Pseudo algebraically closed rings

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March 5, 1999

1 Introduction

In this paper we continue the presentation started in [2] of the results of [3]. The reader is expected to be familiar with the definitions and notations of [2], specially the Local-Gobal Principle with respect to Henselisations (LGPH) which is a variant of the nonsingular Hasse principle, and the notion of the residual (semi-)lattice $L(A)$ attached to a ring A. Basic facts taken from [15] about r-rings $(A, \leq)$ and their residual lattice are collected in section 3.

In this paper we introduce the notion of pseudo algebraically closed (PAC) domains. It appears to be a rather special case of rings satisfying LGPH, namely the case of such rings which have a pseudo algebraically closed fraction field. This allows us to apply to them in the second and third section the results of the general study made in [2]. In particular PAC domains prove to form an elementary class. They have nice properties of relative existential closure which characterize them in terms of rings as well as r-domains. We prove for example in the fourth section:

Theorem 1.1 *Let A be a domain such that $\{\mathbf{m} \in \mathrm{Specmax}(A) \,/\, a \notin \mathbf{m}\}$ is compact for every non zero $a \in A$ (this is a condition appearing in the definition of LGPH as well as PAC for technical reasons). Then A is PAC if and only if $A \leq_{\exists} B$ for every domain B containing A such that $\mathrm{Frac}(B)$ is a regular extension of $\mathrm{Frac}(A)$ and every maximal ideal of A can be lifted to a maximal ideal of B.*

It appears that in order to deal with non necessarily PAC fraction fields it is convenient to considere a larger class of domains that we call PAC* domains. We can define them by using a variant of the above theorem: a PAC* domain is a domain A such that $\{\mathbf{m} \in \mathrm{Specmax}(A) \,/\, a \notin \mathbf{m}\}$ is compact for every non zero $a \in A$, and $A \leq_{\exists} B$ for every domain B such that $\mathrm{Frac}(A) \leq_{\exists} \mathrm{Frac}(B)$ and every maximal ideal of A can be lifted to a maximal ideal of B.

When considering theories of PAC* domains the following result of model completeness is representative of the very flexible criterions which can be obtained when applying these notions to theories of r-domains.

Theorem 1.2 *Let T_l be a model-complete theory of lattices, T_f a model-complete theory of fields. Then the theory of r-domains $(A, \leq)$ such that A is PAC*, the jacobson radical of A is (0) (resp. is not (0)) the radical relation of $(A, \leq)$ is defined by the maximal spectrum of A and:*

$$\mathrm{Frac}(A) \models T_f \quad \textit{and} \quad L(A) \models T_l$$

is model-complete.

However this preliminary theorem does not answer the question of knowing wether or not an embedding between two particular PAC or PAC* domains is elementary, because one can have $\mathrm{Frac}(A) \preceq \mathrm{Frac}(A')$ whitout $\mathrm{Frac}(A)$ having a model-complete theory. This is a much more difficult question in general. We answer it in section six for all PAC domains (regardless of their characteristic) and for PAC* domains having a fraction field in a certain class we call RC-local fields. This class of fields contains the usual examples of fields satisfying a local-global principle, namely PAC fields, pseudo real closed fields (PRC) and pseudo p-adically closed fields (PpC). The study of those classical examples shows that the elementary equivalence between two PAC, PRC or PpC fields is essentially determined by their absolute Galois group in terms of CDM-representaton (see section 6 for a more precise statement). We assume that this property still holds in the larger class of RC-local fields, a fact which is well known by the experts but has never been completely published. We then obtain the following:

Theorem 1.3 *Let A, A' be PAC* domains with RC-local fraction fields (this contains the case of PAC domains) such that the Jacobson radical $J(A) = \bigcap \mathrm{Specmax}(A)$ is zero if and only if $J(A')$ is so.*

1. *If $A \subseteq A'$ then $A \preceq A'$ if and only if $\mathrm{Frac}(A) \preceq \mathrm{Frac}(A')$ and $L(A) \preceq L(A')$.*

2. *A similar criterion is given for elementary equivalence, which boyls down in positive characteristic to the very simple following one: $A \equiv A'$ if and only if $\mathrm{Frac}(A) \equiv \mathrm{Frac}(A')$ and $L(A) \equiv L(A')$.*

All the results of this sixth section are based on a deep and somewhat intricate embedding theorem whose proof is the subject of the whole section five. In the seventh and last section of this paper we apply this to natural rings such as the the ring of all absolutely p-adic integers, that is the ring of algebraic integers which belong to every p-adic closure of $\mathbf{Q}$. Typically we prove:

Theorem 1.4 *The ring of all absolutely p-adic integers is decdable, that is there exists an algorithm which can prove every first order theorem which holds in this ring.*

We also considere the case of various overrings of the ring of all absolutely real integers, that is the ring of algebraic integers whichbelong to every real closure of $\mathbf{Q}$. Although that ring has been proved to have an undecidable complete theory by J. Robinson, its fraction field is decdable. We prove that moreover its localisation by any recursive (for example any finite) nonempty set of noninvertible integers has a decidable complete theory. Eventually we considere some rings of positive characteristic.

Let us mention a natural application to algebra of our model theoretical methods:

Theorem 1.5 *Let A be a PAC domain with algebraically closed fraction fields and a dense residual lattice. Then for every $f_1, \ldots, f_m \in A[X_1, \ldots, X_n, Y]$ the following assertions are equivalent:*

1. *For every $\alpha \in A^n$ the $f_i(\alpha, Y)$'s have a common root in A.*

2. *There exists a non-constant $f \in A[X_1, \ldots, X_n, Y]$ monic in Y which divides simultaneously $f_1, \ldots, f_m$.*

Remark 1.6 All the results published in this paper are part of the doctoral thesis of the author, made under the direction of F. Delon (university Paris 7) and the co-direction of L. Moret-Bailly (university Rennes 1). While this work was achieved, other people were studying very closed topics. The author has to mention first Y.-L. Ershov, whose work on boolean families of valuation rings has been a source of inspiration for him (see [7]). It is not surprising then that the methods of [3] and those developped by Y.-L. Ershov after [7] lead to similar, although not identical, results. In particular Y.-L. Ershov has informed us in february 1998 that the decidability of the ring of all absolutely p-adic integers was part of his own recent results. Another notion of PAC domains has also been introduced independantly by A. Razon in [17], which eventually appears to coincide with a special case of our PAC domains, namely those which have a dense residual lattice. Y.-L. Ershov, A. Razon and the author shared a common inspiration taken from the fondamental papers [15] and [4] as well as their ancestor [18] and the litterature on local-global principle for fields which has grown after the doctoral thesis of L. van den Dries. However they used different methods and achieved different goals. The results concerning overrings of the absolutely real integers, for example, are specific to the author as well as the above application to algebra (for which no algebraic proof has been found until now).

2 Pseudo algebraically closed domains and variants

Recall that for any ring A and $a \in A$ we denote by $M_A(a) = \{\mathbf{m} \in \mathrm{Specmax}(A) \,/\, a \in \mathbf{m}\}$ and $M_A(a)^c = \mathrm{Specmax}(A) \setminus M(a)$ the subscript A being omitted whenever it is not confusing. The set of valuation rings centered at a maximal ideal of a domain A is denoted by W_A.

Let us say that a ring A is **relatively pseudo algebraically closed (PAC*)** if and only if A is a domain satisfying the two following properties:

1. $\forall a \in A \setminus \{0\}$, $M(a)^c$ is compact (with respect to the Zariski topology).
2. Every variety V defined over the fraction field K of A satisfies:

$$V^\circ(A) \neq \emptyset \quad \iff \quad \begin{cases} V^\circ(K) \neq \emptyset \ \text{ which we call condition } (*), \\ \\ \forall R \in W_A,\ V^\circ(R^a) \neq \emptyset. \end{cases}$$

The definition of **pseudo algebraically closed (PAC)** domains is obtained from the above one by removing the $(*)$ condition. Observe that if A is a field, property 1 is trivial and property 2 without $(*)$ is exactly the usual definition of pseudo algebraically closed fields, which substantiates our terminology.

In practice PAC* domains are of some interest only if condition $(*)$ is not too hard to deal with, for example if the fraction field is pseudo real closed or pseudo p-adically closed. We will prove further in this section that PAC domains are exactly PAC* domains with a PAC fraction field (one implication is inevident, the other requires to check that property 2 remains true in a PAC* domain if V is replaced by an affine open subset defined over K).

The LGPH introduced in [2] can also give rise to a $(*)$ version: a ring A satisfies LGPH* if and only if it is a domain, given any non zero element a of A the set of all maximal ideals of A which do not contain a is compact and for every variety V defined over the fraction field K of A:

$$V^\circ(A) \neq \emptyset \quad \iff \quad \begin{cases} V^\circ(K) \neq \emptyset, \\ \forall R \in W_A,\ V^\circ(R^h) \neq \emptyset. \end{cases}$$

Since for every valuation ring R one can embed R^h in R^a over R (usually we will simply considere that R^a contains R^h) it is immediate that PAC* domains satisfy LGPH*. As a consequence all the properties derived from LGPH in [2] still hold for PAC domains.

Proposition 2.1 *Let A a be PAC* domain, then:*

1. *A is a Bezout domain.*
2. *A has the block factorisation property, hence any constructible subset of the maximal spectrum of A is of the form $M(a)$ or $M(a)^c$ for some non zero $a \in A$.*
3. *A has the block approximation property.*
4. *Any two $R \neq R'$ in W_A are independent valuation rings.*
5. *A is dense in $A^h_{\mathbf{m}}$, for every maximal ideal $\mathbf{m}$ of A.*

Proof: See propositions 3.1 to 3.7, and proposition 4.7 of [2]. They were proved with LGPH in place of LGPH*, but the adaptations of the proofs are immediate.
■

The relation between PAC* (resp. PAC) and LGPH* (resp. LGPH) can actually be made more precise. We need two lemmas.

Lemma 2.2 *If a domain A which is not a field is PAC*, then $R^h = R^s$ for every R in W_A.*

Proof: Let A be a PAC* domain which is not a field, let K denote its fraction field and choose an element R_0 of W_A. We have to prove that R_0^h has a separably closed fraction field, hence that any monic separable non constant polynomial $f \in R_0[X]$ has a zero in R_0^h. By density of A inside R_0 it is sufficient to prove it for $f \in A[X]$.

Let $\varepsilon \in A \setminus \{0\}$, and V be the variety defined over K by $f(X) - \varepsilon Y$. This polynomial generates a prime ideal of $K^a[X, Y]$ and V is smooth. The point $(1, f(1)/\varepsilon)$ belongs to $V(K)$, and for every R in W_A since f is monic it has a zero x_R in R^a hence $(x_R, 0) \in V^\circ(R^a)$. By PAC* we get $(a, b) \in V^\circ(A)$, which satisfies $|f(a)|_{R_0} = |\varepsilon b|_{R_0} \leq |\varepsilon|_{R_0}$. it follows that 0 belongs to the topological closure of $f(A)$ inside $\mathrm{Frac}(R_0^h)$. Since f is separable we know that $f(R_0^h)$ is closed inside $\mathrm{Frac}(R_0^h)$ and it contains $f(A)$ hence $0 \in f(R_0^h)$.
■

Lemma 2.3 *Let V be a variety defined over the fraction field F of a valuation ring R. Then $V^\circ(R^s) \neq \emptyset$ if and only if $V^\circ(R^a) \neq \emptyset$.*

Only the implication $V^\circ(R^a) \neq \emptyset \Rightarrow V^\circ(R^s) \neq \emptyset$ has to be proved, the other one being trivial. We can actually prove a stronger result:

Lemma 2.4 *With the above notations, suppose that are given a smooth point x^a of $V(R^a)$ and arbitrary polynomials $h_1, \ldots, h_r$ with coefficients in R^s. Then there exists $x \in V^\circ(R^s)$ such that for every $i \leq r$:*

$$h_i(x) \in m(R^s) \iff h_i(x^a) \in m(R^a).$$

Remark 2.5 Observe that in 2.4 R^a can be replaced by any valuation ring containing R^s such that $R' \cap \mathrm{Frac}(R^s) = R^s$. Indeed, if a smooth point y of V is initially given in any such valuation ring R' then $R^a \preceq R'^a$ because the theory of non trivial valuation rings with an algebraically closed fraction field is model complete (the case of trivial valuation rings is simpler: separably closed fields are PAC hence ,2.4 is immediate in that case). It follows that y can be replaced by a point of R^a to which 2.4 applies.

