},\ |\bar{x}|_{\bar{R}} \leq |x|_{\bar{R}}\}$$

is a non zero prime ideal of $\bar{R}$ (such an elementary extension always exists by the compactness theorem). Since $R \preceq \bar{R}$ it is sufficient to solve the problem in $\bar{R}$ because any such solution gives rise to a solution in R by existential closeness of R inside $\bar{R}$ (in the language of valuation rings).

From $R \preceq \bar{R}$ it follows that $\bar{R}$ has a separably closed fraction field, and from $\mathbf{p} \neq (0)$ we conclude that $\bar{R}_{\mathbf{p}}$ is a non trivial valuation ring with a separably closed fraction field. Hence its residue field k is algebraically closed, and as it contains obviously F it also contains F^a.

The variety V is defined over F, hence a fortiori over $\bar{F}$. By reducing modulo $\mathbf{p}$ it remains a variety defined over k because k contains F. Moreover $x^a \in V^\circ(k)$, and since $\bar{R}_{\mathbf{p}}$ is henselian we can find $\bar{x} \in V^\circ(\bar{R}_{\mathbf{p}})$ whose residual image in k is precisely x^a.

From this it follows that in a common extension of $\bar{R}_{\mathbf{p}}$ and R^a, for example in $\bar{R}^a_{\mathbf{p}}$, the point $\bar{x} - x^a$ has coordinates in $m(\bar{R}^a_{\mathbf{p}})$, and since x^a belongs to $V(\bar{R}^a)$ we conclude that $\bar{x}$ has coordinates in $\bar{R}^a$, hence $\bar{x} \in V^\circ(\bar{R}^a \cap \bar{F}) = V^\circ(\bar{R})$. Moreover for any polynomial h with coordinates in R, and in particular for the given h_i's, the Taylor formula implies that $h(\bar{x}) - h(x^a)$ belongs to the ideal generated in $\bar{R}^a$ by the coordinates of $\bar{x} - x^a$ hence to $m(\bar{R}^a_{\mathbf{p}}) \cap \bar{R}^a$ and a fortiori to $m(\bar{R}^a)$ from which we conclude that:

$$\begin{aligned} h(\bar{x}) \in m(\bar{R}) &\iff h(\bar{x}) \in m(\bar{R}^a) \\ &\iff h(x^a) \in m(\bar{R}^a) \\ &\iff h(x^a) \in m(R^a). \end{aligned}$$

□

By combining the two above lemmas we conclude:

Proposition 4. *A ring A is a PAC* domain if and only if it satisfies LGPH* and either it is a field or $R^h = R^s$ for every $R \in W_A$.*

Obviously, the same holds without $(*)$, that is with PAC* and LGPH* replaced by PAC and LGPH respectively.

Corollary 1. *Let A be a domain with fraction field $K \neq A$.*

1. *If A is a PAC* domain then for every monic non-constant polynomial $f \in A[T]$ and any non-zero ε in A there is $a \in A$ such that $f(a) \in \varepsilon A$.*
2. *Conversely, if A satisfies LGPH* (resp. LGPH) and the above property, then A is PAC* (resp. PAC).*

It follows that the class of PAC (resp. PAC) domains is recursively axiomatisable.*

Proof. 1. Consider the smooth variety V defined over $K = \mathrm{Frac}(A)$ by $f(T) = \varepsilon Y$. Obviously $(0, \varepsilon^{-1} f(0)) \in V^\circ(K)$ and since f is monic and R^a is integrally closed, any root t of f in an algebraically closed field belongs to R^a hence $(t, 0) \in V^\circ(R^a)$. Choose then for a the first coordinate of any point of $V^\circ(A)$ given by PAC*.
2. For the converse we have to prove, by proposition 4, that $A^h_{\mathbf{m}} = A^s_{\mathbf{m}}$ for every maximal ideal $\mathbf{m}$ of A. Fix $\mathbf{m}$ in $\mathrm{Specmax}(A)$, since A is dense in $A^h_{\mathbf{m}}$ by proposition 3 it is sufficient to prove that any monic non-constant separable polynomial in $A[T]$ has a root in $A^h_{\mathbf{m}}$. Choose any such $f \in A[T]$, the hypothesis implies that 0 belongs to the topological closure of $f(A)$ inside $\mathrm{Frac}(A^h_{\mathbf{m}})$. Since f is separable, $f(A^h_{\mathbf{m}})$ is closed by lemma 7.5 of [20] hence $0 \in f(A^h_{\mathbf{m}})$ that is f has a root in $A^h_{\mathbf{m}}$.

The conclusion now follows from proposition 4 because the class of PAC* domains is the class of domains A satisfying:

- LGPH*, which has been recursively axiomatised in [2] (actually it is LGPH which has been axiomatised in [2], but it is clear that condition $(*)$ can be added without additional problem because the fraction field of a domain is uniformly interpretable).

- The theory $\{\varphi \vee \psi_d \,/\, d \in \mathbf{N}^*\}$ where φ is the formula which says that every non-zero element is invertible (case of fields) and ψ_d says that "for every monic non-constant separable polynomial $f \in A[T]$ of degree d and any non-zero ε in A there is $a \in A$ such that $f(a) \in \varepsilon A$".
  $\square$

Another immediate application of proposition 4 is the natural relation existing between PAC and PAC* domains.

Corollary 2. *Let A be a PAC (resp. a PAC*) domain. Let V be a variety defined over $K = \mathrm{Frac}(A)$ such that $V^\circ(A^s_{\mathbf{m}})$ is nonempty for every $\mathbf{m} \in \mathrm{Specmax}(A)$ (resp. and $V^\circ(K)$ is nonempty).*

Then for every $\mathbf{m} \in \mathrm{Specmax}(A)$, $V^\circ(A)$ is dense in $V^\circ(A^s_{\mathbf{m}})$ with respect to the product topology defined by $A^s_{\mathbf{m}}$. As a consequence, PAC domains are exactly those PAC domains whose fraction field is PAC.*

Proof. By proposition 4 the density statement follows from proposition 5.1 of [2] (as usually we leave to the reader to check that the versions with and without $(*)$ of the arguments are essentially the same). It is immediate from the definitions that a PAC* domain with a PAC fraction field is itself a PAC domain. Conversely let A be a PAC domain, it is PAC*, let us prove that its fraction field K is PAC.

Let V be a nonempty variety defined over K and H be a hypersurface defined over K which is K-birationnally equivalent to V. Let $\varphi : H \mapsto V$ be such a birationnal morphism and $U \subseteq H$ an open affine subset on which φ is defined, U being defined as the complement of the zero set of a polynomial $u(X)$ (X is of course a multi-variable). Let H^* be the projective cone of H, that is the affine variety defined over K by a homogenization $h^*(X, Y)$ of a polynomial equation $h(X)$ of H (h^* is irreducible in $K^a[X, Y]$ by usual arguments) and U^* be the affine open subset of H^* defined by $Y u^*(X, Y) \neq 0$. Since V is nonempty so is H hence by Zariski-density U contains a smooth point in K^a which gives rise by reduction of denominators to a smooth point (x^a, y^a) of U^* with coordinates in the integral closure A^a of A in K^a. For every $\mathbf{m} \in \mathrm{Specmax}(A)$ this point belongs to $H^*(A^a_{\mathbf{m}})$. The PAC property then gives a smooth point (x, y) in $H^*(A)$ arbitrarily closed to (x^a, y^a) (with respect to an arbitrary valuation $A^a_{\mathbf{m}_0}$). In particular since $(x^a, y^a) \in U^*(A^a)$ by continuity of polynomial functions one can ensure that $(x, y) \in U(A)$. Then $\varphi(y^{-1}x) \in V(K) \neq \emptyset$ hence K is PAC. $\square$

Corollary 3. *Let A be a PAC* domain with fraction field $K \neq A$. Every R^a ($R \in W_A$) is endowed with the radical relation induced by its maximal spectrum, and so are R^h $(= R^s)$ and A.*
Let $f = (f_1, \ldots, f_m) \in A[X]^m$ with $X = (X_1, \ldots, X_n)$ be a system of equations of a variety V defined over K, and g a fixed element

of $A[X] \setminus I(V)$. *For every* $i \in \{1, \ldots, u\}$ *and every* $j \in \{1, \ldots, v\}$ *fix some* p_i, p'_i, q_j, q'_j *in* $A[X]$ *with the* p_i*'s not belonging to* $I(V)$. *Suppose that* $V^\circ(K) \neq \emptyset$ *and that for every maximal ideal* $\mathbf{m}$ *of* A *the variety* V *has a smooth point* $x_{\mathbf{m}}$ *with coordinates in* $A^a_{\mathbf{m}}$ *such that for every* $i \leq u$*:*

$$A^a_{\mathbf{m}} \models p_i(x_{\mathbf{m}}) \leq p'_i(x_{\mathbf{m}}) \qquad (i \leq u).$$

Suppose in addition that for each $j \leq v$ *there is a maximal ideal* $\mathbf{m}(j)$ *such that:*

$$A^a_{\mathbf{m}(j)} \not\models q_j(x_{\mathbf{m}(j)}) \leq q'_j(x_{\mathbf{m}(j)}).$$

Then V *has a smooth point* a *with coordinates in* A *such that* $g(a) \neq 0$, $A \models p_i(x) \leq p'_i(x)$ *for every* $i \leq u$ *and* $A \not\models q_j(x) \leq q'_j(x)$ *for every* $j \leq v$.

Proof. Use lemma 3 to transfer each local solution $x_{\mathbf{m}}$ from $A^a_{\mathbf{m}}$ to $A^s_{\mathbf{m}} = A^h_{\mathbf{m}}$. Then apply theorem 5.3 of [2] (with LGPH replaced by LGPH*, using the hypothesis $V^\circ(K) \neq \emptyset$). □

PAC* domains now clearly appear as those domains which satisfy condition (v) of [19] section 2. Moreover we have proved that the additional conditions on the radical relations appearing in (v) of [19] section 2 could be eliminated.

We announced in [2] that the Local-Global Principles introduced by van den Dries, Macintyre, Prestel and Schmid in [4], [18] and [19] were essentially particular cases of the LGPH. They are actually particular cases of the PAC and PAC* properties, the difference being that in our PAC* domains the local-global principle is considered without additional hypothesis (until now) on the fraction field or the residual lattice. On the other hand the henselisations of the localisations of a PAC* domain (which we call here its fibers) must have a separably closed fraction field by proposition 4. Hence LGPH itself is more general since no condition at all is required on the fibers.

In [2] we applied the study of LGPH to the case of a domain with a single p-adic fiber in order to axiomatize the integral closure of the ring of integers in any p-adically closed field. In this paper we consider the case of separably closed fibers. The case of possibly infinitely many fibers of finitely many different kind requires additional refinement, see [3] chapter V. However the PAC* notion is a trick which allows to deal partially with this last case without using these refinements: we will do this in the last section of this paper.

4. Relative existential closure

In this section we provide alternative characterisations of PAC and PAC* domains in terms of relative existential closure, for rings as well as for r-domains. This includes a somewhat surprising algebraic criterion (condition 4 in the next proposition) and a result of model-completeness.

Theorem 8. *For any domain A with fraction field K such that $M(a)^c$ is compact for every $a \in A$, the following assertions are equivalent.*

1. *A is PAC.*
2. *For any variety V defined over K we have:*

$$V(A) \neq \emptyset \quad \Longleftrightarrow \quad \forall R \in W_A,\ \ V(R^a) \neq \emptyset.$$

3. *$A \leq_\exists B$ for any domain B containing A such that: $L = \mathrm{Frac}(B)$ is a regular extension of K and for every $\mathbf{m} \in \mathrm{Specmax}(A)$ there is $\mathbf{n} \in \mathrm{Specmax}(B)$ such that $\mathbf{n} \cap A = \mathbf{m}$.*
4. *A variety V defined over K has no point in A if and only if there exists $f \in A[X_1, \dots, X_n]$ and $\alpha \in A \setminus A^*$ such that $\alpha f - 1$ belongs to the ideal of V.*

Proof. 1⇒3. Remember that $A \leq_\exists B$ if and only if $L|K$ is regular, which is assumed here, and for any variety V having a smooth point $b \in B^n$ and any $f \in A[X_1, \dots, X_n]$ such that $f(b) \neq 0$ there is a point $a \in V(A)$ such that $f(a) \neq 0$. Given such a triple (V, b, f) choose for each $\mathbf{m} \in \mathrm{Specmax}(A)$ a valuation ring R with fraction field L containing B and centered at a maximal ideal $\mathbf{n}$ of B such that $\mathbf{n} \cap A = \mathbf{m}$. Then $R \cap K = A_\mathbf{m}$ hence $A^a_\mathbf{m}$ embeds in R^a over A. Since $b \in V^\circ(R^a)$ and $A^a_\mathbf{m} \leq_\exists R^a$ there is $x_\mathbf{m} \in V^\circ(A^s_\mathbf{m})$ such that $f(a_\mathbf{m}) \neq 0$ (here we use the model-completeness of the theory of non trivial valuation rings with algebraically closed fraction field, or the theory of algebraically closed field if $A^a_\mathbf{m}$ is trivial). Then PAC gives a (smooth) point $a \in V^\circ(A)$ and one can choose a so that $f(a) \neq 0$ by corollary 2 (density).

3⇒4. One implication is clear: if $\alpha f - 1 \in I(V)$ then $V(A) = \emptyset$ since otherwise $a \in V(A)$ would imply $\alpha f(a) = 1$ hence $\alpha \in A^*$ a contradiction. Conversely suppose that $V(A) = \emptyset$, let $x_V \in K[V]$ be a generic point of V over K and let $B = A[x_V]$. The extension $K(V)|K$ is regular but $A \not\leq_\exists B$ since $V(B) \neq \emptyset$, hence there exists $\mathbf{m} \in \mathrm{Specmax}(A)$ such that $\mathbf{m}B = B$ that is $1 \in \mathbf{m}B$ (otherwise (3) would give a point in $V(A)$ hence a contradiction). Let $\alpha \in \mathbf{m}$ and $f \in A[X]$ be such that $1 = \alpha f(x_V)$. Since x_V is generic it follows that $\alpha f - 1 \in I(V)$.

4⇒2. Suppose that for every $\mathbf{m} \in \mathrm{Specmax}(A)$ there exists $x_\mathbf{m}$ in $V(A^a_\mathbf{m})$. Then for every $f \in A[X]$ and every $\alpha \in A \setminus A^*$ there is $\mathbf{m} \in \mathrm{Specmax}(A)$ such that $\alpha \in \mathbf{m}$ hence $\alpha f(x_\mathbf{m}) - 1 \neq 0$ because α is not invertible in $A^a_{\mathbf{m}}$. In particular $\alpha f - 1 \notin I(V)$ and this being true for every f and α it follows that $V(A) \neq \emptyset$.

2⇒1. If a variety V defined over K has a smooth point in R^a for every $R \in W_A$ then by (2) it has a point in A. This point may not be smooth. However, this Local-Global Principle holds for any variety defined over K, and it is obviously sufficient to prove all the properties we deduced from our initial nonsingular principle. Indeed the smoothness of the global point is useless when proving the properties enumerated in section 3, except corollary 2 whose conclusion becomes: $V(A)$ in dense in $V^\circ(R^s)$ for every $R \in W_A$. Now given any $R \in W_A$ we have $V^\circ(R^a) \neq \emptyset$ by assumption, hence $V^\circ(R^s) \neq \emptyset$ by lemma 2. The implicit function theorem at a point $x \in V^\circ(R^s)$ gives a neighborhood of x in $V(R^s)$ which contains only smooth points. Then a point of $V(A)$ close enough to x must be a smooth point hence $V^\circ(A) \neq \emptyset$. □

Remark 5. This elimination of the smoothness conditions in the Local-Global Principle is specific to PAC domains but cannot be extended to a domain satisfying LGPH whose some fiber has a non separably closed fraction field.

The following type of extensions of r-domains will be of special interest. A domain A is given as well as an extension L of its fraction field K. We equip A with the radical relation defined by its maximal spectrum. For each $\mathbf{m} \in \mathrm{Specmax}(A)$ a valuation ring $R(\mathbf{m})$ containing A is given with fraction field L such that $m(R(\mathbf{m})) \cap A = \mathbf{m}$. The family of all the $R(\mathbf{m})$'s is denoted by W. Let $B := \bigcap\{R(\mathbf{m}) \,/\, \mathbf{m} \in \mathrm{Specmax}(A)\}$ and $\mathcal{Q}' := \{m(R(\mathbf{m})) \cap B \,/\, \mathbf{m} \in \mathrm{Specmax}(A)\}$. Let $\mathcal{Q} := \mathcal{Q}' \cup \{(0)\}$ if $(0) \in \mathrm{Spec}(A, \leq)$ and $\mathcal{Q} := \mathcal{Q}'$ if not. Equip B with the radical relation induced by $\mathcal{Q}$. Then $(B, \leq)$ is an extension of $(A, \leq)$ by proposition 1. We call it the extension of $(A, \leq)$ **induced by** W.

Lemma 4. *With the above notations, if A is PAC (resp. PAC*) and the extension $L|K$ is regular (resp. $K \leq_\exists L$) then $(A, \leq) \leq_\exists (B, \leq)$.*

Proof. If A is PAC then it is PAC* with a PAC fraction field hence $L|K$ is regular implies that $K \leq_\exists L$: the first case boils down to the second one. We can then suppose that A is PAC* and $K \leq_\exists L$.

Let $\exists x \varphi(x)$ be an existential formula in the language of r-rings with parameters in A which holds in B. We can suppose φ of the following form:

$$\varphi \equiv \bigwedge_{i=1}^{m} f_i = 0 \wedge g \neq 0 \wedge \bigwedge_{i=1}^{u} p_i \leq p'_i \wedge \bigwedge_{j=1}^{v} q_j \not\leq q'_j$$

with $f_i, g, p_i, p'_i, q_j, q'_j$ in $A[X]$ for every i, j ($X = (X_1, \ldots, X_n)$). We have to prove that $(A, \leq) \models \exists x \varphi(x)$ hence to find a in A^n such that $\varphi(a)$

holds in $(A,\leq)$. Choose $b\in B^n$ such that $(B,\leq)\models\varphi(b)$. Since $L|K$ is a regular extension so is $K(b)|K$ hence b can be thought of as the generic point of a variety V defined over K. Since every polynomial h with coefficients in A such that $h(b)=0$ vanishes identically on V it is sufficient to find a point $a\in V(A)$ such that $g(a)\neq 0$ and $(A,\leq)\models\varphi_r(a)$ where φ_r is the conjunction of radical conditions appearing in φ:

$$\varphi_r\equiv\bigwedge_{i=1}^{u} p_i\leq p_i'\wedge\bigwedge_{j=1}^{v} q_j\not\leq q_j'.$$

Step 1: Changing φ_r. Let $I:=\{i\leq u\,/\,p_i(b)\neq 0\}$ and $I':=\{1,\dots,u\}\setminus I$, let $J:=\{j\leq v\,/\,\exists\mathbf{m}\in\mathrm{Specmax}(A),\ q_j(b)\in m(R(\mathbf{m}))$ and $q_j'(b)\notin m(R(\mathbf{m}))\}$ and $J':=\{1,\dots,v\}\setminus J$. There are two cases:

- If $(0)\in\mathrm{Spec}(A,\leq)$ then by construction it belongs to $\mathrm{Spec}(B,\leq)$. For any $i\in I'$ we have $0=p_i(b)\leq p_i'(b)$ which implies $p_i'(b)=0$ hence any $a\in V(A)$ will satisfy $0=p_i(a)=p_i'(a)$ and a fortiori $p_i(a)\leq p_i'(a)$. Moreover for any $j\in J'$ the definition of the radical relation of $(B,\leq)$ and the fact that $q_j(b)\not\leq q_j'(b)$ implies that $q_j(b)=0$ and $q_j'(b)\neq 0$ hence any $a\in V(A)$ such that $q_j'(a)\neq 0$ will satisfy $q_j(a)=0\not\leq q_j'(a)$. Hence it is sufficient to find $a\in V(A)$ such that $G(a)\neq 0$, with $G=g\cdot\prod_{j\in J'}q_j'$, and:

$$\psi(a)\equiv\bigwedge_{i\in I}p_i(a)\leq p_i'(a)\wedge\bigwedge_{j\in J}q_j(a)\not\leq q_j'(a)$$

  holds in $(A,\leq)$.
- If (0) does not belong to $\mathrm{Spec}(A,\leq)$ then $J'=\emptyset$ and there is a non zero $\alpha\in A$ such that $0\leq\alpha$, hence it is sufficient to find $a\in V(A)$ such that $g(a)\neq 0$ and:

$$\psi(a)\equiv\bigwedge_{i\in I}p_i(a)\leq p_i'(a)\wedge\bigwedge_{i\in I'}\alpha\leq p_i'(a)\wedge\bigwedge_{j\in J}q_j(a)\not\leq q_j'(a)$$

  holds in $(A,\leq)$.

As in both cases $\psi(b)$ holds in $(B,\leq)$ it follows that, replacing if necessary g by G and φ_r by ψ we can suppose that $I'=J'=\emptyset$.

Step 2: Transfer. $V^\circ(K)\neq\emptyset$ because $K\leq_\exists L$ and $b\in V^\circ(L)$. Let $h_1,\dots,h_r$ an enumeration of all the p_i's, q_i's, p_i''s and q_i''s. Fix $\mathbf{m}\in\mathrm{Specmax}(A)$ and let $R:=R(\mathbf{m})$. Since A is a Prüfer domain we have $A_{\mathbf{m}}=R\cap K$ hence $A_{\mathbf{m}}^a$ embeds naturally into R^a. Since $b\in V^\circ(R^a)$ and $A_{\mathbf{m}}^a\leq_\exists R^a$ there is $x_{\mathbf{m}}\in V^\circ(A_{\mathbf{m}}^a)$ such that for any $i\leq r$:

$$h_i(x_{\mathbf{m}})\in m(A_{\mathbf{m}}^a)\iff h_i(b)\in m(R(\mathbf{m})).$$

In particular, since $p_i(b) \leq p'_i(b)$ we have $A^a_{\mathbf{m}} \models p_i(x_{\mathbf{m}}) \leq q_i(x_{\mathbf{m}})$ for every $i \leq u$. By step 1, g and the p_i's do not belong to the ideal of V because they do not vanish at b. Moreover there is for each $j \leq v$ some $\mathbf{m}(j) \in \operatorname{Specmax}(A)$ such that $A^a_{\mathbf{m}(j)} \not\models q_j(x_{\mathbf{m}(j)}) \leq q'_i(x_{\mathbf{m}(j)})$. Eventually corollary 3 gives a point $a \in V^\circ(A)$ such that $g(a) \neq 0$, $p_i(a) \leq p'_i(a)$ for every $i \leq u$ and $q_j(a) \not\leq q'_j(a)$ for every $j \leq v$, that is $\varphi(a)$ holds in $(A, \leq)$. $\square$

Theorem 9. *Let A be a PAC (resp. PAC*) domain, and A' an r-extension of A such that:*

1. *If $J(A) = (0)$ then $J(A') = (0)$.*
2. $\mathbf{L}(A) \leq_\exists \mathbf{L}(A')$.
3. $\operatorname{Frac}(A')|\operatorname{Frac}(A)$ *is a regular extension (resp.* $\operatorname{Frac}(A) \leq_\exists \operatorname{Frac}(A')$*).*

Then $(A, \leq_A) \leq_\exists (A', \leq_{A'})$.

Remark 6. The above three conditions are also necessary. For the first one, observe that $J(A) = (0)$ if and only if $\exists a(0 \leq a \wedge 0 \neq a)$ holds in $(A, \leq_A)$. For the other two, remember that $(A, \leq_A) \leq_\exists (A', \leq_{A'})$ if and only if there is an embedding φ of $(A', \leq_{A'})$ over A into an elementary extension $(B, \leq_B)$ of $(A, \leq_A)$. Then $\mathbf{L}(A) \preceq \mathbf{L}(B)$ because $\mathbf{L}(A) = L(A)$ is interpretable in A, and similarly $\operatorname{Frac}(A) \preceq \operatorname{Frac}(B)$. So $\bar{\varphi}$ embeds $\mathbf{L}(A')$ over $L(A)$ into an elementary extension of $\mathbf{L}(A)$ hence $\mathbf{L}(A) \leq_\exists \mathbf{L}(A')$, and similarly $\operatorname{Frac}(A) \leq_\exists \operatorname{Frac}(A')$.