Proof: Replacing R by R^s we can suppose F to be separably closed, hence $R = R^s$ is henselian. Let $\bar{R}$ be an elementary extension of R with infinitesimal elements, that is such that:

$$\mathbf{p} := \{\bar{x} \in \bar{R} \,/\, \forall x \in R,\ |\bar{x}|_{\bar{R}} \leq |x|_{\bar{R}}\}$$

is a non zero prime ideal of $\bar{R}$ (such an elementary extension always exists by the compactness theorem). Since $R \preceq \bar{R}$ it is sufficiant to solve the problem in $\bar{R}$ because any such solution gives rise to a solution in R by existential closeness of R inside $\bar{R}$ (in the language of valuation rings).

From $R \preceq \bar{R}$ it follows that $\bar{R}$ has a separably closed fraction field, and from $\mathbf{p} \neq (0)$ we conclude that $\bar{R}_{\mathbf{p}}$ is a non trivial valuation ring with a separably closed fraction field. Hence its residue field k is algebraically closed, and as it contains obviously F it also contains F^a.

The variety V is defined over F, hence a fortiori over $\bar{F}$. By reducing modulo $\mathbf{p}$ it remains a variety defined over k because k contains F. Moreover $x^a \in V^\circ(k)$, and since $\bar{R}_{\mathbf{p}}$ is henselian we can find $\bar{x} \in V^\circ(\bar{R}_{\mathbf{p}})$ whose residual image in k is precisely x^a.

From this it follows that in a common extension of $\bar{R}_{\mathbf{p}}$ and R^a, for exemple in $\bar{R}^a_{\mathbf{p}}$, the point $\bar{x} - x^a$ has coordinates in $m(\bar{R}^a_{\mathbf{p}})$, and since x^a belongs to $V(\bar{R}^a)$ we conclude that $\bar{x}$ has coordinates in $\bar{R}^a$, hence $x \in V^\circ(\bar{R}^a \cap \bar{F}) = V^\circ(\bar{R})$. Moreover for any polynomial h with coordinates in R, and in particular for the given h_i's, the Taylor formula implies that $h(\bar{x}) - h(x^a)$ belongs to the ideal generated in $\bar{R}^a$ by the coordinates of $\bar{x} - x^a$ hence to $m(\bar{R}^a_{\mathbf{p}}) \cap \bar{R}^a$ and a fortiori to $m(\bar{R}^a)$ from which we conclude that:

$$h(\bar{x}) \in m(\bar{R}) \iff h(\bar{x}) \in m(\bar{R}^a) \iff h(x^a) \in m(\bar{R}^a) \iff h(x^a) \in m(R^a).$$

■

By combining the two above lemmas we conclude:

Proposition 2.6 *A ring A is a PAC* domain if and only if it satisfies LGPH* and either it is a field or $R^h = R^s$ for every $R \in W_A$.*

Remark 2.7 Obviously, the same holds without $(*)$, that is with PAC* and LGPH* replaced by PAC and LGPH respectively.

Corollary 2.8 *Let A be a domain with fraction field $K \neq A$.*

1. *If A is a PAC* domain then for every monic non-constant separable polynomial $f \in A[T]$ and any non-zero ε in A there is $a \in A$ such that $f(a) \in \varepsilon A$.*

2. *Conversely, if A satisfies LGPH* (resp. LGPH) and the above property, then A is PAC* (resp. PAC).*

It follows that the class of PAC (resp. PAC) domains is recursively axiomatisable.*

Proof:
1. Considere the smooth variety V defined over $K = \mathrm{Frac}(A)$ by $f(T) = \varepsilon Y$ (we need f to be separable in order to ensure that V is really defined over K, that is geometrically reduced). Obviously $(0, \varepsilon^{-1}f(0)) \in V^\circ(K)$ and since f is monic and R^a is integrally closed, any root t of f in an algebraically closed field belongs to R^a hence $(t, 0) \in V^\circ(R^a)$. Choose then for a the first coordinate of any point of $V^\circ(A)$ given by PAC*.
2. For the converse we have to prove, by 2.6, that $A^h_{\mathbf{m}} = A^s_{\mathbf{m}}$ for every maximal ideal $\mathbf{m}$ of A. Fix $\mathbf{m}$ in $\mathrm{Specmax}(A)$, since A is dense in $A^h_{\mathbf{m}}$ by 2.1 it is sufficient to prove that any monic non-constant separable polynomial in $A[T]$ has a root in $A^h_{\mathbf{m}}$. Choose any such $f \in A[T]$, the hypothesis implies that 0 belongs to the topological of $f(A)$ inside $\mathrm{Frac}(A^h_{\mathbf{m}})$. Since f is separable $f(A^h_{\mathbf{m}})$ is closed, hence $0 \in f(A^h_{\mathbf{m}})$ that is f has a root in $A^h_{\mathbf{m}}$.

The conclusion now follows from 2.6 because the class of PAC* domains is the class of domains A satisfying:

- LGPH*, which has been recursively axiomatised in [2] (actually it is LGPH which has been axiomatised in [2], but it is clear that condition $(*)$ can be added without additional problem because the fraction field of a domain is uniformally interpretable).
- The theory $\{\varphi \vee \psi_d \,/\, d \in \mathbf{N}^*\}$ where φ is the formula which says that every non-zero element is invertible (case of fields) and ψ_d says that "for every monic non-constant separable polynomial $f \in A[T]$ of degree d and any non-zero ε in A there is $a \in A$ such that $f(a) \in \varepsilon A$".

■

Another immediate application of 2.6 and its following remark is the natural relation existing between PAC and PAC* domains.

Corollary 2.9 *Let A be a PAC (resp. a PAC*) domain. Let V be a variety defined over $K = \mathrm{Frac}(A)$ such that $V^\circ(A^s_{\mathbf{m}})$ is nonempty for every $\mathbf{m} \in \mathrm{Specmax}(A)$ (resp. and $V^\circ(K)$ is nonempty).*

Then for every $\mathbf{m} \in \mathrm{Specmax}(A)$, $V^\circ(A)$ is dense in $V^\circ(A^s_{\mathbf{m}})$ with respect to the product topology defined by $A^s_{\mathbf{m}}$. As a consequence, PAC domains are exactly those PAC domains whose fraction field is PAC.*

Proof: By 2.6 the density statement follows from proposition 5.1 of [2] (as usually we leave to the reader to check that the versions with and without $(*)$ of the arguments are essentially the same). It is immediate from the definitions that a PAC* domain with a PAC fraction field is itself a PAC domain. Conversely let A be a PAC domain, it is PAC*, let us prove that its fraction field K is PAC.

For any given variety V defined over K by a system f of polynomial equations in $K[X_1, \ldots, X_n]$ we have to prove that $V(K) \neq \emptyset$. Let f^* be the corresponding system of homogeneous equations in $K[X_1, \ldots, X_n, Y]$. If V is non empty then $V^\circ(K^s) \neq \emptyset$ hence

by reduction of denominators the variety V^* defined over K by f^* has a smooth point (a^s, b^s) in the integral closure of A in K^s such that $b^s \neq 0$. The above density argument for an arbitrary $\mathbf{m} \in \mathrm{Specmax}(A)$ then gives $(a, b) \in V^{*\circ}(A)$ closed enough to (a^s, b^s) so that $b \neq 0$ hence $x = a/b \in V^\circ(K)$ and K is PAC.
$\blacksquare$

3 PAC* domains with a radical relation

From now on, the reader is supposed to be familiar with the notion of ring with a radical relation or r-ring $(A, \leq)$ introduced in [15] and specially with the role of $\mathrm{Spec}(A, \leq)$ when dealing with extensions of r-rings (lemma 2.7 of [15]). A r-ring such that $(0) \in \mathrm{Spec}(A, \leq)$ is called a $\mathrm{r}_\circ$-domain. The r-ring associated to a ring A is the r-ring $(A, \leq_A)$ where $\leq_A$ is the radical relation defined on A by $\mathrm{Specmax}(A)$, that is:

$$a \leq_A b \quad \iff \quad \forall \mathbf{m} \in \mathrm{Specmax}(A),\ (a \in \mathbf{m} \Rightarrow b \in \mathbf{m}).$$

Its residual lattice (actually an upper semi-lattice) $L(A, \leq_A)$ is denoted by $L(A)$.

Remark 3.1 We will constantly use the trivial observation that $(0) \in \mathrm{Spec}(A, \leq_A)$ if and only if A has a zero jacobson radical: $J(A) = \bigcap \mathrm{Specmax}(A) = (0)$. See [15] for the definition of $\mathrm{Spec}(A, \leq)$

Recall that an embedding of r-rings $\varphi : (A, \leq) \to (A', \leq)$ is an embedding of rings wich preserves the radical relation, that is:

$$\forall a, b \in A, \qquad a \leq b \iff \varphi(a) \leq \varphi(b).$$

It is equivalent to require that $\bar{\varphi} : P(a) \mapsto P(\varphi(a))$ is a well-defined embedding of $L(A, \leq)$ into $L(A', \leq)$ (see [15] lemma 2.10.e). If φ is the inclusion map we write $(A, \leq) \subseteq (A', \leq)$.

We extend this definition in the following way: given any r-ring $(A, \leq)$ we denote by $\mathbf{L}(A, \leq)$ the **full residual lattice** of all subsets of $\mathrm{Spec}(A, \leq)$ of type $P(a_1) \cap \ldots \cap P(a_n)$ for any finite set $a_1, \ldots, a_n$ of elements of A. Then $\varphi : (A, \leq) \to (A', \leq)$ is a **full embedding** if and only if $\bar{\varphi} : \bigcap_{i \leq n} P(a_i) \mapsto \bigcap_{i \leq n} P(\varphi(a_i))$ is a well-defined embedding of $\mathbf{L}(A, \leq)$ into $\mathbf{L}(A' \leq)$. The full residual lattice of a ring A is the full residual lattice of its associated r-ring, and we write $\mathbf{L}(A)$ for $\mathbf{L}(A, \leq_A)$.

A (full) extension of a r-ring $(A, \leq)$ is simply a r-ring $(B, \leq)$ such that A is a subring of B and the inclusion is a (full) embedding of r-rings. Observe that if this happens then $L(A, \leq) \subseteq L(B, \leq)$ (resp. $\mathbf{L}(A, \leq) \subseteq \mathbf{L}(B, \leq)$). These notions make sense for rings: we call (full) r-embedings those embeddings of rings which are (full) embeddings between the associated r-rings, and similarly for (full) r-extensions.

Remark 3.2 All the lattices or semi-lattices we considere in this paper are distributive with an upper and a least bound, hence we require all our morphisms of lattices or semi-lattices to preserve these bounds, as well as sup and inf whenever they exist.

Lemma 3.3 *Let A be a domain such that for every non zero a and every b in A there is c in A such that $M(a) \setminus M(b) = M(c)$ then $L(A) = \mathbf{L}(A)$ is a relatively complemented lattice, that is:*

$$\forall \alpha, \beta \in \mathbf{L}(A), \quad (\alpha \neq P(0) \ \Rightarrow\ \exists \gamma \in \mathbf{L}(A),\ \alpha \setminus \beta = \gamma)$$

and one of the following occurs:

1. *If* $J(A) \neq (0)$ *then* $\mathrm{Spec}(A, \leq_A) = \mathrm{Specmax}(A)$ *and* $\mathbf{L}(A)$ *is a boolean algebra.*

2. *If* $J(A) = (0)$ *then* $\mathrm{Spec}(A, \leq_A) = \mathrm{Specmax}(A) \cup \{(0)\}$ *and, provided* A *is not a field,* $\mathbf{L}(A)$ *is not a boolean algebra.*

Remark 3.4 If A is a PAC* domain then the hypothesis of this lemma is satisfied by 2.1 hence $L(A)$ is relatively complemented.

Proof: We first check that $P(a) = M(a)$ for any non zero element of A, that is every non zero $\mathbf{p} \in \mathrm{Spec}(A, \leq_A)$ is a maximal ideal. Suppose this is not so, let $\mathbf{m}$ be a maximal ideal containing $\mathbf{p}$, choose $a \in \mathbf{p} \setminus \{0\}$ and $b \in \mathbf{m} \setminus \mathbf{p}$. By hypothesis there is $c \in A$ such that $M(a) \setminus M(b) = M(c)$. Then $a \leq_A bc$ but since neither b (by construction) nor c (because $c \notin \mathbf{m}$) belongs to $\mathbf{p}$ it follows that $bc \notin \mathbf{p}$ and this contradicts $\mathbf{p} \in \mathrm{Spec}(A, \leq_A)$.

Since $M(a) \cap M(b) = M(a) \setminus (M(a) \setminus M(b))$ the hypothesis implies that $L(A) = \mathbf{L}(A)$ and it is relatively complemented (the case of zero elements is left to the reader). The point 1 follows, using remark 3.1. In order to finish the proof of point 2 suppose that $L(A)$ is a boolean algebra and A is not a field. Take $a \in A \setminus (A^* \cup \{0\})$ and $b \in A$ such that $P(b)^c = P(a)$. Then $0 \leq_A ab$ and $b \neq 0$ because $P(a)$ is nonempty, hence $ab \neq 0$ and $ab \in J(A)$.

■

The following theorem compares PAC* with the local-global principle introduced by Prestel and Schmid in [15] and generalised in [16].

Theorem 3.5 *Let* A *be a PAC** *domain with fraction field* $K \neq A$. *Every* R^a $(R \in W_A)$ *is endowed with the radical relation induced by its maximal spectrum, and so are* R^h $(= R^s)$ *and* A.
Let $f = (f_1, \ldots, f_m) \in A[X]^m$ *with* $X = (X_1, \ldots, X_n)$ *be a system of equations of a variety* V *defined over* K, *and* g *a fixed element of* $A[X] \setminus I(V)$. *For every* $i \in \{1, \ldots, u\}$ *and every* $j \in \{1, \ldots, v\}$ *fix some* p_i, p'_i, q_j, q'_j *in* $A[X]$ *with the* p_i*'s not belonging to* $I(V)$. *Suppose that* $V^\circ(K) \neq \emptyset$ *and that for every maximal ideal* $\mathbf{m}$ *of* A *the variety* V *has a smooth point* $x_{\mathbf{m}}$ *with coordinates in* $A^a_{\mathbf{m}}$ *such that for every* $i \leq u$:

$$A^a_{\mathbf{m}} \models p_i(x_{\mathbf{m}}) \leq p'_i(x_{\mathbf{m}}) \qquad (i \leq u).$$

Suppose in addition that for each $j \leq v$ *there is a maximal ideal* $\mathbf{m}(j)$ *such that:*

$$A^a_{\mathbf{m}(j)} \not\models q_j(x_{\mathbf{m}(j)}) \leq q'_j(x_{\mathbf{m}(j)}).$$

Then V *has a smooth point* a *with coordinates in* A *such that* $g(a) \neq 0$, $A \models p_i(x) \leq p'_i(x)$ *for every* $i \leq u$ *and* $A \not\models q_j(x) \leq q'_j(x)$ *for every* $j \leq v$.

Proof: Use 2.4 to transfer each local solution $x_{\mathbf{m}}$ from $A^a_{\mathbf{m}}$ to $A^s_{\mathbf{m}} = A^h_{\mathbf{m}}$. Then apply theorem 5.3 of [2] (with LGPH replaced by LGPH*, using the hypothesis $V^\circ(K) \neq \emptyset$).

■

Remark 3.6 PAC* domains now clearly appear as those domains which satisfy condition (v) of [16] section 2. Moreover we have proved that the additionnal conditions on the radical relations appearing in (v) of [16] section 2 could be eliminated.

We annouced in [2] that the notions introduced by van den Dries, Macintyre, Prestel and Schmid were essentially particular cases of the LGPH. They are actually particular cases of the PAC and PAC* properties, the difference being that in this paper we considere rings with the local-global principle and no additional hypothesis (until now) on the fraction field or the residual lattice. On the other hand the henselisations of the localisations of a PAC* domain (which we call here its fibers) must have a separably closed fraction field by 2.6. Hence LGPH itself is much more general since no condition at all is required on the fibers.

In [2] we applied the LGPH to the case of a domain with a single p-adic fiber in order to axiomatize the integral closure of the ring of integer in any p-adically closed field. In this paper we consider the case of separably closed fibers. The case of possibly infinitely many fibers of finitely many different kind requires additionnal refinement, see [3] chapter V. However there is a trick to deal partially with this last case without using these refinements: we will do this in the last section of this paper.

4 Relative existential closure

In this section we provide alternative characterisations of PAC and PAC* domains in terms of relative existential closure, for rings as well as for r-domains. This includes a surprising algebraic criterion (condition 4 in the next proposition) and a result of model-completeness.

Proposition 4.1 *For any domain A such that $M(a)^c$ is quasi compact for every $a \in A$, the following assertions are equivalent.*

1. *A is PAC.*
2. *For any variety V defined over* $\mathrm{Frac}(A)$ *we have:*
$$V(A) \neq \emptyset \quad \iff \quad \forall R \in W_A,\ V(R^a) \neq \emptyset.$$
3. *$A \leq_\exists B$ for any domain B containing A such that: $L = \mathrm{Frac}(B)$ is a regular extension of K and for every $\mathbf{m} \in \mathrm{Specmax}(A)$ there is $\mathbf{n} \in \mathrm{Specmax}(B)$ such that $\mathbf{n} \cap A = \mathbf{m}$.*
4. *A variety V defined over K has no point in A if and only if there exists $f \in A[X_1, \ldots, X_n]$ and $\alpha \in A \setminus A^*$ such that $\alpha f - 1$ belongs to the ideal of V.*

Proof:

1⇒3. Remember that $A \leq_\exists B$ if and only if $L|K$ is regular, which is assumed here, and for any variety V having a smooth point $b \in B^n$ and any $f \in A[X_1, \ldots, X_n]$ such that $f(b) \neq 0$ there is a point $a \in V(A)$ such that $f(a) \neq 0$. Given such a triple (V, b, f) choose for each $\mathbf{m} \in \mathrm{Specmax}(A)$ a valuation ring R with fraction field L containing B and centered at a maximal ideal $\mathbf{n}$ of B such that $\mathbf{n} \cap A = \mathbf{m}$. Then $R \cap K = A_\mathbf{m}$ hence $A_\mathbf{m}^a$ embeds in R^a over A. Since $b \in V^\circ(R^a)$ and $A_\mathbf{m}^s = R^a \cap K^s$ by 2.5 there is $x_\mathbf{m} \in V^\circ(A_\mathbf{m}^s)$ such that $f(a_\mathbf{m}) \neq 0$. Then PAC gives a (smooth) point $a \in V^\circ(A)$ and one can choose A so that $f(a) \neq 0$ by 2.9 (density).

3⇒4. One implication is clear: if $\alpha f - 1 \in I(V)$ then $V(A) = \emptyset$ since otherwise $a \in V(A)$ would imply $\alpha f(a) = 1$ hence $\alpha \in A^*$ a contradiction. Conversely suppose that $V(A) = \emptyset$, let $x_V \in K[V]$ be a generic point of V over K and let $B = A[x_V]$. The extension $K(V)|K$ is regular but $A \not\leq_\exists B$ since $V(B) \neq \emptyset$, hence there exists $\mathbf{m} \in \mathrm{Specmax}(A)$ such that $\mathbf{m}B = B$ that is $1 \in \mathbf{m}B$ (otherwise (3) would give a point in $V(A)$ hence a contradiction). Let $\alpha \in \mathbf{m}$ and $f \in A[X]$ be such that $1 = \alpha f(x_V)$. Since x_V is generic it follows that $\alpha f - 1 \in I(V)$.

4⇒2. Suppose that for every $\mathbf{m} \in \mathrm{Specmax}(A)$ there exists $x_{\mathbf{m}} \in V(A^a_{\mathbf{m}}) \neq \emptyset$. Then for every $f \in A[X]$ and every $\alpha \in A \setminus A^*$ there is $\mathbf{m} \in \mathrm{Specmax}(A)$ such that $\alpha \in \mathbf{m}$ hence $\alpha f(x_{\mathbf{m}}) - 1 \neq 0$ because α is not invertible in $A^a_{\mathbf{m}}$. In particular $\alpha f - 1 \notin I(V)$ and this being true for every f and α it follows that $V(A) \neq \emptyset$.

2⇒1. If a variety V defined over K has a smooth point in R^a for every $R \in W_A$ then by (2) it has a point in A. This point may not be smooth. However, this Local-Global Principle holds for any variety de fined over K, and it is obviously sufficient to prove all the properties we deduced from our initial nonsingular principle. Indeed the smoothness of the global point is useless when proving the properties enumerated in section 2, except 2.9 whose conclusion becomes: $V(A)$ in dense in $V^\circ(R^s)$ for every $R \in W_A$. Now given any $R \in W_A$ we have $V^\circ(R^a) \neq \emptyset$ by assumption, hence $V^\circ(R^s) \neq \emptyset$ by 2.4. The implicit function theorem at a point $x \in V^\circ(R^s)$ gives a neighborhood of x in $V(R^s)$ which contains only smooth points. Then a point of $V(A)$ close enough to x must be a smooth point hence $V^\circ(A) \neq \emptyset$.

■

Remark 4.2 This elimination of the smoothness conditions in the Local-Global Principle is a curiosity of PAC domains but it cannot be generalized to the LGPH for example.

The following type of extensions of r-domains will be of special interest. A domain A is given as well as an extension L of its fraction field K. We equip A with the radical relation defined by its maximal spectrum. For each $\mathbf{m} \in \mathrm{Specmax}(A)$ a valuation ring $R(\mathbf{m})$ containing A is given with fraction field L such that $m(R(\mathbf{m})) \cap A = \mathbf{m}$. Let $B := \bigcap\{R(\mathbf{m}) \,/\, \mathbf{m} \in \mathrm{Specmax}(A)\}$ and $\mathcal{Q}' := \{m(R(\mathbf{m})) \cap B \,/\, \mathbf{m} \in \mathrm{Specmax}(A)\}$. Let $\mathcal{Q} := \mathcal{Q}' \cup \{(0)\}$ if $(0) \in \mathrm{Spec}(A, \leq)$ and $\mathcal{Q} := \mathcal{Q}'$ if not. Equip B with the radical relation induced by $\mathcal{Q}$. Then $(B, \leq)$ is an extension of $(A, \leq)$ (see [15] lemma 2.7) we call it the extension of $(A, \leq)$ **induced by** W.

Proposition 4.3 *With the above notations, if A is PAC (resp. PAC*) and the extension $L|K$ is regular (resp. $K \leq_\exists L$) then $(A, \leq) \leq_\exists (B, \leq)$.*

Proof: If A is PAC then it is PAC* with a PAC fraction field hence $L|K$ is regular implies that $K \leq_\exists L$: the first case boils down to the second one. We can then suppose that A is PAC* and $K \leq_\exists L$.