The PAC case being a particular case, we only have to prove the theorem when A is PAC* and $K \leq_\exists L$. The demonstration requires the following lemma.

Lemma 5. *Let $(A'', \leq)$ and $(A', \leq)$ be two extensions of a r-domain $(A, \leq)$. Suppose that an embedding σ is given of $\mathbf{L}(A', \leq)$ into $\mathbf{L}(A'', \leq)$ whose restriction to $\mathbf{L}(A, \leq)$ is identity. Then:*

1. $\forall \mathbf{p}'' \in \operatorname{Spec}(A'', \leq)$, $\hat{\sigma}(\mathbf{p}'') := \{a' \in A' / \mathbf{p}'' \in \sigma(P(a'))\}$ *is a prime ideal of A' which belongs to* $\operatorname{Spec}(A', \leq)$*. Moreover $\mathbf{p}'' \cap A = \hat{\sigma}(\mathbf{p}'') \cap A$ and any element of* $\operatorname{Spec}(A', \leq)$ *which is maximal in* $\operatorname{Spec}(A', \leq)$ *with respect to the inclusion lies in the image of $\hat{\sigma}$.*
2. *If the radical relations of A' and A'' are defined by their respective maximal spectra and if their fraction fields K' and K'' are regular extensions of the fraction field K of A, then for each maximal ideal $\mathbf{m}''$ of A'' there is a valuation ring $R(\mathbf{m}'')$ with fraction field* $\operatorname{Frac}(K' \otimes_K K'')$ *containing both A' and A'' such that:*

$$m(R(\mathbf{m}'')) \cap A'' = \mathbf{m}'' \quad \textit{and} \quad m(R(\mathbf{m}'')) \cap A' = \hat{\sigma}(\mathbf{m}'').$$

Proof. 1. Let $\mathbf{p}'' \in \mathrm{Spec}(A'', \leq)$ and $\mathbf{p}' = \hat{\sigma}(\mathbf{p}'')$. We have $\mathbf{p}'' \in P(0) = \sigma(P(0))$ hence $0 \in \mathbf{p}'$, and $\mathbf{p}'' \notin P(1) = \sigma(P(1))$ hence $1 \notin \mathbf{p}'$. For any $a', b' \in A'$, if $a' \in \mathbf{p}'$ and $a' \leq b'$ then $\mathbf{p}'' \in \sigma(P(a')) \subseteq \sigma(P(b'))$ hence $b' \in \mathbf{p}'$. A fortiori for any $a' \in \mathbf{p}'$ and any $b' \in A'$ since $a' \leq a'b'$ we have $a'b' \in \mathbf{p}'$. Conversely let $a', b' \in A'$, if $a'b' \in \mathbf{p}'$ then:

$$\mathbf{p}'' \in \sigma(P(a'b')) = \sigma(P(a')) \cup \sigma(P(b'))$$

hence $\mathbf{p}'' \in \sigma(P(a'))$ or $\mathbf{p}'' \in \sigma(P(b'))$ that is $a' \in \mathbf{p}'$ or $b' \in \mathbf{p}'$. If $a', b' \in \mathbf{p}'$ then since $P(a') \cap P(b') \subseteq P(a' + b')$ we have:

$$\mathbf{p}'' \in \sigma(P(a')) \cap \sigma(P(b')) = \sigma(P(a') \cap P(b')) \subseteq \sigma(P(a' + b'))$$

hence $a' + b' \in \mathbf{p}'$ which ends the proof that $\mathbf{p}' \in \mathrm{Spec}(A', \leq)$. Moreover the restriction of σ to $\mathbf{L}(A, \leq)$ being identity it is clear that $\mathbf{p}'' \cap A = \mathbf{p}' \cap A$.

Let now be given any maximal element $\mathbf{p}'$ of $\mathrm{Spec}(A', \leq)$ and let $I'' := \{a'' \in A'' \,/\, \exists a' \in \mathbf{p}',\ \sigma(P(a')) \subseteq P(a'')\}$. It is an easy exercise of the same type as above to check that I'' is a proper ideal of A'' and that for any $a'' \in I''$ and any $b'' \in A''$ such that $a'' \leq b''$ we have $b'' \in I''$. It is proved in [18] lemma 2.4, that such an ideal is the intersection of the family of $\mathbf{p}'' \in \mathrm{Spec}(A'', \leq)$ which contain I''. For any such $\mathbf{p}''$ we have obviously $\mathbf{p}' \subseteq \hat{\sigma}(\mathbf{p}'')$ hence $\mathbf{p}' = \hat{\sigma}(\mathbf{p}'')$ by maximality of $\mathbf{p}'$.

2. Observe first that the extensions of K being regular $K' \otimes_K K''$ is a domain, hence it is meaningful to consider $K^\circ := \mathrm{Frac}(K' \otimes_K K'')$ as a common extension of K' and K'' over K. Fix any $\mathbf{m}'' \in \mathrm{Specmax}(A'')$ and let $\mathbf{p}' := \hat{\sigma}(\mathbf{m}'')$. Choose any valuation ring $R'' \subseteq K''$ dominating $A''_{\mathbf{m}''}$ and $R' \subseteq K'$ dominating $A'_{\mathbf{p}'}$, and denote by φ'' and φ' the associated places. Because $\mathbf{m}'' \cap A = \mathbf{p}' \cap A$ we can manage to ensure that $R'' \cap K = R' \cap K$. Then $\varphi' \otimes \varphi''$ is a partial morphism of K° extending both φ' and φ''. Any maximal extension of this partial morphism defines a place on K° whose associated valuation ring $R(\mathbf{m}'')$ has the required properties. □

Remark 7. The argument used in the proof of (2) above is simpler than the one used in [3], lemme d'amalgamation IV.3.1. It was given to me by A. Razon during private conversions in Oberwolfach, fall 1998.

We can now prove theorem 9.

Proof. Let $(A'', \leq)$ be a $|A'|^+$-saturated elementary extension of $(A, \leq)$ (remember that the radical relations are those defined by the maximal spectra: it was assumed for A and A', and it is true also for A'' because it is a first order property hence preserved by elementary extensions). Since A'' is PAC* it is Bezout hence $\mathbf{L}(A'') = L(A'')$ is interpretable in $(A'', \leq)$. Since $\mathbf{L}(A) \leq_\exists \mathbf{L}(A')$ this provides an embedding σ of $\mathbf{L}(A')$ into $\mathbf{L}(A'')$ whose restriction to $\mathbf{L}(A)$ is identity (if $L(A)$ is finite the three lattices are the same and we take for σ the identity, if not then $L(A'')$ inherits the saturation property of $(A'', \leq)$ and this provides plenty of such embeddings σ). For each maximal ideal $\mathbf{m}''$ of A'' let $R(\mathbf{m}'')$ be a valuation ring with fraction field $\mathrm{Frac}(K' \otimes_K K'')$ such that $m(R(\mathbf{m}'')) \cap A'' = \mathbf{m}''$ and $m(R(\mathbf{m}'')) \cap A' = \hat{\sigma}(\mathbf{m}'')$ (see lemma 5). Eventually let $(A^\circ, \leq)$ be the extension of $(A'', \leq)$ induced by $\{R(\mathbf{m}'') \;/\; \mathbf{m}'' \in \mathrm{Specmax}(A'')\}$.

We claim that $(A^\circ, \leq)$ is an extension of $(A', \leq)$. Indeed, for any $\mathbf{m}'' \in \mathrm{Specmax}(A'')$ we have $A' \subseteq R(\mathbf{m}'')$ hence $A' \subseteq A^\circ$. Moreover the radical relation of $(A^\circ, \leq)$ is defined by a set $\mathcal{P}$ of prime ideals such that:

- For any $\mathbf{p} \in \mathcal{P}$, either there is $\mathbf{m}'' \in \mathrm{Specmax}(A'')$ such that $\mathbf{p} = A^\circ \cap m(R(\mathbf{m}''))$ and then $\mathbf{p} \cap A' = \hat{\sigma}(\mathbf{m}'')$ belongs to $\mathrm{Spec}(A', \leq)$, or $\mathbf{p} = (0)$ and $(0) \in \mathrm{Spec}(A'', \leq)$ but in that case $(0) \in \mathrm{Spec}(A, \leq)$ because this an elementary property, hence $(0) \in \mathrm{Spec}(A', \leq)$ by hypothesis and once again $\mathbf{p} \cap A'$ belongs $\mathrm{Spec}(A', \leq)$.
- For any $\mathbf{m}' \in \mathrm{Specmax}(A')$ by lemma 5 there is a $\mathbf{m}'' \in \mathrm{Specmax}(A'')$ such that $\mathbf{m}' = \hat{\sigma}(\mathbf{m}'')$ hence there is $\mathbf{p} = m(R(\mathbf{m}'')) \cap A^\circ \in \mathcal{P}$ such that $\mathbf{p} \cap A' = \mathbf{m}'$.

This proves the claim by proposition 1. Now $(A'', \leq) \leq_\exists (A^\circ, \leq)$ by lemma 4 hence $(A, \leq) \leq_\exists (A^\circ, \leq)$ and as $(A', \leq) \subseteq (A^\circ, \leq)$ we conclude that $(A, \leq) \leq_\exists (A', \leq)$. □

Corollary 4. *Let T_l a model-complete theory of lattices, T_f a model-complete theory of fields. Then the theory of $r_\circ$-domains (resp. of r-domains which are not $r_\circ$-domains) $(A, \leq)$ such that A is PAC*, the radical relation of $(A, \leq)$ is defined by the maximal spectrum of A, $\mathrm{Frac}(A) \models T_f$ and $L(A) \models T_l$ is model-complete.*

Proof. This is an elementary class because the above conditions are elementary (only the axiomatisation of PAC* needs some efforts, and it has been done in corollary 1). It is model-complete by theorem 9. □

5. The embedding theorem

Ershov introduced in [7] the following notion: a field K is **regularly closed relatively to a family** $\mathcal{F}$ of algebraic extensions if and only if it is existentially closed in any regular extension K' such that $F \leq_\exists \mathrm{Frac}(F \otimes_K K')$ for every $F \in \mathcal{F}$. We will say that a field is **RC-local** if it is regularly closed with respect to the family of all its real closures and its p-adic closures (in the sense of [17] where p-adically closed fields are exactly those fields which are elementary equivalent to some finite extension of $\mathbf{Q}_p$). This notion contains most of the usual local-global principle for fields, for example pseudo real closed fields (resp. pseudo p-adically closed fields) are exactly those fields which are regularly closed with respect to the family of all their real closed (resp. p-adically closed of rank 1) algebraic extensions. A field is pseudo algebraically closed if and only if it is regularly closed relatively to $\emptyset$.

We consider local fields whose theories have been completely determined, this is one of the reasons why local fields of positive characteristics are avoided. On the other hand, observe that the above definition does not exclude completely the positive characteristic case: a field of positive characteristic is RC-local if and only if it is PAC.

The aim of this section is to prove the following theorem, which is a key result of this paper:

Theorem 10 (Embedding). *Let $(A', \leq)$, $(A'', \leq)$ be two extensions of an r-domain $(A, \leq)$. Denote by K, K', K'' their respective fraction fields and $\mathbf{L}, \mathbf{L}', \mathbf{L}''$ their full residual lattices. Suppose that:*

- *K' and K'' are regular extensions of K.*
- *If K has positive characteristic p then $[K' : K'^p] \leq [K'' : K''^p]$.*
- *The radical relations of $(A', \leq)$ and $(A'', \leq)$ are defined by their respective maximal spectra, and if $J(A') = (0)$ then $J(A'') = (0)$.*
- *An embedding $\sigma : \mathbf{L}' \to \mathbf{L}''$ is given whose restriction to $\mathbf{L}$ is identity.*
- *An epimorphism $\pi : G(K'') \to G(K')$ is given which commutes with the restriction epimorpisms from $G(K')$ and $G(K'')$ to $G(K)$.*
- *A'' is PAC*, $|A'|^+$-saturated and K'' is RC-local.*

Then there is an embedding $\Theta : K'^s \to K''^s$ over K^s such that:

1. *K'' is a regular extension of $\Theta(K')$.*
2. *$\tau \circ \Theta = \Theta \circ \pi(\tau)$ for every τ in $G(K'')$.*
3. *The restriction φ of Θ to A' embeds $(A', \leq)$ into $(A'', \leq)$ over $(A, \leq)$ so that $\bar{\varphi} : \mathbf{L}' \to \mathbf{L}''$ is exactly the given embedding σ.*

The proof of the embedding theorem 10 starts now and will continue until the end of this section. It requires the construction of several auxiliary objects.