Let $\exists x \varphi(x)$ be an existential formula in the language of r-rings with parameters in A which holds in B. We can suppose φ of the following form:

$$\varphi \equiv \bigwedge_{i=1}^{m} f_i = 0 \wedge g \neq 0 \wedge \bigwedge_{i=1}^{u} p_i \leq p'_i \wedge \bigwedge_{j=1}^{v} q_j \not\leq q'_j$$

with $f_i, g, p_i, p'_i, q_j, q'_j$ in $A[X]$ for every i, j ($X = (X_1, \ldots, X_n)$). We have to prove that $(A, \leq) \models \exists x \varphi(x)$ hence to find a in A^n such that $\varphi(a)$ holds in $(A, \leq)$. Choose $b \in B^n$ such that $(B, \leq) \models \varphi(b)$. Since $L|K$ is a regular extension so is $K(b)|K$ hence b can be thought of as the generic point of a variety V defined over K. Since every polynomial h with coefficients in A such that $h(b) = 0$ vanishes identically on V it is sufficient to find a point $a \in V(A)$ such that $g(a) \neq 0$ and $(A, \leq) \models \varphi_r(a)$ where φ_r is the conjunction of radical conditions appearing in φ:

$$\varphi_r \equiv \bigwedge_{i=1}^{u} p_i \leq p'_i \wedge \bigwedge_{j=1}^{v} q_j \not\leq q'_j.$$

Step 1: Changing φ_r. Let $I := \{i \leq u \,/\, p_i(b) \neq 0\}$ and $I' := \{1,\ldots,u\} \setminus I$, let $J := \{j \leq v \,/\, \exists \mathbf{m} \in \mathrm{Specmax}(A),\ q_j(b) \in m(R(\mathbf{m}))$ and $q'_j(b) \notin m(R(\mathbf{m}))\}$ and $J' := \{1,\ldots,v\} \setminus J$. There are two cases:

- If (0) belongs to $\mathrm{Spec}(A,\leq)$ by construction it also belongs to $\mathrm{Spec}(B,\leq)$. For any $i \in I'$ we have $0 = p_i(b) \leq p'_i(b)$ which implies $p'_i(b) = 0$ hence any $a \in V(A)$ will satisfy $0 = p_i(a) = p'_i(a)$ and a fortiori $p_i(a) \leq p'_i(a)$. Moreover for any $j \in J'$ the definition of the radical relation of $(B,\leq)$ and the fact that $q_j(b) \not\leq q'_j(b)$ implies that $q_j(b) = 0$ and $q'_j(b) \neq 0$ hence any $a \in V(A)$ such that $q'_j(a) \neq 0$ will satisfy $q_j(a) = 0 \not\leq q'_j(a)$. Hence it is sufficient to find $a \in V(A)$ such that $G(a) \neq 0$, with $G = g \times \prod_{j\in J'} q'_j$, and:

  $$\psi(a) \equiv \bigwedge_{i\in I} p_i(a) \leq p'_i(a) \wedge \bigwedge_{j\in J} q_j(a) \not\leq q'_j(a)$$

  holds in $(A,\leq)$.

- If (0) does not belong to $\mathrm{Spec}(A,\leq)$ then $J' = \emptyset$ and there is a non zero $\alpha \in A$ such that $0 \leq \alpha$, hence it is sufficient to find $a \in V(A)$ such that $g(a) \neq 0$ and

  $$\psi(a) \equiv \bigwedge_{i\in I} p_i(a) \leq p'_i(a) \wedge \bigwedge_{i\in I'} \alpha \leq p'_i(a) \wedge \bigwedge_{j\in J} q_j(a) \not\leq q'_j(a)$$

  holds in $(A,\leq)$.

As in both cases $\psi(b)$ holds in $(B,\leq)$ it follows that, replacing if necessary g by G and φ_r by ψ we can suppose that $I' = J' = \emptyset$.

Step 2: Transfer. $V^\circ(K) \neq \emptyset$ because $K \leq_\exists L$ and $b \in V^\circ(L)$. Let $h_1,\ldots,h_r$ an enumeration of all the p_i's, q_i's, p'_i's and q'_i's. Fix $\mathbf{m} \in \mathrm{Specmax}(A)$ and let $R := R(\mathbf{m})$. Since A is a Prüfer domain we have $A_{\mathbf{m}} = R \cap K$ hence $A^a_{\mathbf{m}}$ embeds naturally into R^a. Since $b \in V^\circ(R^a)$ and $A^a_{\mathbf{m}} \leq_\exists R^a$ there is $x_{\mathbf{m}} \in V^\circ(A^a_{\mathbf{m}})$ such that for any $i \leq r$:

$$h_i(x_{\mathbf{m}}) \in m(A^a_{\mathbf{m}}) \iff h_i(b) \in m(R(\mathbf{m})).$$

In particular, since $p_i(b) \leq p'_i(b)$ we have $A^a_{\mathbf{m}} \models p_i(x_{\mathbf{m}}) \leq q_i(x_{\mathbf{m}})$ for every $i \leq u$. By step 1, g and the p_i's do not belong to the ideal of V because they do not vanish at b. Moreover there is for each $j \leq v$ some $\mathbf{m} \in \mathrm{Specmax}(A)$ such that $A^a_{\mathbf{m}} \not\models q_j(x_{\mathbf{m}}) \leq q'_i(x_{\mathbf{m}})$. Eventually 3.5 gives a point $a \in V^\circ(A)$ such that $g(a) \neq 0$, $p_i(a) \leq p'_i(a)$ for every $i \leq u$ and $q_j(a) \not\leq q'_j(a)$ for every $j \leq v$, that is $\varphi(a)$ holds in $(A,\leq)$.
■

Theorem 4.4 *Let A be a PAC (resp. PAC*) domain, and A' a full r-extension of A such that:*

1. *If $J(A) = (0)$ then $J(A') = (0)$.*
2. *$\mathbf{L}(A) \leq_\exists \mathbf{L}(A')$.*
3. *$\mathrm{Frac}(A')$ is a regular extension of $\mathrm{Frac}(A)$ (resp. $\mathrm{Frac}(A) \leq_\exists \mathrm{Frac}(A')$).*

Then $(A,\leq_A) \leq_\exists (A',\leq_{A'})$.

Remark 4.5 Observe that if $(A, \leq_A) \leq_\exists (A', \leq_{A'})$ then conditions 1 and 3 above are fullfilled (for 1 observe that $J(A) = (0)$ if and only if $\exists a(0 \leq a \wedge 0 \neq a)$ holds in $(A, \leq_A)$) and moreover $L(A) \leq_\exists L(A')$. Hence if $\mathbf{L}(A') = L(A')$ then the above three sufficient conditions are also necessary. This happens for example if A' is Bezout, a fortiori if it is PAC*.

The PAC case being a particular case, we only have to prove the theorem when A is PAC* and $K \leq_\exists L$. The demonstration requires the following lemma.

Lemma 4.6 *Let $(A'', \leq)$ and $(A', \leq)$ be two full extensions of a r-domain $(A, \leq)$. Suppose that an embedding σ is given of $\mathbf{L}(A', \leq)$ into $\mathbf{L}(A'', \leq)$ whose restriction to $\mathbf{L}(A, \leq)$ is identity. Then:*

1. *$\forall \mathbf{p}'' \in \mathrm{Spec}(A'', \leq)$, $\hat{\sigma}(\mathbf{p}'') := \{a' \in A' \,/\, \mathbf{p}'' \in \sigma(P(a'))\}$ is a prime ideal of A' which belongs to $\mathrm{Spec}(A', \leq)$. Moreover $\mathbf{p}'' \cap A = \hat{\sigma}(\mathbf{p}'') \cap A$ and any element of $\mathrm{Spec}(A', \leq)$ which is maximal in $\mathrm{Spec}(A', \leq)$ with respect to the inclusion lies in the image of $\hat{\sigma}$.*

2. *If the radical relations of A' and A'' are defined by their respective maximal spectra and if their fraction fields K' and K'' are regular extensions of the fraction field K of A, then for each maximal ideal $\mathbf{m}''$ of A'' there is a valuation ring $R(\mathbf{m}'')$ with fraction field $\mathrm{Frac}(K' \otimes_K K'')$ containing both A' and A'' such that:*

$$m(R(\mathbf{m}'')) \cap A'' = \mathbf{m}'' \quad \textit{and} \quad m(R(\mathbf{m}'')) \cap A' = \hat{\sigma}(\mathbf{m}'').$$

Proof:
1. Let $\mathbf{p}'' \in \mathrm{Spec}(A'', \leq)$ and $\mathbf{p}' = \hat{\sigma}(\mathbf{p}'')$. We have $\mathbf{p}'' \in P(0) = \sigma(P(0))$ hence $0 \in \mathbf{p}'$, and $\mathbf{p}'' \notin P(1) = \sigma(P(1))$ hence $1 \notin \mathbf{p}'$. For any $a', b' \in A'$, if $a' \in \mathbf{p}'$ and $a' \leq b'$ then $\mathbf{p}' \in \sigma(P(a')) \subseteq \sigma(P(b'))$ hence $b' \in \mathbf{p}'$. A fortiori for any $a' \in \mathbf{p}'$ and any $b' \in A'$ since $a' \leq a'b'$ we have $a'b' \in \mathbf{p}'$. Eventually let $a', b' \in \mathbf{p}'$ since $P(a') \cap P(b') \subseteq P(a' + b')$ we have:

$$\mathbf{p}'' \in \sigma(P(a')) \cap \sigma(P(b')) = \sigma(P(a') \cap P(b')) \subseteq \sigma(P(a' + b'))$$

hence $a' + b' \in \mathbf{p}'$ which proves that $\mathbf{p}' \in \mathrm{Spec}(A', \leq)$. Moreover the restriction of σ to $\mathbf{L}(A, \leq)$ being identity it is clear that $\mathbf{p}'' \cap A = \mathbf{p}' \cap A$.

Let now be given any maximal element $\mathbf{p}'$ of $\mathrm{Spec}(A', \leq)$ and let $I'' := \{a'' \in A'' \,/\, \exists a' \in \mathbf{p}',\ \sigma(P(a')) \subseteq P(a'')\}$. It is an easy exercice of the same type as above to check that I'' is a proper ideal of A'' and that for any $a'' \in I''$ and any $b'' \in A''$ such that $a'' \leq b''$ we have $b'' \in I''$. It is proved in [15] lemma 2.4, that such an ideal is the intersection of the family of $\mathbf{p}'' \in \mathrm{Spec}(A'', \leq)$ which contain I''. For any such $\mathbf{p}''$ we have obviously $\mathbf{p}' \subseteq \hat{\sigma}(\mathbf{p}'')$ hence $\mathbf{p}' = \hat{\sigma}(\mathbf{p}'')$ by maximality of $\mathbf{p}'$.
2. Observe first that the extensions of K being regular $K' \otimes_K K''$ is a domain, hence it is meaningfull to consider $K^\circ := \mathrm{Frac}(K' \otimes_K K'')$ as a common extension of K' and K'' over K. Fix any $\mathbf{m}'' \in \mathrm{Specmax}(A'')$ and let $\mathbf{p}' := \hat{\sigma}(\mathbf{m}'')$. Choose any valuation ring $R'' \subseteq K''$ dominating $A''_{\mathbf{m}''}$ and $R' \subseteq K'$ dominating $A'_{\mathbf{p}'}$, and denote by φ'' and φ' be the associated places. Because $\mathbf{m}'' \cap A = \mathbf{p}' \cap A$ we can manage to ensure that $R'' \cap K = R' \cap K$. Then $\varphi' \otimes \varphi''$ is a partial morphism of K° extending both φ' and φ''. Any maximal extension of this partial morphism defines a place on K° whose associated valuation ring $R(\mathbf{m}'')$ has the required properties.
∎

Remark 4.7 The argument used in the proof of (2) above is simpler than the one used in [3], lemme d'amalgamation IV.3.1. It was given to me by A. Razon during private conversions in Oberwolfach, fall 1998.

We can now prove 4.4.

Proof: Let $(A'', \leq)$ be a $|A'|^+$-saturated elementary extension of $(A, \leq)$ (remember that the radical relations are those defined by the maximal spectra: it was assumed for A and A', and it is true also for A'' because it is a first order property hence preserved by elementary extensions). Since A'' is PAC* it is Bezout hence $\mathbf{L}(A'') = L(A'')$ is interpretable in $(A'', \leq)$. Since $\mathbf{L}(A) \leq_\exists \mathbf{L}(A')$ this provides an embedding σ of $\mathbf{L}(A')$ into $\mathbf{L}(A'')$ whose restriction to $\mathbf{L}(A)$ is identity (if $L(A)$ is finite the three lattices are the same and we take for σ the identity, if not then $L(A'')$ inherits the saturation property of $(A'', \leq)$ and this provides plenty of such embeddings σ). For each maximal ideal $\mathbf{m}''$ of A'' let $R(\mathbf{m}'')$ be a valuation ring with fraction field $\mathrm{Frac}(K' \otimes_K K'')$ such that $m(R(\mathbf{m}'')) \cap A'' = \mathbf{m}''$ and $m(R(\mathbf{m}'')) \cap A' = \hat{\sigma}(\mathbf{m}'')$ (see 4.6). Eventually let $(A^\circ, \leq)$ be the extension of $(A'', \leq)$ induced by $\{R(\mathbf{m}'') \,/\, \mathbf{m}'' \in \mathrm{Specmax}(A'')\}$.

We claim that $(A^\circ, \leq)$ is an extension of $(A', \leq)$. Indeed, for any $\mathbf{m}'' \in \mathrm{Specmax}(A'')$ we have $A' \subseteq R(\mathbf{m}'')$ hence $A' \subseteq A^\circ$. Moreover the radical relation of $(A^\circ, \leq)$ is defined by a set $\mathcal{P}$ of prime ideals such that:

- For any $\mathbf{p} \in \mathcal{P}$, either there is $\mathbf{m}'' \in \mathrm{Specmax}(A'')$ such that $\mathbf{p} = A^\circ \cap m(R(\mathbf{m}''))$ and then $\mathbf{p} \cap A' = \hat{\sigma}(\mathbf{m}'')$ belongs to $\mathrm{Spec}(A', \leq)$, or $\mathbf{p} = (0)$ and $(0) \in \mathrm{Spec}(A'', \leq)$ but in that case $(0) \in \mathrm{Spec}(A, \leq)$ because this an elementary property, hence $(0) \in \mathrm{Spec}(A', \leq)$ by hypothesis and once again $\mathbf{p} \cap A'$ belongs $\mathrm{Spec}(A', \leq)$.
- For any $\mathbf{m}' \in \mathrm{Specmax}(A)$ by 4.6 there is a $\mathbf{m}'' \in \mathrm{Specmax}(A'')$ such that $\mathbf{m}' = \hat{\sigma}(\mathbf{m}'')$ hence there is $\mathbf{p} \in \mathcal{P}$, namely $\mathbf{p} = m(R(\mathbf{m}'')) \cap A^\circ$ such that $\mathbf{p} \cap A' = \mathbf{m}'$.

This proves the claim by [15] lemma 2.7. By 4.3 $(A'', \leq) \leq_\exists (A^\circ, \leq)$ hence $(A, \leq) \leq_\exists (A^\circ, \leq)$ and as $(A', \leq) \subseteq (A^\circ, \leq)$ we conclude that $(A, \leq) \leq_\exists (A', \leq)$.

■

Corollary 4.8 *Let T_l a model-complete theory of lattices, T_f a model-complete theory of fields. Then the theory of $r_\circ$-domains (resp. of r-domains which are not $r_\circ$-domains) $(A, \leq)$ such that A is PAC*, the radical relation of $(A, \leq)$ is defined by the maximal spectrum of A and:*

1. $\mathrm{Frac}(A) \models T_f$;
2. $L(A) \models T_l$.

is model-complete.

Proof: This is an elementary class because the above conditions are elementary (only the axiomatisation of PAC* needs some efforts, and it has been done in 2.8). It is model-complete by 4.4.

■

5 The embedding theorem

Ershov introduced in [7] the following notion: a field K is **regularly closed relatively to a family** $\mathcal{F}$ of algebraic extensions if and only if it is existentially closed in any regular extension K' such that $F \leq_{\exists} \mathrm{Frac}(F \otimes_K K')$ for every $F \in \mathcal{F}$. This notion contains every usual local-global principle for field, for example pseudo real closed fields (resp. pseudo p-adically closed fields) are exactly those fields wich are regularly closed with respect to a family of real closed (resp. p-adically closed) extensions. A field is pseudo algebraically closed if and only if it is regularly closed relatively to $\emptyset$.

We will say that a field K is **RC-local** if K is regularly closed relatively to a family $\mathcal{F}$ of algebraic extensions such that: there is a finite set of fields $F_1, \dots, F_n$ taken among $\mathbf{R}$ and the finite extensions of the fields $\mathbf{Q}_p$ such that each F in $\mathcal{F}$ is elementary equivalent to one of the F_i's. This contains of course PAC fields (with $n = 0$), PRC fields (with $n = 1$, $F_1 = \mathbf{R}$) and PpC fields (with $n = 1$, $F_1 = \mathbf{Q}_p$). Moreover the class of RC-local fields is preserved by taking finite extensions.

Remark 5.1 In the above definition we required the complete theories of the elements of $\mathcal{F}$ to range among a fixed *finite* set (namely the theories of the F_i's) only to ensure that the class of RC-local fields is preserved by elementary equivalence and elementary extensions.

We considere local fields whose theories have been completely determined, this is the reason why local fields of positive characteristics are avoided. On the other hand, observe that the above definition does not exclude completely the positive characteristic case: a field of positive characteristic is RC-local if and only if it is PAC.

For any field K let $G(K)$ denote the absolute Galois group of K, that is the group of K-automorphisms of the separable closure K^s of K.

The aim of this section is to prove the following theorem, wich is a key result of this paper:

Theorem 5.2 (embedding theorem) *Let $(A', \leq)$, $(A'', \leq)$ be two full extensions of a r-domain $(A, \leq)$. Denote by K, K', K'' their respective fraction fields and $\mathbf{L}, \mathbf{L}', \mathbf{L}''$ their full residual lattices. Suppose that:*

- *K' and K'' are regular extensions of K.*
- *If K has positive characteristic p then $[K' : K'^p] \leq [K'' : K''^p]$.*
- *The radical relations of $(A', \leq)$ and $(A'', \leq)$ are defined by their respective maximal spectra, and if $J(A') = (0)$ then $J(A'') = (0)$.*
- *An embedding $\sigma : \mathbf{L}' \to \mathbf{L}''$ is given whose restriction to $\mathbf{L}$ is identity.*
- *An epimorphism $\pi : G(K'') \to G(K')$ is given which commutes with the restriction epimormorpisms from $G(K')$ and $G(K'')$ to $G(K)$.*
- *A'' is PAC*, $|A'|^+$-saturated and K'' is RC-local.*

Then there is an embedding $\Theta : K'^s \to K''^s$ over K^s such that:

1. *K'' is a regular extension of $\Theta(K')$.*

2. *$\tau \circ \Theta = \Theta \circ \pi(\tau)$ for every τ in $G(K'')$.*

3. *The restriction φ of Θ to A' is a full embededding $(A', \leq)$ into $(A'', \leq)$ over $(A, \leq)$ such that the induced embedding $\bar{\varphi} : \mathbf{L}' \to \mathbf{L}''$ is exactly the given embedding σ.*

The proof of 5.2 starts now and will continue until the end of this section. It requires the construction of several auxiliary objects.

Remark 5.3 If A' is finite, since $K'|K$ is regular we would have $K' = K$ hence $A' = A$ and the theorem is proved. In what follows we can then assume A' to be infinite, and the saturation hypothesis on A'' then implies that A'' also is infinite.

Construction 1.
Since $K' \otimes_K K''$ is a domain, we can embed all our fields in an algebraic closure Ω of $\mathrm{Frac}(K' \otimes_K K'')$. Observe that if F, F', F'' are any algebraic extensions of K, K', K'' respectively, such that the extensions $F'|F$ and $F''|F$ are regular then the compositum $F'F''$ inside Ω is F-isomorphic to $\mathrm{Frac}(F' \otimes_F F'')$. The Galois group of $K'^s K''^s$ over $K'K''$ is $G(K') \times_{G(K)} G(K'')$. Let $G := \{\langle \pi(\tau), \tau \rangle \ / \ \tau \in G(K'')\}$ and $K^\circ := (K'^s K''^s)^G$.

For any $\mathbf{m}'' \in \mathrm{Specmax}(A'')$ the valuation ring $R(\mathbf{m}'')$ given by 4.6 applied to σ extends to a valuation ring $R^\circ(\mathbf{m}'')$ with fraction field K°. As in the proof of 4.4 this provides a common extension $(A^\circ, \leq)$ of $(A', \leq)$ and $(A'', \leq)$. By construction A° contains the integral closure of $A'A''$ inside K° hence $\mathrm{Frac}(A^\circ) = K^\circ$.

Lemma 5.4 *$K'' \leq_\exists K^\circ$ and $(A'', \leq) \leq_\exists (A^\circ, \leq)$.*

Proof: K'' is relatively separably algebraically closed in K° because for every τ in $G(K'')$ the field $K^\circ \cap K''^s$ is fixed by $\langle \pi(\tau), \tau \rangle_{|K''^s} = \tau$. Moreover $K'^s K''^s$ is a separable extension of $K'K''$ and $K'K'' \simeq \mathrm{Frac}(K' \otimes_K K'')$ is a regular extension of K'', hence $K'^s K''^s$ and a fortiori K° is a separable extension of K''. We conclude that $K^\circ|K''$ is regular.

Let $\mathcal{F}$ be the given family of real closed and p-adically closed fields relatively to which K'' is regularly closed. If $\mathcal{F}$ is empty then K'' is PAC hence $K^\circ|K''$ is regular implies that $K'' \leq_\exists K^\circ$. This case being solved we suppose now that $\mathcal{F}$ is non-empty and in particular that the characteristic is zero, and we considere algebraic closures in place of separable closures.

For every $F'' \in \mathcal{F}$ let $F = F'' \cap K^a$. Since $F''^a = F''\mathbf{Q}^a = F''K^a$ the restriction epimorphism from $G(F'')$ onto $G(F)$ is an isomorpism. The same arguments holds for the fixed field F' of $\pi(G(F''))$ inside K'^a. Observe that $F'' \cap K^a = F' \cap K^a = F$.

If F is real closed then the absolute galois group of F, F' and F'' is $\mathbf{Z}/2\mathbf{Z}$ hence all of them are real closed. It follows that $F \preceq F'$ and $F \preceq F''$.

If F is elementary equivalent to a finite extension of some $\mathbf{Q}_p$ then once again $F \preceq F'$ and $F \preceq F''$ because F, F' and F'' have the same absolute Galois group and $F'' \cap \mathbf{Q}^a = F' \cap \mathbf{Q}^a = F \cap \mathbf{Q}^a$ (see [12] theorem 4.1 and its following comment).