If A' is finite, since $K'|K$ is regular we have $K' = K$ hence $A' = A$ and the theorem is proved. In what follows we can then assume A' to be infinite, and the saturation hypothesis on A'' then implies that A'' also is infinite.

Construction 1.
Since $K' \otimes_K K''$ is a domain, we can embed all our fields in an algebraic closure Ω of $\mathrm{Frac}(K' \otimes_K K'')$. Observe that if F, F', F'' are any algebraic extensions of K, K', K'' respectively, such that the extensions $F'|F$ and $F''|F$ are regular then the compositum $F'F''$ inside Ω is F-isomorphic to $\mathrm{Frac}(F' \otimes_F F'')$. The Galois group of $K'^s K''^s$ over

$K'K''$ is $G(K') \times_{G(K)} G(K'')$. Let $G := \{\langle \pi(\tau), \tau \rangle \ / \ \tau \in G(K'')\}$ and $K^\circ := (K'^s K''^s)^G$.

For any $\mathbf{m}'' \in \mathrm{Specmax}(A'')$ the valuation ring $R(\mathbf{m}'')$ given by lemma 5 applied to σ extends to a valuation ring $R^\circ(\mathbf{m}'')$ with fraction field K°. As in the proof of theorem 9 this provides a common extension $(A^\circ, \leq)$ of $(A', \leq)$ and $(A'', \leq)$. By construction A° contains the integral closure of $A'A''$ inside K° hence $\mathrm{Frac}(A^\circ) = K^\circ$.

Lemma 6. *$K'' \leq_\exists K^\circ$ and $(A'', \leq) \leq_\exists (A^\circ, \leq)$.*

Proof. K'' is relatively separably algebraically closed in K° because for every τ in $G(K'')$ the field $K^\circ \cap K''^s$ is fixed by $\langle \pi(\tau), \tau \rangle_{|K''^s} = \tau$. Moreover $K'^s K''^s$ is a separable extension of $K'K''$ and $K'K'' \simeq \mathrm{Frac}(K' \otimes_K K'')$ is a regular extension of K'', hence $K'^s K''^s$ and a fortiori K° is a separable extension of K''. We conclude that $K^\circ|K''$ is regular.

Let $\mathcal{F}$ be the family of all real and p-adic closures of K''. If $\mathcal{F}$ is empty then K'' is PAC hence $K^\circ|K''$ is regular implies that $K'' \leq_\exists K^\circ$. This case being solved we suppose now that $\mathcal{F}$ is non-empty, in particular the characteristic is zero so we consider algebraic closures in place of separable closures.

For every $F'' \in \mathcal{F}$ let $F = F'' \cap K^a$ and F' the fixed field of $\pi(G(F''))$ inside K'^a. Observe that $F'' \cap K^a = F' \cap K^a = F$. Moreover, since $F''^a = F''\mathbf{Q}^a = F''K^a$ the restriction epimorphism from $G(F'')$ onto $G(F)$ is an isomorpism, and the same arguments holds for F' hence $G(F') = G(F'')$.

If F'' is real closed then the absolute Galois group of F' is $\mathbf{Z}/2\mathbf{Z}$ hence it is real closed, and so is F because it is relativally algebraically closed in F''. It follows that $F \preceq F'$ and $F \preceq F''$.

If F'' is p-adically closed then so is F' by theorem 4.1 of [13], and since F is relatively algebraically closed inside F' and F'' it follows from theorem 3.4 and theorem 5.1 of [17] that once again $F \preceq F'$ and $F \preceq F''$.

A general model theoretical argument allows us to amalgamate any two elementary extensions R' and R'' of a ring R in a common elementary extension $\bar{R}$ in such a way that any tuple of elements of R' which are linearly independent over R remain so in $\bar{R}$ over R'' (see for example [15] proof of lemma 4). Applying this to F, F', F'' we can embed $\mathrm{Frac}(F' \otimes_F F'') \simeq F'F''$ over F into an elementary extension of F'' hence $F'' \leq_\exists F'F''$. Moreover for any $\tau \in Gal(K'^a K''^a | F'F'')$ we have $(\pi(\tau_{|K''^a}))_{|K^a} = (\tau_{|K'^a})_{|K^a}$ and since the restriction epimorphism from $G(F')$ to $G(F)$ is an isomorphism $\pi(\tau_{|K''^a}) = \tau_{|K'^a}$. It follows that $\tau = \left\langle \pi(\tau_{|K''^a}), \tau_{|K''^a} \right\rangle$ belongs to G, in other words $Gal(K'^a K''^a | F'F'') \subseteq G$ hence $K^\circ \subseteq F'F''$ and $F'' \leq_\exists F''K^\circ$. Applying this to any $F'' \in \mathcal{F}$ and using the hypothesis that K'' is regularly closed relatively to $\mathcal{F}$ we conclude that $K'' \leq_\exists K^\circ$.

$(A'', \leq) \leq_\exists (A^\circ, \leq)$ then follows from lemma 4. □

Since we have embedded $(A', \leq)$ into $(A^\circ, \leq)$ over A and $(A'', \leq) \leq_\exists (A^\circ, \leq)$, the $|A'|^+$-saturation property of $(A'', \leq)$ provides plenty of embeddings of $(A', \leq)$ into $(A'', \leq)$ over A. In order to construct a particular embedding having the required additional properties we have to introduce some refinements.

Construction 2.

• For any $a' \in A'$ choose $a''_{a'} \in A''$ such that $P(a''_{a'}) = \sigma(P(a'))$ and $a' = 0$ if and only if $a''_{a'} = 0$. This is possible since $\sigma(P(0)) = P(0)$, and if $a' \neq 0$ and $P(a') = P(0)$ then $(0) \notin \operatorname{Spec}(A', \leq)$ hence by hypothesis $(0) \notin \operatorname{Spec}(A', \leq)$ and we can find a non-zero α in A such that $0 \leq \alpha$, hence we can set $a''_{a'} = \alpha$.

Observe that for every $a' \in A'$ we have in A°:

$$a''_{a'} \leq a' \leq a''_{a'}.$$

Indeed, for any $\mathbf{m}'' \in \operatorname{Specmax}(A'')$ we have:

$$\begin{aligned} a''_{a'} \in m(R(\mathbf{m}'')) \cap A^\circ &\iff a''_{a'} \in \mathbf{m}'' \\ &\iff a' \in \hat{\sigma}(\mathbf{m}') \\ &\iff a' \in m(R(\mathbf{m}'')) \cap A^\circ \end{aligned}$$

Since the radical relation of A° has been defined by the family of all $m(R(\mathbf{m}'')) \cap A^\circ$ as $\mathbf{m}''$ ranges over $\operatorname{Specmax}(A'')$ (plus maybe (0) but the case of zero is solved by our construction since $a''_{a'} = 0$ if and only if $a' = 0$) it follows that $a''_{a'} \leq a' \leq a''_{a'}$.

Let $\mathcal{A} = \{a''_{a'} \,/\, a' \in A'\}$ and choose an elementary substructure $K_{0,0}$ of K'' containing $\mathcal{A}$ such that $K(\mathcal{A}) \subseteq K_{0,0}$ and $|K_{0,0}| \leq |K'|$.

• Let $(e''_\alpha)_{\alpha<\beta}$ be a basis of K''^s as a K''-vector space. Since K° is a regular extension of K'' it remains linearly independent over K°. Moreover $Gal(K'^s K''^s | K^\circ) \simeq G = Gal(K''^s | K'')$ hence $K^\circ K''^s = K'^s K''^s$ that is $(e''_\alpha)_{\alpha<\beta}$ generates $K'^s K''^s$ as a K°-vector space. In particular any $x' \in K'^s$ writes uniquely:

$$x' = \sum_{i=1}^{n(x')} \lambda_i(x') e''_{\alpha_i(x')}, \quad \text{with } \lambda_i(x') \in K^\circ \setminus \{0\}.$$

Write each $\lambda_i(x') = c_i(x')/d_i(x')$ with $c_i(x'), d_i(x') \in A^\circ \setminus \{0\}$ and let $\mathcal{C} = \{c_i(x'), d_i(x') \,/\, x' \in K'^s,\ i \leq n(x')\}$. Eventually let $K_{1,0} = K' K_{0,0}(\mathcal{C})$. Observe that $|K_{1,0}| = |K'|$ since $|\mathcal{C}| \leq |K'|$ and $|K_{0,0}| \leq |K'|$.

• We cannot be sure that $K_{1,0}$ and K'' are linearly disjoint over $K_{0,0}$. The following construction will allow us to reduce to this case. Let $n \in \mathbf{N}$

and suppose that we have already build the following tower of extensions (arrows are inclusion maps, the diagram is commutative):

$$\begin{array}{ccccccccccc} & & K_{0,0} & \preceq & K_{0,1} & \preceq \cdots \preceq & K_{0,n} & \preceq & K'' \\ & \nearrow & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ K & \to & K_{1,0} & \to & K_{1,1} & \to \cdots \to & K_{1,n} & \to & K^\circ \end{array}$$

such that $K_{0,n}$ and $K_{1,n}$ have cardinality less than or equal to $|A'|$, and for every $k < n$ every tuple of elements of $K_{1,k}$ which is K''-linearly dependent is $K_{0,k}$-linearly dependent. Then for any tuple x of elements of $K_{1,n}$ which is K''-linearly dependent let ν_x be the set of coefficients of one particular dependence relation, and let $\mathcal{N}$ be the union of all the sets ν_x. Then $|\mathcal{N}| \leq |K_{1,n}| \leq |A'|$ hence there exists $K_{0,n+1} \preceq K''$ such that $K_{0,n}(\mathcal{N}) \subseteq K_{0,n+1}$ and $|K_{0,n}| \leq |A'|$. Let then $K_{1,n+1} = K_{0,n+1}K_{1,n}$, and iterate the construction.

• Eventually let $K_0 = \bigcup_{n\in\mathbf{N}} K_{0,n}$ and $K_1 = \bigcup_{n\in\mathbf{N}} K_{1,n}$. Obviously $K_0 \preceq K''$, $K_0 \subseteq K_1$, $|K_1| = |A'|$ and K_1 is linearly disjoint from K'' over K_0.

The proof of the embedding theorem eventually boils down to the following lemma. Let $A_0 = A'' \cap K_0 = A^\circ \cap K_0$ and $A_1 = A^\circ \cap K_1$. Equip them with the radical relations induced by $(A^\circ, \leq)$.

Lemma 7. *There exists an embedding Θ of $(A_1, \leq)$ into $(A'', \leq)$ over A_0 such that K'' is a separable extension of* $\mathrm{Frac}(\Theta(A_1))$.

Proof. Observe that $(A_1, \leq) \subseteq (A^\circ, \leq)$ and $(A'', \leq) \leq_\exists (A^\circ, \leq)$. Since $|A_1| = |A'|$ the existence of A_0-embeddings of $(A_1, \leq)$ into $(A'', \leq)$ then follows from the saturation property of $(A'', \leq)$. Hence there is nothing to prove in zero characteristic.

Let us now consider the case of positive characteristic p. Let $(x'_i)_{i\in I'}$ be a p-basis of K', that is a basis of K' as a K'^p-vector space. We have to find an embedding of $(A_1, \leq)$ into $(A'', \leq)$ over A_0 whose extension to K_1 maps $(x'_i)_{i\in I'}$ onto a p-free family of K''. An easy compactness argument proves that it is sufficient to find for any finite subset $\{x'_1, \ldots, x'_n\}$ of $(x'_i)_{i\in I'}$ and for any $(A'_1, \leq) \subseteq (A_1, \leq)$ finitely generated over $(A_0, \leq)$, an A_0-embedding Θ' of $(A'_1, \leq)$ into $(A'', \leq)$ whose extension to $K'_1 = \mathrm{Frac}(A'_1)$ maps $\{x'_1, \ldots, x'_n\}$ onto a p-free family.