A general model theoretical argument allows us to amalgamate any two elementary extensions R' and R'' of a ring R in a common elementary extension $\bar{R}$ in such a way that any tuple of elements of R' which are linearly independant over R remain so in $\bar{R}$ over R'' (see for example [13] proof of lemma 4). Applying this to F, F', F'' we can embed $\mathrm{Frac}(F' \otimes_F F'') \simeq F'F''$ over F into an elementary extension of F'' hence $F'' \leq_\exists F'F''$. Moreover for any $\tau \in Gal(K'^a K''^a | F'F'')$ we have $(\pi(\tau_{|K''^a}))_{|K^a} = (\tau_{|K'^a})_{|K^a}$ and since the restriction epimorphism from $G(F')$ to $G(F)$ is an isomorphism $\pi(\tau_{|K''^a}) = \tau_{|K'^a}$. It follows that $\tau = \langle \pi(\tau_{|K''^a}), \tau_{|K''^a} \rangle$ belongs to G, in other words $Gal(K'^a K''^a | F'F'') \subseteq G$ hence $K^\circ \subseteq F'F''$ and $F'' \leq_\exists F''K^\circ$. Applying this to any $F'' \in \mathcal{F}$ and using the hypothesis that K'' is regularly closed relatively to $\mathcal{F}$ we conclude that $K'' \leq_\exists K^\circ$.

$(A'', \leq) \leq_\exists (A^\circ, \leq)$ then follows from 4.3.

■

Since we have embedded $(A', \leq)$ into $(A^\circ, \leq)$ over A and $(A'', \leq) \leq_\exists (A^\circ, \leq)$, the $|A'|^+$-saturation property of $(A'', \leq)$ provides plenty of embedings of $(A', \leq)$ into $(A'', \leq)$ over A. In order to construct a particular embedding having the required additional properties we have to introduce some refinements.

Construction 2.

- For any $a' \in A'$ choose $a''_{a'} \in A''$ such that $P(a''_{a'}) = \sigma(P(a'))$ and:

  1. If $a' \in A$ then $a''_{a'} = a'$.

  2. $a' = 0$ if and only if $a''_{a'} = 0$. This is possible since $\sigma(P(0)) = P(0)$, and if $a' \neq 0$ and $P(a') = P(0)$ then $(0) \notin \mathrm{Spec}(A', \leq)$ hence by hypothesis $(0) \notin \mathrm{Spec}(A', \leq)$ and we can find a non-zero α in A such that $0 \leq \alpha$, hence we can set $a''_{a'} = \alpha$.

Let $\mathcal{A} = \{a''_{a'} \,/\, a' \in A'\}$ and choose an elementary substructure $K_{0,0}$ of K'' containing $\mathcal{A}$ such that $K(\mathcal{A}) \subseteq K_{0,0}$ and $|K_{0,0}| \leq |K'|$.

- Let $(e''_\alpha)_{\alpha<\beta}$ be a basis of K''^s as a K^s-vector space. Since K° is a regular extension of K'' it remains linearly independant over K°. Moreover $Gal(K'^s K''^s | K^\circ) \simeq G = Gal(K''^s | K'')$ hence $K^\circ K''^s = K'^s K''^s$ that is $(e''_\alpha)_{\alpha<\beta}$ generates $K'^s K''^s$ as a K°-vector space. In particular any $x' \in K'^s$ writes uniquely:

$$x' = \sum_{i=1}^{n(x')} \lambda_i(x') e''_{\alpha_i(x')}, \quad \text{with } \lambda_i(x') \in K^\circ \setminus \{0\}.$$

Write each $\lambda_i(x') = c_i(x')/d_i(x')$ with $c_i(x'), d_i(x') \in A^\circ \setminus \{0\}$ and let $\mathcal{C} = \{c_i(x'), d_i(x') \,/\, x' \in K'^a,\ i < n(x')\}$. Eventually let $K_{1,0} = K' K_{0,0}(\mathcal{C})$. Observe that $|K_{1,0}| = |K'|$ since $|\mathcal{C} \leq |K'|$ and $|K_{0,0}| \leq |K'|$.

- We cannot be sure that $K_{1,0}$ and K'' are linearly disjoint over $K_{0,0}$. The following construction will allow us to reduce to this case. Let $n \in \mathbf{N}$ and suppose that we have already build the following tower of extensions (arrows are inclusion maps, the diagram is commutative):

$$\begin{array}{ccccccccc}
 & & K_{0,0} & \preceq & K_{0,1} & \preceq \cdots \preceq & K_{0,n} & \preceq & K'' \\
 & \nearrow & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
K & \to & K_{1,0} & \to & K_{1,1} & \to \cdots \to & K_{1,n} & \to & K^\circ
\end{array}$$

such that $K_{0,n}$ and $K_{1,n}$ have cardinality less than or equal to $|A'|$, and for every $k < n$ every tuple of elements of $K_{1,k}$ which is K''-linearly dependant is $K_{0,k}$-linearly dependant. Then for any tuple x of elements of $K_{1,n}$ which is K''-linearly dependant let ν_x be the set of of coefficients of one particular dependance relation, and let $\mathcal{N}$ be set of every such ν_x as x varies. Then $|\mathcal{N}| \leq |K_{1,n}| \leq |A'|$ hence their exists $K_{0,n+1} \preceq K''$ such that $K_{0,n}(\mathcal{N}) \subseteq K_{0,n+1}$ and $|K_{0,n}| \leq |A'|$. Let then $K_{1,n+1} = K_{0,n+1} K_{1,n}$, and iterate the construction.

• Eventually let $K_0 = \bigcup_{n\in\mathbf{N}} K_{0,n}$ and $K_1 = \bigcup_{n\in\mathbf{N}} K_{1,n}$. Obviously $K_0 \preceq K''$, $K_0 \subseteq K_1$, $|K_1| = |A'|$ and K_1 is linearly disjoint from K'' over K_0.

The proof of the embedding theorem eventually boyls down to the following lemma. Let $A_0 = A'' \cap K_0 = A^\circ \cap K_0$ and $A_1 = A^\circ \cap K_1$. Equip them with the radical relations induced by $(A^\circ, \leq)$.

Lemma 5.5 *There exists an embedding* Θ *of* $(A_1, \leq)$ *into* $(A'', \leq)$ *over* A_0 *such that* K'' *is a separable extension of* $\mathrm{Frac}(\Theta(A_1))$.

Proof: Observe that $(A_1, \leq) \subseteq (A^\circ, \leq)$ and $(A'', \leq) \leq_\exists (A^\circ, \leq)$. Since $|A_1| = |A'|$ the existence of A_0-embeddings $(A_1, \leq)$ into $(A'', \leq)$ then follows from the saturation property of $(A'', \leq)$. Hence there is nothing to prove in zero characteristic.

Let us now considere the case of positive characteristic p. Let $(x'_i)_{i\in I'}$ be a p-basis of K', that is a basis of K' as a K'^p-vector space. We have to find an embedding of $(A_1, \leq)$ into $(A'', \leq)$ over A_0 whose extension to K_1 maps $(x'_i)_{i\in I'}$ onto a p-free family of K''. An easy compactness argument proves that it is sufficient to find for any finite subset $\{x'_1, \ldots, x'_n\}$ of $(x'_i)_{i\in I'}$ and for any $(A'_1, \leq) \subseteq (A_1, \leq)$ finitely generated over $(A_0, \leq)$, an A_0-embedding Θ' of $(A'_1, \leq)$ into $(A'', \leq)$ whose extension to $K'_1 = \mathrm{Frac}(A'_1)$ maps $\{x'_1, \ldots, x'_n\}$ onto a p-free family.

Tamagawa's construction.
In order to construct Θ' we follow the method of the proof of Tamagawa's lemma given in [10] proposition 10.11. We follow this reference as closely as possible (this is [3] lemme IV.4.1).

Observe first that $K^\circ|K'$ being a separable extension, $\{x'_1, \ldots, x'_n\}$ remains p-free over $(K^\circ)^p$. Let $K''_1 = K''K'_1$ and $L = K''K'^p_1(x'_1, \ldots, x'_n)$. Relabel the x'_i's so that $(x'_1, \ldots, x'_s)$ is a p-basis of L over $K''K'^p_1$ (for some $s \leq n$). Let $(x_i)_{i\in I}$ be a p-bases of K'':

$$L = K''K'^p_1[x'_1, \ldots, x'_s] = K''^p_1[(x_i)_{i\in I}, \, x'_1, \ldots, x'_s].$$

It follows that there exists $x_1, \ldots, x_r$ among the x_i's such that $x'_{s+1}, \ldots, x'_n \in K''^p_1[x_1, \ldots, x_r, x'_1, \ldots, x'_s]$. By hypothesis $[K' : K'^p] \leq [K'' : K''^p]$ hence $|I| \geq |I'| \geq n$ (this is clear if $[K' : K'^p]$ is finite, otherwise it follows from the saturation property of K'' inherited from A''). In particular we can take $r \geq n$. Relabeling the x_i's if necessary the exchange lemma then gives:

$$x_{r-(n-s)+1}, \ldots, x_r \in K''^p_1[x_1, \ldots, x_{r-(n-s)}, x'_1, \ldots, x'_n]. \tag{1}$$

Let $\eta_i = (x_{r-n+i}x'_i)^{1/p}$ for $1 \leq i \leq s$ and let $K''_2 = K''_1[\eta_1, \ldots, \eta_s]$. The field extension $K''_2|K''$ is exactly the extension $F'|K$ of [10] proposition 10.11, hence the arguments given there apply to our situation: $K''_2|K''$ is a regular extension.

Remember that $(A^\circ, \leq)$ is the extension of $(A'', \leq)$ induced by a family of valuation rings $R(\mathbf{m}'')$ with fraction field K° indexed on $\mathrm{Specmax}(A'')$. Extend each $R(\mathbf{m}'')$ to a valuation ring with fraction field $K^\circ[\eta_1, \ldots, \eta_s]$ and let $(B, \leq)$ be the induced extension of $(A^\circ, \leq)$. Equip $A''_2 = B \cap K''_2$ with the radical relation inherited from $(B, \leq)$. Since K'' is RC-local with positive characteristic it is PAC, hence $K'' \leq_\exists K''_2$ and 4.3 implies: $(A'', \leq) \leq_\exists (A''_2, \leq)$.

Denote by $\mathcal{I}$ the set of all applications from $\{1, \ldots, r+s\}$ to $\{0, \ldots, p-1\}$. For with i with $r-n+s+1 \leq i \leq r$, the relation (1) gives coefficients $(v_{i,j})_{j\in\mathcal{I}}$ in K''_1 such that:

$$x_i = \sum_{j\in\mathcal{I}} v^p_{i,j} x_1^{j(1)} \ldots x_{r-n+s}^{j(r-n+s)} x_1'^{j(r-n+s+1)} \ldots x_n'^{j(r+s)}.$$

The field K'_2 generated over K'_1 (and inside K''_2) by those coefficients $v_{i,j}$, plus $x_1,\dots,x_r$ and $\eta_1,\dots,\eta_s$ has the same cardinality as K'_1. Equip $A'_2 = A''_2 \cap K''_2$ with the radical relation inherited from $(A''_2,\leq)$. Eventually fix any finitely generated extension A'_0 of A_0 inside $A''\cap A'_2$ whose fraction field contains $x_1,\dots,x_r$. The saturation property of $(A'',\leq)$ provides an A'_0-embedding Θ' of $(A'_2,\leq)$ into $(A'',\leq)$. Its extension to K'_2 is still denoted by Θ', for every $r-n+s+1\leq i\leq r$ it satisfies:

$$x_i = \Theta'(x_i) = \sum_{j\in\mathcal{I}} \Theta'(v_{i,j})^p x_1^{j(1)}\dots x_{r-n+s}^{j(r-n+s)}\Theta'(x'_1)^{j(r-n+s+1)}\dots\Theta'(x'_n)^{j(r+s)}. \qquad (2)$$

For any $1\leq k\leq s$ we have:

$$x_{r-n+k} = \Theta'(x_{r-n+k}) = \Theta'(\eta_k^p x_k'^{-1}) = \Theta'(\eta_k)^p\Theta'(x'_k)^{-1}$$

hence by (2):

$$x_1,\dots,x_n \in K''^p(x_1,\dots,x_{r-n},\Theta'(x'_1),\dots,\Theta'(x'_n)).$$

But $(x_1,\dots,x_n)$ is p-free over K''^p hence so is $(x_1,\dots,x_{r-n},\Theta'(x'_1),\dots,\Theta'(x'_n))$ in particular $(\Theta'(x'_1),\dots,\Theta'(x'_n))$ is p-free over K''^p as we wanted.