Tamagawa's construction.
In order to construct Θ' we follow the method of the proof of Tamagawa's lemma given in [10] proposition 10.11. We follow this reference as closely as possible (this is [3] lemme IV.4.1).

Observe first that $K^\circ|K'$ being a separable extension, $\{x'_1, \ldots, x'_n\}$ remains p-free over $(K^\circ)^p$. Let $K''_1 = K''K'_1$ and $L = K''K'^p_1(x'_1, \ldots, x'_n)$. Relabel the x'_i's so that $(x'_1, \ldots, x'_s)$ is a p-basis of L over $K''K'^p_1$ (for some

$s \leq n$). Let $(x_i)_{i\in I}$ be a p-basis of K'':

$$L = K''K_1'^p[x_1', \dots, x_s'] = K_1''^p[(x_i)_{i\in I},\ x_1', \dots, x_s'].$$

It follows that there are elements $x_1, \dots, x_r$ among the x_i's such that $x_{s+1}', \dots, x_n' \in K_1''^p[x_1, \dots, x_r, x_1', \dots, x_s']$. By hypothesis $[K' : K'^p] \leq [K'' : K''^p]$ hence $|I| \geq |I'| \geq n$ (this is clear if $[K' : K'^p]$ is finite, otherwise it follows from the saturation property of K'' inherited from A''). In particular we can take $r \geq n$. Relabeling the x_i's if necessary the exchange lemma then gives:

$$x_{r-(n-s)+1}, \dots, x_r \in K_1''^p[x_1, \dots, x_{r-(n-s)}, x_1', \dots, x_n']. \qquad (1)$$

Let $\eta_i = (x_{r-n+i}x_i')^{1/p}$ for $1 \leq i \leq s$ and let $K_2'' = K_1''[\eta_1, \dots, \eta_s]$. The field extension $K_2''|K''$ is exactly the extension $F'|K$ of [10] proposition 10.11, hence the arguments given there apply to our situation: $K_2''|K''$ is a regular extension.

Remember that $(A^\circ, \leq)$ is the extension of $(A'', \leq)$ induced by a family of valuation rings $R(\mathbf{m}'')$ with fraction field K° indexed on $\operatorname{Specmax}(A'')$. Extend each $R(\mathbf{m}'')$ to a valuation ring with fraction field $K^\circ[\eta_1, \dots, \eta_s]$ and let $(B, \leq)$ be the induced extension of $(A^\circ, \leq)$. Equip $A_2'' = B \cap K_2''$ with the radical relation inherited from $(B, \leq)$. Since K'' is RC-local with positive characteristic it is PAC, hence $K'' \leq_\exists K_2''$ and lemma 4 implies:

$$(A'', \leq) \leq_\exists (A_2'', \leq).$$

Let $\mathcal{I}$ be the set of applications from $\{1, \dots, r+s\}$ to $\{0, \dots, p-1\}$. For with i with $r - n + s + 1 \leq i \leq r$, the relation (1) gives coefficients $(v_{i,j})_{j\in\mathcal{I}}$ in K_1'' such that:

$$x_i = \sum_{j\in\mathcal{I}} v_{i,j}^p x_1^{j(1)} \dots x_{r-n+s}^{j(r-n+s)} x_1'^{j(r-n+s+1)} \dots x_n'^{j(r+s)}.$$

The field K_2' generated over K_1' (and inside K_2'') by those coefficients $v_{i,j}$, plus $x_1, \dots, x_r$ and $\eta_1, \dots, \eta_s$ has the same cardinality as K_1'. Equip $A_2' = A_2'' \cap K_2''$ with the radical relation inherited from $(A_2'', \leq)$. Eventually fix any finitely generated extension A_0' of A_0 inside $A'' \cap A_2'$ whose fraction field contains $x_1, \dots, x_r$. The saturation property of $(A'', \leq)$ provides an A_0'-embedding Θ' of $(A_2', \leq)$ into $(A'', \leq)$. Its extension to K_2' is still denoted by Θ', for every $r - n + s + 1 \leq i \leq r$, $x_i = \Theta'(x_i)$ is equal to:

$$\sum_{j\in\mathcal{I}} \Theta'(v_{i,j})^p x_1^{j(1)} \dots x_{r-n+s}^{j(r-n+s)} \Theta'(x_1')^{j(r-n+s+1)} \dots \Theta'(x_n')^{j(r+s)}. \quad (2)$$

For any $1 \leq k \leq s$ we have:

$$x_{r-n+k} = \Theta'(x_{r-n+k}) = \Theta'(\eta_k^p x_k'^{-1}) = \Theta'(\eta_k)^p \Theta'(x_k')^{-1}$$

hence by (2):

$$x_1, \ldots, x_n \in K''^p(x_1, \ldots, x_{r-n}, \Theta'(x'_1), \ldots, \Theta'(x'_n)).$$

But the tuple of elements $(x_1, \ldots, x_n)$ is p-free over K''^p hence so is $(x_1, \ldots, x_{r-n}, \Theta'(x'_1), \ldots, \Theta'(x'_n))$. In particular $(\Theta'(x'_1), \ldots, \Theta'(x'_n))$ is p-free over K''^p as we wanted. $\square$

The embedding theorem is now at hand: let Θ be the A_0-embedding of $(A_1, \leq)$ into $(A'', \leq)$ over A_0 given by lemma 7. Then Θ extends to an embedding of K_1 into K'' over K. Observe that K° is a regular extension of K'' and $K''K_1 \subseteq K^\circ$ hence $K''K_1$ is linearly disjoint from K''^s over K''. By construction K'' is linearly disjoint from K_1 over K_0, hence K_1 is linearly disjoint from K''^s over K_0. This allows to extend this embedding to a homomorphism of $K_0''^s$-algebra $K''^s[K_1] \to K''^s$ which we still denote by Θ. By construction $K'^s \subseteq K''^s(\mathcal{C}) \subseteq K''^s[K_1]$ hence we can define the restriction of the homomorphism Θ to K'^s, which is an embedding since K'^s is a field.

$K''|\Theta(K_1)$ is separable, so is $K_1|K'$ because $K_1 \subseteq K^\circ$ and $K^\circ|K'$ is separable hence $K''|\Theta(K')$ is separable.

For every $x' = \sum_{i=1}^{n(x')} \lambda_i(x') e''_{\alpha_i(x')} \in K'^s$ and every $\tau \in G(K'')$ we have:

$$\begin{aligned}
\Big(\pi(\tau)\Big)(x') &= \Big\langle \pi(\tau), \tau \Big\rangle (x') \\
&= \sum_{i=1}^{n(x')} \Big\langle \pi(\tau), \tau \Big\rangle (\lambda_i(x')). \Big\langle \pi(\tau), \tau \Big\rangle (e''_{\alpha_i(x')}) \\
&= \sum_{i=1}^{n(x')} \lambda_i(x') \tau(e''_{\alpha_i(x')})
\end{aligned}$$

It follows that:

$$\Theta\Big(\pi(\tau)(x')\Big) = \sum_{i=1}^{n(x')} \Theta(\lambda_i(x'))\tau(e''_{\alpha_i(x')}) = \tau(\Theta(x'))$$

The last equality comes from the facts that $\tau_{|K''} = Id_{K''}$ and the $\lambda_i(x')$'s belong to K_1 hence there image by Θ lies in K''. This proves part 2 of the conclusion, and since $K''|\Theta(K')$ is separable part 1 follows.

By construction $\varphi = \Theta_{|A'}$ embeds $(A', \leq)$ into $(A'', \leq)$ over A. It remains to prove that $\bar{\varphi}$ equals σ. Since both of them commute with $\cap$ it is sufficient for this to check that they coincide on $L(A')$. Let a' be any element of A'. In $(A'', \leq)$ we have $P(a''_{a'}) = \sigma(P(a'))$ hence by construction

$a''_{a'} \leq a' \leq a''_{a'}$ in $(A^\circ, \leq)$. Since $(A'_1, \leq) \subseteq (A^\circ, \leq)$ this remains true in $(A'_1, \leq)$, hence $\Theta(a''_{a'}) \leq \Theta(a') \leq \Theta(a''_{a'})$. Observe that $\Theta(a''_{a'}) = a''_{a'}$ because $a''_{a'} \in A_0$ hence in $(A'', \leq)$:

$$a''_{a'} \leq \Theta(a') = \varphi(a') \leq a''_{a'} \;\Rightarrow$$
$$\bar{\varphi}(P(a')) = P(\varphi(a')) = P(a''_{a'}) = \sigma(P(a')).$$

The embedding theorem is proved.

6. Elementary embeddings, elementary equivalence

For any structure E in a language $\mathcal{L}$, and any subset F of E we denote by $\mathcal{L}(F)$ the expansion of $\mathcal{L}$ obtained by adding new constant symbols for each element of F, and (E, F) the natural $\mathcal{L}(F)$-structure. Observe that if F is a substructure of E then:

$$F \preceq E \iff (F, F) \equiv (E, F).$$

The notion of CDM-representation, introduced by Cherlin, van den Dries and Macintyre [1] plays an essential role in the model-theory of PAC, PRC and PpC fields. We recall its main properties, as reported for example in [6].

There is a class of structures Σ, elementary with respect to a many-sorted language $\mathcal{L}$, and a contravariant functor Φ sending profinite groups to elements of Σ such that:

1. For each $\mathcal{L}$-sentence $\varphi_{\mathcal{L}}$ there is a sentence φ in the language of rings such that a field K models φ if and only if $\Phi G(K)$ models $\varphi_{\mathcal{L}}$;
2. If K is κ-saturated, then $\Phi G(K)$ is κ-saturated;
3. Every epimorphism of profinite groups $\psi : H \to G$ induces an embedding of $\mathcal{L}$-structures $\Phi\psi : \Phi G \to \Phi H$, and conversely every $\mathcal{L}$-embedding of ΦG into ΦH is of this form.

If K is relatively separably closed in an extension field L, the restriction $\rho : G(L) \to G(K)$ induces a monomorphism $\Phi\rho : \Phi G(K) \to \Phi G(L)$. The expanded structure $(\Phi G(L), \Phi\rho(\Phi G(K)))$ will simply be noted $(\Phi G(L), K)$. When given any two regular extensions K_1, K_2 of K we have:

$$(K_1, K) \equiv (K_2, K) \Rightarrow (\Phi G(K_1), K) \equiv (\Phi G(K_2), K).$$

We define the class of **restricted RC-local** fields as those RC-local fields K such that any elementary extension of K is RC-local. Observe that an elementary extension of a restricted RC-local field is itself a restricted RC-local field. Classical examples of restricted RC-local fields are PAC, PRC and PpC fields, as well as PC$_M$ fields (which generalise PpC fields in the sense that p varies in the finite set M of prime numbers, however the p-adic valuations must have rank 1 which is not very convenient) see for example [1], [6], [14]. Other examples (allowing finite ramification) will be given in the applications.

Theorem 11. *Let A', A'' be two PAC* domains with restricted RC-local fraction fields K' and K'' containing a given domain A with fraction field K. Equip A' and A'' with the radical relations associated to their maximal spectra. Suppose that the restrictions to A of these two radical relations coincide and equip A with this relation. If K', K'' are regular extension of K then $(A', A) \equiv (A'', A)$ if and only if the following conditions are fullfilled:*

1. $J(A') = (0) \iff J(A'') = (0)$.
2. $(K', K) \equiv (K'', K)$.
3. $(\mathbf{L}(A'), \mathbf{L}(A)) \equiv (\mathbf{L}(A''), \mathbf{L}(A))$.

Remark 8. If A' is a non-trivial valuation ring with algebraically closed fraction field (the simplest example of a PAC domain which is not a field) and $A'' = \mathrm{Frac}(A')$ then conditions (2) and (3) above are obviously satisfied although $A' \not\equiv A''$. Condition (1) is necessary only to avoid this case, because if neither A' nor A'' are fields then condition (3) will imply condition (1) by proposition 2.

Proof. One direction is immediate: if $(A', A) \equiv (A'', A)$ then $(A', \leq, A) \equiv (A'', \leq, A)$ because these radical relations are definable hence $(A', \leq)$ and $(A'', \leq)$ have A-isomorphic elementary extensions. Such an isomorphism induces a K-isomorphism between elementary extensions of K' and K'' hence $(K', K) \equiv (K'', K)$ and similarly $(\mathbf{L}(A'), \mathbf{L}(A)) \equiv (\mathbf{L}(A''), \mathbf{L}(A))$. By the above remark this implies $J(A') = (0) \iff J(A'') = (0)$.

In order to prove the converse we are going to build A-isomorphic r-extensions of A' and A'' by using inductively the embedding theorem. Let $A_0 = A$, $A_1 = A'$ and A_2 be a $|A'|^+$-saturated elementary extension of A'' (we can suppose all our rings, except maybe A, to be infinite because otherwise the condition 2 implies that $A' = A''$ and there is nothing to prove). Given an integer n suppose that we have already build a sequence $(A_k)_{k\leq n+2}$ of PAC* domains with restricted RC-local fraction fields $(K_k)_{k\leq n+2}$ such that for $1 \leq k \leq n$, $A_k \preceq A_{k+2}$ and for $k < n$ an r-embedding φ_{k+1} is given from A_{k+1} into A_{k+2} whose restriction to A_k is identity, which makes K_{k+2} a regular extension of $\mathrm{Frac}(\varphi_{k+1}(A_{k+1}))$ and $\mathbf{L}(A_{k+2})$ an elementary extension of $\bar{\varphi}_{k+1}(\mathbf{L}(A_{k+1}))$, so that the following diagram is commutative:

$$\begin{tikzcd}
A_0 \arrow[r] \arrow[dr] & A_2 \arrow[r, phantom, "\preceq"] \arrow[dr] & A_4 \arrow[r, phantom, "\cdots\preceq"] & A_{n+1} & \\
& A_1 \arrow[u] \arrow[r, phantom, "\preceq"] & A_3 \arrow[u] \arrow[r, phantom, "\cdots\preceq"] & A_n \arrow[u] \arrow[r, phantom, "\preceq"] & A_{n+2}
\end{tikzcd}$$

the vertical and diagonal arrows being the φ_k's. Suppose in addition that, identifying A_k with its image $\varphi_k(A_k)$ whenever it is convenient, we have:

1. $J(A_{n+1}) = (0) \iff J(A_{n+2}) = (0)$.
2. $(\Phi G(K_{n+1}), K_n) \equiv (\Phi G(K_{n+2}), K_n)$.
3. $(\mathbf{L}(A_{n+1}), \mathbf{L}(A_n)) \equiv (\mathbf{L}(A_{n+2}), \mathbf{L}(A_n))$.
4. A_{n+2} is $|A_{n+1}|^+$-saturated.

The extension $K_{n+1}|K_n$ is regular by hypothesis, and $K_{n+2}|K_n$ is regular because $K_n \preceq K_{n+2}$.

Since $(K', K) \equiv (K'', K)$ the fields K' and K'' have the same characteristic, and if it is positive they have the same imperfect degree hence so do K_{n+1} and K_{n+2} because this property is elementary hence preserved by taking elementary extensions. It follows that int he characteristic p case $[K_{n+1} : K_{n+1}^p] \leq [K_{n+2} : K_{n+2}^p]$. Indeed, equality holds if the imperfect degree is finite, and otherwise the inequality follows from the saturation property of A_{n_2}.

If $\mathbf{L}(A_{n+2})$ is infinite it is $|\mathbf{L}(A_{n+1})|^+$-saturated hence condition 3 provides an elementary embedding σ of $\mathbf{L}(A_{n+1})$ into $\mathbf{L}(A_{n+2})$ whose restriction to $\mathbf{L}(A_n)$ is identity. If $\mathbf{L}(A_{n+2})$ is finite then $\mathbf{L}(A_{n+1}) = \mathbf{L}(A_{n+2})$ by condition 3 hence we can take for σ the identity map.

Eventually condition 2 and the saturation property gives an elementary embedding $\Phi\pi$ of $\Phi G(K_{n+1})$ into $\Phi G(K_{n+2})$ over $\Phi G(K_n)$, hence an epimorphism π from $G(K_{n+2})$ onto $G(K_{n+1})$ which commutes to restrictions to K_n^s.

The embedding theorem then applies: it gives an r-embedding φ_{n+1} of A_{n+1} into A_{n+2} over A_n, such that K_{n+2} is a regular extension of the fraction field of $\varphi_{k+1}(A_{k+1})$, $\Phi G(K_{n+2})$ is an elementary extension of $\Phi G(\mathrm{Frac}(\varphi_{k+1}(A_{k+1}))) = \Phi\pi(\Phi G(K_{n+1})) \simeq \Phi G(K_{n+1})$, and $\bar{\varphi}_{n+1} = \sigma$ is an elementary embedding of $\mathbf{L}(A_{n+1})$ into $\mathbf{L}(A_{n+2})$. Let A_{n+3} be any $|A_{n+2}|^+$-saturated extension of A_{n+1}. Its fraction field K_{n+3} is an elementary extension of K_{n+1} hence it is a restricted RC-local field. The induction hypothesis 1 and 4 are clearly satisfied at the rank $n+1$. So is condition 3 because $\mathbf{L}(A_{n+1}) \preceq \mathbf{L}(A_{n+2})$ and $\mathbf{L}(A_{n+1}) \preceq \mathbf{L}(A_{n+3})$ by construction. Moreover $\Phi G(K_{n+1}) \preceq \Phi G(K_{n+2})$ and $\Phi G(K_{n+1}) \preceq \Phi G(K_{n+3})$ hence condition 2 is satisfied: $(\Phi G(K_{n+2}), K_{n+1}) \equiv (\Phi G(K_{n+3}), K_{n+1})$.

Repeat this construction ω times and let $B' = \bigcup_{n\in\mathbf{N}} A_{2n+1}$, $B'' = \bigcup_{n\in\mathbf{N}} A_{2n}$, $\varphi = \bigcup_{n\in\mathbf{N}} \varphi_{2n+1}$ and $\psi = \bigcup_{n\in\mathbf{N}} \varphi_{2n}$. By construction

$A' \preceq B'$, $A'' \preceq B''$ and φ, ψ are reciprocal A-isomorphisms between B' and B''. The conclusion follows. $\square$

Corollary 5. *Let $A \subseteq A'$ be two PAC* domains with restricted RC-local fraction fields K, K'. Suppose that $J(A) = (0)$ if and only if $J(A') = (0)$. Then:*

$$A \preceq A' \quad \Longleftrightarrow \quad K \preceq K' \ \textit{and}\ \ L(A) \preceq L(A').$$

Remark 9. Observe that if $A \preceq A'$ then $(A, \leq_A) \preceq (A', \leq_{A'})$ because the radical relation induced by the maximal spectrum is definable in the ring, hence the above equivalence makes sense.

Proof. Let $A'' = A$ and apply theorem 11. $\square$

The algebraic part of a domain A, that is the subring of elements of A which are algebraic over the prime subfield of $\operatorname{Frac}(A)$, is usually denoted by $\operatorname{Alg}(A)$. Given an r-domain $(A, \leq)$ with residual semi-lattice L and full residual lattice $\mathbf{L}$ we denote by $\operatorname{Alg}(A, \leq)$ the r-domain obtained by adjoining to $\operatorname{Alg}(A)$ the radical relation induced by the one of $(A, \leq)$, and $\operatorname{Alg}(L) = L(\operatorname{Alg}(A, \leq))$, $\operatorname{Alg}(\mathbf{L}) = \mathbf{L}(\operatorname{Alg}(A, \leq))$.

Corollary 6. *Let A and A' be two PAC* domains with restricted RC-local fraction fields K and K' respectively. Equip them with the radical relation induced by their maximal spectra and let L, L' denote their respective residual lattices. Then $A \equiv A'$ if and only if:*

1. *$J(A) = (0)$ if and only if $J(A') = (0)$.*
2. *$K \equiv K'$.*
3. *$\operatorname{Alg}(A, \leq) \simeq \operatorname{Alg}(A', \leq)$ and after identification $(L, \operatorname{Alg}(L)) \equiv (L', \operatorname{Alg}(L))$.*

Proof. One direction is obvious: if $A \equiv A'$ then they have isomorphic elementary extensions and any such isomorphism maps algebraic parts to algebraic parts. The above three conditions follow.

Conversely if these conditions are fulfilled then $K \equiv K'$ implies that $(K, \operatorname{Alg}(K)) \equiv (K', \operatorname{Alg}(K'))$ and theorem 11 applies. $\square$

In positive characteristic, the above criterion for elementary equivalence is somewhat simpler:

Corollary 7. *Let A and A' be two PAC domains of positive characteristic such that $J(A) = (0)$ if and only if $J(A') = (0)$, then:*

$$A \equiv A' \quad \Longleftrightarrow \quad \operatorname{Frac}(A) \equiv \operatorname{Frac}(A') \ \textit{and}\ \ L(A) \equiv L(A').$$

Proof. In positive characteristic $\mathrm{Alg}(A) = \mathrm{Alg}(\mathrm{Frac}(A))$ is a field, hence $\mathrm{Frac}(A) \equiv \mathrm{Frac}(A')$ implies that $\mathrm{Alg}(A) \simeq \mathrm{Alg}(A')$. Moreover the radical relation induced by A on $\mathrm{Alg}(A)$ is necessarily the one defined by $\mathrm{Spec}(\mathrm{Alg}(A)) = \{(0)\}$ and it is the same for A' hence $\mathrm{Alg}(A, \leq) \simeq \mathrm{Alg}(A', \leq)$. Since $\mathrm{Alg}(L(A)) = \{P(0), P(1)\} = \mathrm{Alg}(L'(A))$ is already in the language of lattices we get $L(A) \equiv L(A')$. All its hypothesis being satisfied, corollary 6 then gives the conclusion. □

7. Applications to decidability and algebra

In this section we apply our study to various examples of PAC* domains provided by théorème 1.3 of L. Moret-Bailly [16]. We call it **the fundamental theorem**. The same examples can be deduced from the simpler corollary 3.4 of B. Green, F. Pop and P. Roquette [11].

Given a finite family $\mathcal{F}_0$ of fields taken among $\mathbf{R}$ and the finite extensions of the $\mathbf{Q}_p$'s, let:

$$\mathcal{F} = \{\mathrm{Alg}(F)^\tau \,/\, \tau \in G(\mathbf{Q}),\ F \in \mathcal{F}_0\}.$$

Define $\mathbf{Q}_\mathcal{F}$ as the field of all algebraic numbers whose minimal polynomial splits in linear factor in $F[X]$ for every $F \in \mathcal{F}_0$ or equivalently:

$$\mathbf{Q}_\mathcal{F} = \bigcap\{F \,/\, F \in \mathcal{F}\}.$$

The integral closure of $\mathbf{Z}$ inside $\mathbf{Q}_\mathcal{F}$ is denoted by $\mathbf{Z}_\mathcal{F}$. For any $F \in \mathcal{F}$ we let $R(F) = F$ if F is real closed, and the unique p-valuation ring of F if it is p-adically closed. Define:

$$W_\mathcal{F} = \{R(F) \cap \mathbf{Q}_\mathcal{F} \,/\, F \in \mathcal{F}\}$$

Eventually we let $\mathcal{P} = \{p_1, \ldots, p_r\}$ be the finite set of distinct primes p such that $\mathcal{F}_0$ contains at least one finite extension of $\mathbf{Q}_p$, and $m = \prod_{p \in \mathcal{P}} p$.

We will keep these notations throughout this section. Among other things the fundamental theorem implies that:

Theorem 12. *Let W be any family of valuation rings with fraction field $\mathbf{Q}_\mathcal{F}$ and let $A = \bigcap W$. Assume that $\mathbf{R} \notin \mathcal{F}_0$ or there is a prime $p \notin \mathcal{P}$ invertible in A. Then a variety V defined over $\mathbf{Q}_\mathcal{F}$ has a smooth point in A if and only if:*

1. $\forall R \in W \setminus W_\mathcal{F},\ V^\circ(R^a) \neq \emptyset$.
2. $\forall F \in F,\ V^\circ(F) \neq \emptyset$ *and if* $R(F) \cap \mathbf{Q}_\mathcal{F} \in W,\ V^\circ(R(F)) \neq \emptyset$.