■

The embedding theorem is now at hand: let Θ be the A_0-embedding of $(A_1,\leq)$ into $(A'',\leq)$ over A_0 given by 5.5. Then Θ extends to an embedding of K_1 into K'' over K. Observe that K° is a regular extension of K'' and $K''K_1\subseteq K^\circ$ hence $K''K_1$ is linearly disjoint from K''^a over K''. By construction K'' is linearly disjoint from K_1 over K_0, hence K_1 is linearly disjoint from K''^a over K_0. This allows to extend this embedding to a homomorphism of $K_0''^a$-algebra $K''^a[K_1]\to K''^a$ which we still denote by Θ. By construction $K'^s\subseteq K''^s(\mathcal{C})\subseteq K''^s[K_1]$ hence we can define the restriction of the homomorphism Θ to K'^s, which is an embedding since K'^s is a field.

$K''|\Theta(K_1)$ is separable, so is $K_1|K'$ because $K_1\subseteq K^\circ$ and $K^\circ|K'$ is separable hence $K''|\Theta(K')$ is separable.

For every $x' = \sum_{i=1}^{n(x')}\lambda_i(x')e''_{\alpha_i(x')}\in K'^s$ and every $\tau\in G(K'')$ we have:

$$\begin{aligned}
\big(\pi(\tau)\big)(x') &= \big\langle\pi(\tau),\tau\big\rangle(x')\\
&= \sum_{i=1}^{n(x')}\big\langle\pi(\tau),\tau\big\rangle(\lambda_i(x')).\big\langle\pi(\tau),\tau\big\rangle(e''_{\alpha_i(x')})\\
&= \sum_{i=1}^{n(x')}\lambda_i(x')\tau(e''_{\alpha_i(x')})
\end{aligned}$$

It follows that:

$$\Theta\Big(\pi(\tau)(x')\Big) = \sum_{i=1}^{n(x')}\Theta(\lambda_i(x'))\tau(e''_{\alpha_i(x')}) = \tau(\Theta(x'))$$

The last equality comes from the facts that $\tau_{|K''} = Id_{K''}$ and the $\lambda_i(x')$'s belong to K_1 hence there image by Θ lies in K''. This proves part 2 of the conclusion, and since $K''|\Theta(K')$ is separable part 1 follows.

By construction $\varphi = \Theta_{|A'}$ embeds $(A',\leq)$ into $(A'',\leq)$ over A and it remains to prove that $\bar\varphi$ is a full embedding and that it equals σ. Let a' be any element of A'.

In $(A'', \leq)$ we have $P(a''_{a'}) = \sigma(P(a'))$ hence by construction $a''_{a'} \leq a' \leq a''_{a'}$ in $(A^\circ, \leq)$. Since $(A'_1, \leq) \subseteq (A^\circ, \leq)$ this remains true in $(A'_1, \leq)$, hence $\Theta(a''_{a'}) \leq \Theta(a') \leq \Theta(a''_{a'})$. Observe that $\Theta(a''_{a'}) = a''_{a'}$ because $a''_{a'} \in A''$ hence in $(A'', \leq)$:

$$a''_{a'} \leq \Theta(a') = \varphi(a') \leq a''_{a'} \quad \Rightarrow \quad P(\varphi(a')) = P(a''_{a'}) = \sigma(P(a')). \tag{3}$$

Remember that $\bar{\varphi}(P(a'))$ is defined to be $P(\varphi(a'))$. In order to prove that φ is full it suffices to check that for any $a_1, a_2, b_1, b_2 \in A'$, if $P(a_1) \cap P(a_2)$ equals $P(b_1) \cap P(b_2)$ then $P(\varphi(a_1)) \cap P(\varphi(a_2))$ equals $P(\varphi(b_1)) \cap P(\varphi(b_2))$. But for any such four elements of A' by (3) and because σ is a full embedding we have:

$$\begin{aligned} P(\varphi(a_1)) \cap P(\varphi(a_2)) &= \sigma(P(a_1)) \cap \sigma(P(a_2)) = \sigma(P(a_1) \cap P(a_2)) \\ &= \sigma(P(b_1) \cap P(b_2)) = \sigma(P(b_1)) \cap \sigma(P(b_2)) \\ &= P(\varphi(b_1)) \cap P(\varphi(b_2)). \end{aligned}$$

The embedding theorem is proved.

6 Elementary embeddings, elementary equivalence

For any structure E in a language $\mathcal{L}$, and any subset F of E we denote by $\mathcal{L}(F)$ the expansion of $\mathcal{L}$ obtained by adding new constant symbols for each element of F, and (E, F) the natural $\mathcal{L}(F)$-structure. Observe that if F is a substructure of E then:

$$F \preceq E \quad \Longleftrightarrow \quad (F, F) \equiv (E, F).$$

The notion of CDM-representation, introduced by Cherlin, van den Dries and Macintyre [1] plays an essential role in the model-theory of PAC fields. We recall its main properties, as reported by Ershov [6].

For any field K we denote by $G(K)$ the absolute Galois group of K, that is the group of K-automorphisms of a separable closure K^s of K. There is a class of structures Σ, elementary with respect to a many-sorted language $\mathcal{L}$, and a contravariant foncteur Φ sending profinite groups to elements of Σ such that:

1. For each $\mathcal{L}$-sentence $\varphi_{\mathcal{L}}$ there is a sentence φ in the language of rings such that a field K models φ if and only if $\Phi G(K)$ models $\varphi_{\mathcal{L}}$;

2. If K is κ-saturated, then $\Phi G(K)$ is κ-saturated;

3. Every epimorphism of profinite groups $\psi : H \to G$ induces an embedding of $\mathcal{L}$-structures $\Phi\psi : \Phi G \to \Phi H$, and conversely every $\mathcal{L}$-embedding of ΦG into ΦH is of this form.

If K is relatively separably closed in an extension field L, the restriction $\rho : G(L) \to G(K)$ induces a monomorphism $\Phi\rho : \Phi G(K) \to \Phi G(L)$. The expanded structure $(\Phi G(L), \Phi\rho(\Phi G(K)))$ will simply be noted $(\Phi G(L), K)$. When given any two regular extensions K_1, K_2 of K we have:

$$(K_1, K) \equiv (K_2, K) \Rightarrow (\Phi G(K_1), K) \equiv (\Phi G(K_2), K).$$

Claim. If in addition K_1, K_2 are both RC-local and have the same degree of imperfection, then the converse is true.

This is a well known theorem for PAC, PRC or PpC fields. In the mixed case of RC-local fields it is known as a theorem by the experts, although it is hard to find a published reference for it.

Theorem 6.1 *Let A', A'' be two PAC* domains with RC-local fraction fields K' and K'' containing a given domain A with fraction field K. Equip A' and A'' with the radical relations associated to their maximal spectra. Suppose that the restrictions to A of these two radical relations coincide and equip A with this relation. Assume that $(A, \leq)$ is a full sub-extension of $(A', \leq_{A'})$ and $(A'', \leq_{A''})$. If K', K'' are regular extension of K then $(A', A) \equiv (A'', A)$ if and only if the following conditions are fullfilled:*

1. $J(A') = (0) \iff J(A'') = (0)$.
2. $(K', K) \equiv (K'', K)$.
3. $(\mathbf{L}(A'), \mathbf{L}(A)) \equiv (\mathbf{L}(A''), \mathbf{L}(A))$.

Remark 6.2 If A' is a non-trivial valuation ring with algebraically closed fraction field (the simplest exemple of a PAC domain which is not a field) and $A'' = \mathrm{Frac}(A')$ then conditions 2 and 3 are obviously satisfied although $A' \not\equiv A''$. Condition 1 is necessary only to avoid this case, because if neither A' nor A'' are fields then condition 3 will imply condition 1 by 3.3.

Proof: One direction is immediate: if $(A', A) \equiv (A'', A)$ then $(A', \leq, A) \equiv (A'', \leq, A)$ because these radical relations are definable hence $(A', \leq)$ and $(A'', \leq)$ have A-isomorphic elementary extensions. Such an isomorphism induces a K-isomorphism between elementary extensions of K' and K'' hence $(K', K) \equiv (K'', K)$ and similarly $(\mathbf{L}(A'), \mathbf{L}(A)) \equiv (\mathbf{L}(A''), \mathbf{L}(A))$. By the above remark this impies $J(A') = (0) \iff J(A'') = (0)$.

In order to prove the converse we are going to build A-isomorphic r-extensions of A' and A'' by using inductively the embedding theorem. Since every intermediate extensions of A' and A'' will be PAC* hence Bezout, every r-embeddings are full r-embeddings.

Let $A_0 = A$, $A_1 = A'$ and A_2 be a $|A'|^+$-saturated elementary extension of A'' (we can suppose all our rings, except maybe A, to be infinite because otherwise the condition 2 implies that $A' = A''$ and there is nothing to prove). Given an integer n suppose that we have already build a sequence $(A_k)_{k \leq n+2}$ of PAC* domains with respective RC-local fraction fields $(K_k)_{k \leq n+2}$ such that for $1 \leq k \leq n$, $A_k \preceq A_{k+2}$ and for $k < n$ an r-embedding φ_{k+1} is given from A_{k+1} into A_{k+2} whose restriction to A_k is identity, which makes K_{k+2} a regular extension of $\mathrm{Frac}(\varphi_{k+1}(A_{k+1}))$ and $\mathbf{L}(A_{k+2})$ an elementary extension of $\bar{\varphi}_{k+1}(\mathbf{L}(A_{k+1}))$, so that the following diagramm is commutative:

$$\begin{array}{ccccccccc}
A_0 & \to & A_2 & \preceq & A_4 & \cdots \preceq & A_{n+1} & & \\
 & \searrow & \uparrow & \searrow & \uparrow & & \uparrow & & \\
 & & A_1 & \preceq & A_3 & \cdots \preceq & A_n & \preceq & A_{n+2}
\end{array}$$

the vertical and diagonal arrows being the φ_k's. Suppose in addition that, identifying A_k with its image $\varphi_k(A_k)$ whenever it is convenient, we have:

1. $J(A_{n+1}) = (0) \iff J(A_{n+2}) = (0)$.
2. $(K_{n+1}, K_n) \equiv (K_{n+2}, K_n)$.
3. $(\mathbf{L}(A_{n+1}), \mathbf{L}(A_n)) \equiv (\mathbf{L}(A_{n+2}), \mathbf{L}(A_n))$.
4. A_{n+2} is $|A_{n+1}|^+$-saturated.