Similar results hold in positive characteristic, but we will only use that any integrally closed domain whose fraction field is separably closed and algebraic over $\mathbf{F}_p(T)$ is PAC.

Corollary 8.

1. *$\mathbf{Q}_{\mathcal{F}}$ is regularly closed relatively to $\mathcal{F}$, in particular it is RC-local.*
2. *For any family $\mathcal{N}$ of prime numbers, if either $\mathbf{R} \notin \mathcal{F}_0$ or $\mathcal{N} \nsubseteq \mathcal{P}$ then $\mathbf{Z}_{\mathcal{F}}[1/n]_{n\in\mathcal{N}}$ satisfies LGPH* and $\mathbf{Z}_{\mathcal{F}}[1/mn]_{n\in\mathcal{N}}$ is PAC*.*

Proof.
1. Apply theorem 12 with $W = \emptyset$.
2. Since $\mathbf{Z}_{\mathcal{F}}$ is integral over $\mathbf{Z}$ all its localisation including itself have dimension ≤ 1 hence the non geometric condition of the LGPH* is always verified. Moreover $\mathbf{Z}_{\mathcal{F}}$ is Prüfer and so are its localisations hence it is the intersection of all valuation rings which contain it and those valuation rings which are non trivial are exatly the localisations by its maximal ideals. The geometric condition then follows from the fundamental theorem: for $\mathbf{Z}_{\mathcal{F}}[1/n]_{n\in\mathcal{N}}$ (resp. $\mathbf{Z}_{\mathcal{F}}[1/mn]_{n\in\mathcal{N}}$) apply theorem 12 with W as the set of all valuation rings of $\mathbf{Q}_{\mathcal{F}}$ containing $1/n$ (resp.$1/mn$) for every $n \in \mathcal{N}$. Observe simply that W is exactly the family of valuation rings of $\mathbf{Q}_{\mathcal{F}}$ containing $\mathbf{Z}_{\mathcal{F}}[1/n]_{n\in\mathcal{N}}$ (resp. $\mathbf{Z}_{\mathcal{F}}[1/mn]_{n\in\mathcal{N}}$) and for any such ring R, $R^h \subseteq R^a$ and if $R \in W_{\mathcal{F}}$ then $R^h \subseteq R(F)$ for some $F \in \mathcal{F}$ such that $R(F) \cap \mathbf{Q}_{\mathcal{F}} = R$. $\square$

Remark 10. Relatively complemented lattices look so much similar to boolean algebras that the methods of the firsts apply also to the lasts. This analogy is used in [18] page 193 to convince the reader that the theory of dense relatively complemented lattices has quantifier elimination (using this indication, this theory has been proved to be model-complete in [3] annexe A). With the same philosophy we claim here that every relatively complemented lattice has a decidable theory, and that this can be proved by analogy with the corresponding result for boolean algebras (we thank Y.-L. Ershov who pointed us this fact proved in [5] (in russian)).

Lemma 8. *For any family $\mathcal{N}$ of prime numbers, any domain A elementary equivalent to $B = \mathbf{Z}_{\mathcal{F}}[1/n]_{n\in\mathcal{N}}$ satisfies the following conditions:*

1. *$J(A) = (0)$ if and only if $J(B) = (0)$.*
2. *For all $N \in \mathbf{N}^*$, $0 \leq_A N$ if and only if $0 \leq_B N$.*
3. *A is integrally closed.*
4. *Every element of $\mathcal{N}$ is invertible in A.*
5. *$A \models \forall x,\ f(x) \neq 0$, for every irreducible $f \in \mathbf{Z}[X]$ whose leading coefficient has a prime factor which does not belong to $\mathcal{N}$.*
6. *$K = \mathrm{Frac}(A) \equiv \mathbf{Q}_{\mathcal{F}}$.*
7. *$L(A) \equiv L(B)$.*

Conversely, if A satisfies the above conditions 1 to 6 then $\mathrm{Alg}(A, \leq_A)$ is isomorphic to $(B, \leq_B)$.
Conditions 1, 2, 3 and 7 are recursively axiomatisable, so are 4 and 5 if $\mathcal{N}$ is recursive, and so is 6 if $\mathbf{Q}_{\mathcal{F}}$ is decidable.

Proof. A structure is decidable if and only if its complete theory is recursively axiomatisable. The recursive axiomatisability of conditions 1 to 3 is immediate, as well as 4 and 5 if $\mathcal{N}$ is recursive.

Eventually let us consider condition 7. It is an easy exercise to check that if R is any Dedekind domain whose Picard group is torsion and R' is any integrally closed domain whose fraction field is algebraic over $\mathrm{Frac}(R)$ then for any nonzero $a \in R'$ and any $b \in R'$ there exists $c \in R'$ such that $M_{R'}(a) \setminus M_{R'}(b) = M_{R'}(c)$. So proposition 2 proves that $L(R') = \mathbf{L}(R')$ is locally complemented, and this applies in particular to B since $\mathbf{Z}$ is a Dedekind domain with zero Picard group. Then one can either use remark 10 which guaranties that $L(B)$ has a decidable complete theory whatever it can be, or check that $L(B)$ is dense and derive from this a recursive (actually a finite) axiomatisation of its complete theory.

Since all the above properties are elementary and satisfied by B, any domain elementary equivalent to B must share them.

Conversely let A be a domain satisfying conditions 1 to 6. Since $K \equiv \mathbf{Q}_{\mathcal{F}}$ we have $\mathrm{Frac}(\mathrm{Alg}(A)) = \mathrm{Alg}(K) \simeq \mathbf{Q}_{\mathcal{F}}$. After identifying $\mathrm{Alg}(K)$ with $\mathbf{Q}_{\mathcal{F}}$ we have $B \subseteq \mathrm{Alg}(A)$ because A is integrally closed (condition 3) and contains $1/n$ for every $n \in \mathcal{N}$ (condition 4), and $\mathrm{Alg}(A) = \mathrm{Alg}(A) \cap \mathbf{Q}_{\mathcal{F}} = B$ by condition 5. It remains to prove that the radical relation induced on B by $(A, \leq_A)$ is the radical relation $\leq_B$.

Let $a, b \in B$ be such that $a \leq_B b$. If $a \neq 0$ then for every $\mathbf{m} \in \mathrm{Specmax}(A)$ containing a we have $\mathbf{m} \cap B \in \mathrm{Specmax}(B)$ (because B has dimension 1) hence $b \in \mathbf{m}$ by definition of $\leq_B$, which proves that $a \leq_A b$. If $a = 0$ then either $J(B) = (0)$ and then $b = 0$ hence $a \leq_A b$, or $J(B) \neq (0)$ and then there is some nonzero integer N such that $0 \leq_B N$ hence $0 \leq_A N$ by condition 2. Then $N \leq_B b$ and by the above argument $N \leq_A b$ hence once again $0 = a \leq_A b$.

Conversely suppose $a \not\leq_B b$, that is $M_B(a) \setminus M_B(b) \neq \emptyset$. If $a = 0$ then either $J(B) = (0)$ and $b \neq 0$ hence $J(A) = (0)$ by condition 1 and $a \not\leq_A b$, or $J(A) \neq (0)$ and then replacing a by some nonzero integer N such that $0 \leq_B N$ we have to prove that $N \not\leq_A b$ hence we have reduced to the case $a \neq 0$. If $a \neq 0$ let $c \in B \setminus \{0\}$ be such that $M_B(c) = M_B(a) \setminus M_B(b)$. Since $a \not\leq_B b$, $M_B(c)$ is nonempty hence $c \notin B^* = A^* \cap \mathbf{Q}_{\mathcal{F}}$. Let $\mathbf{m}$ be any maximal ideal of A containing c, $\mathbf{m} \cap B$ is a prime ideal containing c hence a maximal ideal since $c \neq 0$ and B has dimension one. It follows that $a \in \mathbf{m}$ and $b \notin \mathbf{m}$ hence $a \not\leq_A b$. $\square$

Define the **holomorphy ring** $R_{\mathcal{F}}$ of $\mathbf{Z}_{\mathcal{F}}$ as the intersection of all the elements of $W_{\mathcal{F}}$.

Lemma 9. *If* $\mathbf{R} \notin \mathcal{F}_0$ *then* $R_{\mathcal{F}}$ *is definable in* $\mathbf{Q}_{\mathcal{F}}$ *and* $\mathbf{Q}_{\mathcal{F}}$ *is restricted RC-local.*

Proof. The crucial tools here are the notions of π-valuations and $\mathrm{RC}_{\pi,\mathbf{Z}}$ fields introduced by Ershov in [7]. In order to use it we have to find an element π in $\mathbf{Q}_{\mathcal{F}}$ such that every $R \in W_{\mathcal{F}}$ is a π-valuation ring, that is π is a generator for the maximal ideal of R.

First observe that if in $\mathcal{F}_0$ all the fields which contain the same $\mathbf{Q}_p$ are replaced by there intersection (in some algebraic closure of $\mathbf{Q}_p$) this does not change the field $\mathbf{Q}_{\mathcal{F}}$. Hence we can suppose without loss of generality that for each $i \leq n$, $\mathcal{F}_0$ contains a unique finite algebraic extension F_i of $\mathbf{Q}_{p_i}$. Similarly if F_i is replaced by the intersection of all its conjugates over $\mathbf{Q}_{p_i}$ once again we obtain the same field $\mathbf{Q}_{\mathcal{F}}$ hence we can suppose that F_i is a normal extension of $\mathbf{Q}_{p_i}$. Let e_i, f_i be the ramification index and the residue degree of th p_i-adically closed valuation of F_i (see [17] for the definitions).

For any $i \leq r$ let V_i be the variety defined over $\mathbf{Q}_{\mathcal{F}}$ by:

$$\begin{cases} X^{e_i} - p_i Y = 0 & (1) \\ 1 - YZ = 0 & (2) \\ S_j X + T_j p_j = 1, \quad 1 \leq j \leq r,\ j \neq i & (3) \end{cases}$$

For any $F \in \mathcal{F}$, if F is p_i-adically closed it has ramification index e_i so any generator x_F of the maximal ideal of $R(F)$ satisfies $y_F = x_F^{e_i}/p_i \in R(F)^*$ hence:

$$(x_F, y_F, z_F = 1/y_F, \{s_{j,F} = 0\}_{j \neq i}, \{t_{j,F} = 1/p_j\}_{j \neq i}) \in V_i^{\circ}(R(F)).$$

If F is p_k-adically closed for some $k \neq i$ then $p_i \in R(F)^*$ hence:

$$(1, 1/p_i, p_i, \{s_{j,F} = 1\}_{j \neq i}, \{t_{j,F} = 0\}_{j \neq i}) \in V_i^{\circ}(R(F)).$$

Applying the fundamental theorem to V_i with $W = W_{\mathcal{F}}$ then gives a point of $V_i^{\circ}(R_{\mathcal{F}})$. Let x_i be the first coordinate of this point. By construction x_i generates the maximal ideal of any p_i-adic $R(F)$ (because they have ramification index e_i and $x_i^{e_i}/p_i \in R(F)^*$ by (1) and (2)) and is invertible in any other $R(F)$ by (3) hence $\pi = \prod_{i \leq r} x_i$ is an element in $R_{\mathcal{F}}$ which generates the maximal ideal of any $R(F)$ for every $F \in \mathcal{F}$. In other words every $R \in W_{\mathcal{F}}$ is a π-valuation ring.

Now observe that any valuation ring R of $\mathbf{Q}_{\mathcal{F}}$ which does not belong to $W_{\mathcal{F}}$ is PAC* by the fundamental theorem hence $R^h = R^a$. In particular such a valuation ring is not a π-valuation ring. So $W_{\mathcal{F}}$ is exactly the family of all π-valuation rings of $\mathbf{Q}_{\mathcal{F}}$ hence its holomorphy ring is exactly the ring denoted by R_π in [7]. Such a ring has fraction field $\mathbf{Q}_{\mathcal{F}}$ and the π-valuation of $\mathbf{Q}_{\mathcal{F}}$, in other words the elements of $W_{\mathcal{F}}$, are exactly the localisations of $R_{\mathcal{F}}$ by its maximal ideals. By the fundamental theorem the field $\mathbf{Q}_{\mathcal{F}}$ is regularly closed relatively to $\mathcal{F}$ hence K is a $\mathrm{RC}_{\pi,\mathbf{Z}}$ field and by theorem 1 of [7] the ring $R_{\mathcal{F}} = R_\pi$ is definable in $\mathbf{Q}_{\mathcal{F}}$.

Now let K' be any elementary extension of K.We freely use all the theorems on $\mathrm{RC}_{\pi,\mathbf{Z}}$ fields proved in [7]. Then K' is an $\mathrm{RC}_{\pi,\mathbf{Z}}$ field because this is an elementary property, hence it is regularly closed with respect to the family of all its π-adic closures. In order to conclude that K' is RC-local it remains only to check that its π-adic closures are p-adically closed of finite rank, for various prime numbers p.

The π-adic holomorphy ring R'_π of K' is definable in K' by the same formula as $R_{\mathcal{F}}$ in $\mathbf{Q}_{\mathcal{F}}$ hence $R_{\mathcal{F}} \preceq R'_\pi$. Now let R' be any π-valuation ring of K'. Then $\mathbf{m}' = m(R') \cap R'_\pi$ is a maximal ideal of R'_π and $R' = (R'_\pi)_{\mathbf{m}'}$. Since $\prod_{i \leq r} p_i$ belongs to $J(R_{\mathcal{F}})$ and this is an elementary property, it belongs also to $J(R'_\pi)$ hence there is an $i \leq r$ such that $p_i \in \mathbf{m}'$. Since $R_{\mathcal{F}} \models \forall a, \exists b, \prod_{i \leq r}(a^{p_i^{f_i}} - a) = \pi b$, so does R'_π hence there is an index $i \leq r$ such that $\bar{R}'_\pi/\mathbf{m}' = R'/m(R')$ is finite over $\mathbf{F}_{p_i}$ with dimension $\leq f_i$. Moreover $\pi^{e_i}/p_i \in R_{\mathcal{F}} \subseteq R'_\pi$ hence R is a p_i-adic valuation ring of type (e_i, f_i) and its value group is a $\mathbf{Z}$-group because every π-valuation of an $\mathrm{RC}_{\pi,\mathbf{Z}}$ field does so, hence $\mathrm{Frac}(R'^h)$ is a p_i-adically closed of type (e_i, f_i). $\square$

Remark 11. We know that $\mathbf{Q}_{\mathcal{F}}$ is restricted RC-local and $R_{\mathcal{F}}$ is definable in $\mathbf{Q}_{\mathcal{F}}$ in two cases: when $\mathbf{R} \notin \mathcal{F}_0$ by lemma 9 and when $\mathcal{F}_0 = \{\mathbf{R}\}$ because then $R_{\mathcal{F}} = \mathbf{Q}_{\mathcal{F}}$ is pseudo real closed. However it remains probably true for any finite $\mathcal{F}_0$, but there is no reference in the literature for RC-local fields mixing valuations and orders (except for a finite number of them in [12] but this is excluded for $\mathbf{Q}_{\mathcal{F}}$) and it would require some extra efforts to do it.

Theorem 13. *Let $\mathcal{N}$ be any family of prime numbers. If $\mathbf{Q}_{\mathcal{F}}$ is restricted RC-local and its holomorphy ring $R_{\mathcal{F}}$ is definable in $\mathbf{Q}_{\mathcal{F}}$ and if either $\mathbf{R} \notin \mathcal{F}_0$ or there is $p \notin \mathcal{P}$ such that $1/p \in \mathcal{N}$, then the ring $\mathbf{Z}_{\mathcal{F}}[1/n]_{n \in \mathcal{N}}$ is decidable if and only if $\mathbf{Q}_{\mathcal{F}}$ is decidable and $\mathcal{N}$ is recursive.*
In particular:

1. *If $\mathbf{R} \notin \mathcal{F}_0$ then $\mathbf{Z}_{\mathcal{F}}$ is decidable if and only if $\mathbf{Q}_{\mathcal{F}}$ is decidable.*
2. *If $\mathcal{F}_0 = \{\mathbf{R}\}$ then $\mathbf{Z}_{\mathcal{F}}[1/n]_{n \in \mathcal{N}}$ is decidable if and only if $\mathcal{N}$ is recursive and nonempty.*

Proof. One direction is trivial: if $A = \mathbf{Z}_{\mathcal{F}}[1/n]_{n \in \mathcal{N}}$ is decidable then so is its fraction field because it is interpretable in A, and $\mathcal{N}$ is recursive because it is the set of prime numbers which are invertible in A. Moreover if $\mathcal{F}_0 = \{\mathbf{R}\}$ then a famous result of J. Robinson asserts that $\mathbf{Z}_{\mathcal{F}}$ is undecidable hence if A is decidable it implies that $\mathcal{N} \neq \emptyset$.

Suppose now that $\mathbf{Q}_{\mathcal{F}}$ is decidable and $\mathcal{N}$ is recursive. If W denotes the family of all non trivial valuation rings containing A let $W_0 = W \setminus W_{\mathcal{F}}$, $W_1 = W \cap W_{\mathcal{F}}$, $W_2 = W_{\mathcal{F}} \setminus W$ and $A_i = \bigcap W_i$ for $i = 0, 1, 2$. Let m_1 be the product of primes $p \in \mathcal{P} \cap \mathcal{N}$. It is clear that $R_{\mathcal{F}} = A_1 \cap A_2$, that $A = A_0 \cap A_1$ and that $R_{\mathcal{F}}[1/m_1] = A_1$ (for example for the last one: $R_{\mathcal{F}}[1/m_1]$ is the intersection of the family of valuation rings in $W_{\mathcal{F}}$ containing $1/m_1$, that is $W_{\mathcal{F}} \cap W = W_1$).

Since $1/m \in A_0$ by the fundamental theorem $A_0 = \mathbf{Z}_{\mathcal{F}}[1/mn]_{n \in \mathcal{N}}$ is PAC* hence by corollary 6 its complete theory is recursively axiomatised by the theory of PAC* domains and the list of conditions given in lemma 8 hence it is decidable.

Since $\mathbf{Q}_{\mathcal{F}} = \mathrm{Frac}(A_0)$ is interpretable in A_0 and $R_{\mathcal{F}}$ is definable in $\mathbf{Q}_{\mathcal{F}}$ it follows that $A_0 \cap R_{\mathcal{F}}$ is definable in A_0 (as usually $\mathbf{Q}_{\mathcal{F}}$ is interpreted as a quotient of $A_0 \times A_0$ and $A_0 \cap R_{\mathcal{F}}$ is the set of elements a of A_0 such that the image of $(a, 1)$ in this quotient belongs to $R_{\mathcal{F}}$). So $A = A_0 \cap A_1 = A_0 \cap R_{\mathcal{F}}[1/m_1]$ is definable in the decidable ring A_0 hence A is decidable.

The two particular cases then follow by taking $\mathcal{N} = \emptyset$ for the first one, and by observing that for the second one $\mathcal{P} = \emptyset$ hence if $\mathcal{N}$ is non empty it contains a prime number which is obviously not in $\mathcal{P}$. $\square$

Remark 12. Experts agree that since the structure of the absolute Galois group of $\mathbf{Q}_{\{\mathbf{Q}_p\}}$ was elucidated by F. Pop, this field has a decidable theory. More generally $\mathbf{Q}_{\mathcal{F}}$ is decidable whenever $\mathcal{F}_0$ is a finite set of $\mathbf{Q}_p$'s, see [8] theorem 3. It then follows from theorem 13 that $\mathbf{Z}_{\mathcal{F}}$ is decidable.

As the example of $\mathbf{Z}_{\{\mathbf{R}\}}$ shows it is not possible in general to reduce the decidability of a domain to the decidability of its fraction field. However this is a somewhat surprising result, and even more surprising is the fact that not only the fraction field of $\mathbf{Z}_{\{\mathbf{R}\}}$ is decidable but also for example all the intermediate rings $\mathbf{Z}_{\{\mathbf{R}\}}[1/n]$ for n ranging over $\mathrm{N} \setminus \{0, 1\}$ by theorem 13. We turn now to some cases of positive characteristic.

Theorem 14. *A PAC domain of positive characteristic is decidable if and only if its fraction field is decidable.*
In particular any integrally closed ring whose fraction field is separably closed and algebraic over $\mathbf{F}_p(t)$ is decidable.

Proof. Let A be a PAC domain of positive characteristic. By corollary 7 its complete theory is axiomatised by:

- The recursively axiomatisable theory of PAC domains.
- The formula saying if $J(A) = (0)$ or not.
- The axioms describing the complete theory of its fraction field and residual lattice.

Since the residual lattice is relatively complemented it has a decidable complete theory, and so does the fraction field by hypothesis, hence the above list of axioms is recursively enumerable and A is then decidable.

To derive the second assertion observe that by the fundamental theorem any integrally closed ring whose fraction field is separably closed and algebraic over $\mathbf{F}_p(t)$ is a PAC domain. Moreover its fraction field is known to be recursively axiomatised by the axioms of separably closed fields of characteristic p and fixed imperfect degree (0 or 1). The conclusion follows. $\square$

Our last application is purely algebraic. It does not actually use our extensive study of PAC domains but only the particular case of PAC domains with algebraically closed fraction field and a dense residual lattice studied by Prestel and Schmid in [18]. On the other hand it works in arbitrary characteristic, and it is interesting to notice that contrary to theorem 5.3 of [18] no simpler algebraic proof is known by now.

Theorem 15. *Let A be a PAC domain with an algebraically closed fraction fields and a dense residual lattice. Then for every $f_1, \dots, f_m \in A[X_1, \dots, X_n, Y]$ the following assertions are equivalent:*

1. *For every $\alpha \in A^n$ the $f_i(\alpha, Y)$'s have a common root in A.*
2. *There exists a non-constant $f \in A[X_1, \dots, X_n, Y]$ monic in Y which divides simultaneously $f_1, \dots, f_m$.*

Proof. If a non-constant $f \in A[X_1, \dots, X_n, Y]$ monic in Y divides simultaneously $f_1, \dots, f_m$ then for every $\alpha \in A^n$ any root of $f(\alpha, Y)$ will be a common root the $f_i(\alpha, Y)$'s. Since A is integrally closed with algebraically closed fraction field and $f(\alpha, Y)$ is non-constant and monic such a root exists and must belong to A.

Conversely let $\varphi(x, y)$ be the formula with $n + 1$ variables and parameters the coefficients of the f_i's which writes $\bigwedge\!\!\!\bigwedge_{i \le n} f_i(x, y) = 0$ and suppose that $A \models \forall x \exists y \varphi(x, y)$. Let B be the integral closure of $A[X_1, \dots, X_n]$ in an algebraically closed field. Equip B with the radical relation defined by its maximal spectrum. The r-domain $(B, \le_B)$ is an extension of $(A, \le_A)$ and $J(B) = (0)$ if and only if $J(A) = (0)$. The construction of [18] 3.3. apply here: it proves that $(B, \le_B)$ embeds in a r-domain $(C, \le)$ which is PAC, has an algebraically closed fraction field, a dense residual lattice, a radical relation defined by its maximal spectrum, a Jacobson radical which is zero if and only if $J(B)$ is so, and such that every maximal ideal of B can be lifted to a maximal ideal of C. Since the theory of algebraically closed fields and the theory of dense relatively complemented lattices are model-complete, we conclude by theorem 4.3 of [18] (or more generally by corollary 4) that $(A, \le_A) \preceq (C, \le)$.

In particular $(C, \le) \models \forall x \exists y \varphi(x, y)$. Apply this to $(X_1, \dots, X_n) \in C^n$: there exists $\theta \in C$ such that $C \models \varphi(X_1, \dots, X_n, \theta)$. Such a θ is algebraic over $A[X_1, \dots, X_n]$ hence $\theta \in \mathrm{Frac}(B)$. Since B is integrally closed and every maximal ideal of B lifts to C we have $C \cap \mathrm{Frac}(B) = B$ hence $\theta \in B$. Let $f \in A[X_1, \ldots, X_n, Y]$ be the minimal polynomial of θ over $A[X_1, \ldots, X_n]$. It is monic in Y because θ is integral over $A[X_1, \ldots, X_n]$, and it divides every $f_i(X_1, \ldots, X_n, Y)$ since all of them vanish on θ and the conclusion follows. $\square$

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