The extension $K_{n+1}|K_n$ is regular by hypothesis, and $K_{n+2}|K_n$ is regular because $K_n \preceq K_{n+2}$. In the case of positive characteristic p, $K_{n+1} \equiv K_{n+2}$ implies that those fields have the same degree of imperfection, hence $[K_{n+1} : K_{n+1}^p] \leq [K_{n+2} : K_{n+2}^p]$ (equality holds if the imperfect degree is finite, otherwise the inequality is given by saturation since we can replace A_{n+2} by any suitably saturated elementary extension). By condition 1 we have $(0) \in \mathrm{Spec}(A_{n+1}, \leq)$ if and only if $(0) \in \mathrm{Spec}(A_{n+2}, \leq)$. If $\mathbf{L}(A_{n+2})$ is infinite it is $|\mathbf{L}(A_{n+1})|^+$-saturated hence condition 3 provides an elementary embedding σ of $\mathbf{L}(A_{n+1})$ into $\mathbf{L}(A_{n+2})$ whose restriction to $\mathbf{L}(A_n)$ is identity. If $\mathbf{L}(A_{n+2})$ is finite then $\mathbf{L}(A_{n+1}) = \mathbf{L}(A_{n+2})$ by condition 3 hence we can take for σ the identity map. Eventually condition 2 implies that $(\Phi G(K_{n+1}), K_n) \equiv (\Phi G(K_{n+2}), K_n)$ and the saturation property gives an elementary embedding $\Phi\pi$ of $\Phi G(K_{n+1})$ into $\Phi G(K_{n+2})$ over $\Phi G(K_n)$, hence an epimorphism π from $G(K_{n+2})$ onto $G(K_{n+1})$ which commutes to restrictions to K_n^s.

The embedding theorem then applies: it gives a full r-embedding φ_{n+1} of A_{n+1} into A_{n+2} over A_n, such that K_{n+2} is a regular extension of $\mathrm{Frac}(\varphi_{k+1}(A_{k+1}))$, $\Phi G(K_{n+2})$ is an elementary extension of $\Phi G(\mathrm{Frac}(\varphi_{k+1}(A_{k+1}))) = \Phi\pi(\Phi G(K_{n+1})) \simeq \Phi G(K_{n+1})$, and $\bar{\varphi}_{n+1} = \sigma$ is an elementary embedding of $\mathbf{L}(A_{n+1})$ into $\mathbf{L}(A_{n+2})$. Let A_{n+3} be any $|A_{n+2}|^+$-saturated extension of A_{n+1}. The induction hypothesis 1 and 4 are clearly satisfied at the rank $n+1$. So is condition 3 because $\mathbf{L}(A_{n+1}) \preceq \mathbf{L}(A_{n+2})$ and $\mathbf{L}(A_{n+1}) \preceq \mathbf{L}(A_{n+3})$ by construction. Moreover $\Phi G(K_{n+1}) \preceq \Phi G(K_{n+2})$ and $\Phi G(K_{n+1}) \preceq \Phi G(K_{n+3})$ hence condition 2 is satisfied because $(\Phi G(K_{n+2}), K_{n+1}) \equiv (\Phi G(K_{n+3}), K_{n+1})$.

Repeat this construction ω times and let $B' = \bigcup_{n\in\mathbf{N}} A_{2n+1}$, $B'' = \bigcup_{n\in\mathbf{N}} A_{2n}$, $\varphi = \bigcup_{n\in\mathbf{N}} \varphi_{2n+1}$ and $\psi = \bigcup_{n\in\mathbf{N}} \varphi_{2n}$. By construction $A' \preceq B'$, $A'' \preceq B''$ and φ, ψ are reciprocal A-isomorphisms between B' and B''. The conclusion follows.

■

Corollary 6.3 *Let $A \subseteq A'$ be two PAC* domains with RC-local fraction fields K, K'. Suppose that $J(A) = (0)$ if and only if $J(A') = (0)$. Then:*

$$A \preceq A' \iff K \preceq K' \text{ and } L(A) \preceq L(A').$$

Remark 6.4 Observe that if $A \preceq A'$ then $(A, \leq_A) \preceq (A', \leq_{A'})$ because the radical relation induced by the maximal spectrum is definable in the ring, hence the above equivalence makes sense.

Proof: Let $A'' = A$ and apply theorem 6.1.

■

The algebraic part of a domain A, that is the subring of elements of A which are algebraic over the prime subfield of $\mathrm{Frac}(A)$, is usually denoted by $\mathrm{Alg}(A)$. Given a r-domain $(A, \leq)$ with residual semi-lattice L and full residual lattice $\mathbf{L}$ we denote by $\mathrm{Alg}(A, \leq)$ the r-domain obtained by adjoining to $\mathrm{Alg}(A)$ the radical relation induced by the one of $(A, \leq)$, and $\mathrm{Alg}(L) = L(\mathrm{Alg}(A, \leq))$, $\mathrm{Alg}(\mathbf{L}) = \mathbf{L}(\mathrm{Alg}(A, \leq))$.

Proposition 6.5 *For any r-domain $(A, \leq)$ such that $\mathrm{Alg}(A)$ is integrally closed the semi-lattice $L(\mathrm{Alg}(A, \leq))$ is stable by $\cap$ hence $L(\mathrm{Alg}(A, \leq)) = \mathbf{L}(\mathrm{Alg}(A, \leq))$ is a lattice and $(A, \leq)$ is a full extension of $\mathrm{Alg}(A, \leq)$.*

Proof: If $\mathrm{Alg}(A)$ is a field it is in particular a Bezout ring and the conclusion is immediate, hence we can suppose this is not the case. In particular we are in zero characteristic and $\mathrm{Alg}(A)$ is a Prüfer domain of dimension 1. Choose any non-zero $a, b \in A$ (if $a = 0$ then $P(a) \cap P(b) = P(b)$ so we can forget this case) and let $B = \mathrm{Alg}(A) \cap \mathbf{Q}(a, b)$. Since B is a Dedekind domain $M_B(a)$ and $M_B(b)$ are finite. Since $\mathrm{Pic}(B)$ is finite (torsion is enough) for every $\mathbf{m} \in \mathrm{Specmax}(A)$ there is $n \in \mathbf{N}^*$ and $c_{\mathbf{m}} \in B$ such that $\mathbf{m}^n$ is principal with generator $c_{\mathbf{m}}$ hence $M(c_{\mathbf{m}}) = \{\mathbf{m}\}$. Let c be the product of the $c_{\mathbf{m}}$'s for $\mathbf{m}$ ranging over $M_B(b) \cap M_B(a)$, then $M_B(c) = M_B(b) \cap M_B(a)$. The same holds in $\mathrm{Alg}(A)$ and since $\mathrm{Alg}(A)$ has dimension 1 we have in $\mathrm{Alg}(A, \leq)$ that $P(a) = M(a) \cap \mathrm{Spec}(\mathrm{Alg}(A, \leq))$ hence $P(c) = P(b) \cap P(a)$.

■

Remark 6.6 Observe that the finiteness of $M_B(a)$, $M_B(b)$ and $\mathrm{Pic}(B)$ proves as well that for any non-zero a, b in $\mathrm{Alg}(A)$ there exists $c \in \mathrm{Alg}(A)$ such that $M(a) \setminus M(b) = M(c)$. We will use this in the applications.

Corollary 6.7 *Let A and A' be two PAC* domains with RC-local fraction fields K and K' respectively. Equip them with the radical relation induced by their maximal spectra and let L, L' denote their respective residual lattices. Then $A \equiv A'$ if and only if:*

1. *$J(A) = (0)$ if and only if $J(A') = (0)$.*
2. *$K \equiv K'$.*
3. *$\mathrm{Alg}(A, \leq) \simeq \mathrm{Alg}(A', \leq)$ and after identification $(L, \mathrm{Alg}(L)) \equiv (L', \mathrm{Alg}(L))$.*

Proof: One direction is obvious: if $A \equiv A'$ then they have isomorphic elementary extensions and any such isomorphism maps algebraic parts to algebraic parts. The above three conditions follow.

Conversely if these conditions are fullfilled then $K \equiv K'$ implies that $(K, \mathrm{Alg}(K)) \equiv (K', \mathrm{Alg}(K'))$ and by 6.5, theorem 6.1 applies.

■

In positive characteristic, the above criterion for elementary equivalence is somewhat simpler:

Corollary 6.8 *Let A and A' be two PAC domains of positive characteristic such that $J(A) = (0)$ if and only if $J(A') = (0)$, then:*

$$A \equiv A' \iff \mathrm{Frac}(A) \equiv \mathrm{Frac}(A') \quad \textit{and} \quad L(A) \equiv L(A').$$

Proof: In positive characteristic $\mathrm{Alg}(A) = \mathrm{Alg}(\mathrm{Frac}(A))$ is a field, hence $\mathrm{Frac}(A) \equiv \mathrm{Frac}(A')$ implies that $\mathrm{Alg}(A) \simeq \mathrm{Alg}(A')$. Moreover the radical relation induced by A on $\mathrm{Alg}(A)$ is necessarily the one defined by $\mathrm{Spec}(\mathrm{Alg}(A)) = \{(0)\}$ and it is the same for A' hence $\mathrm{Alg}(A, \leq) \simeq \mathrm{Alg}(A', \leq)$. Since $\mathrm{Alg}(L(A)) = \{P(0), P(1)\} = \mathrm{Alg}(L'(A))$ is already in the language of lattices we get $L(A) \equiv L(A')$. All its hypothesis being satisfied, 6.7 then gives the conclusion.

■

7 Applications to decidability and algebra

In this section we apply our study to various examples of PAC* domains provided by théorème 1.3 of L. Moret-Bailly [14]. Another proof is given by B. Green, F. Pop and P. Roquette in [11] corollary 3.4. We call it **the fundamental theorem**.

Given a family $\mathcal{F}$ of finite extensions of $\mathbf{Q}_p$ or $\mathbf{R}$, with p ranging over a finite set $\mathcal{P}$ of primes, define $\mathbf{Q}_\mathcal{F}$ as the field of all algebraic numbers x such that x and all its conjugates over $\mathbf{Q}$ belong to every $F \in \mathcal{F}$:

$$\mathbf{Q}_\mathcal{F} = \bigcap \{F^\tau \;/\; \tau \in Gal(\mathbf{Q}),\ F \in \mathcal{F}\}$$

The integral closure of $\mathbf{Z}$ inside $\mathbf{Q}_\mathcal{F}$ is denoted by $\mathbf{Z}_\mathcal{F}$. We will keep these notations throughout all this section. Among other things the fundamental theorem asserts for example that:

- $\mathbf{Q}_\mathcal{F}$ is regularly closed relatively to $\mathcal{F}$, and in particular it is RC-local.
- Given any integrally closed subring A of $\mathbf{Q}_\mathcal{F}$, if $A \neq \mathbf{Z}_\mathcal{F}$ then A satisfies LGPH*. If moreover every $p \in \mathcal{P}$ is invertible in A then A is PAC*.
- If $\mathbf{R} \notin \mathcal{F}$ then $\mathbf{Z}_\mathcal{F}$ satisfies LGPH and all its overrings satisfy at least LGPH*.
- Similar results hold in positive characteristic, but we will only use that any integrally closed domain whose fraction field is separably closed and algebraic over $\mathbf{F}_p(T)$ is PAC.

Remark 7.1 Relatively complemented lattices are so much similar to boolean algebras that the methods of the firsts apply also to the lasts. This analogy is used in [15] page 193 to convince the reader that the theory of dense relatively complemented lattices has quantifier elimination (this theory is proved to be a model-complete in [3] annexe A). With the same philosophy we claim here that every relatively complemented lattice has a decidable theory, and that this can be proved by a carefull transcription of the proof of the analogeous result for boolean algebras (we thanck Y.-L. Ershov who pointed us this fact proved in [5] (in russian)).

Lemma 7.2 *For any $\mathcal{N} \subseteq \mathbf{N} \setminus \{0,1\}$ every domain A elementary equivalent to $B = \mathbf{Z}_\mathcal{F}[1/n]_{n\in\mathcal{N}}$ satisfies the following conditions:*

1. *$J(A) = (0)$ if and only if $J(B) = (0)$.*
2. *$0 \leq_A N$ if and only if $0 \leq_B N$ for some $N \in \mathbf{N}^*$.*
3. *A is integrally closed.*
4. *Every element of $\mathcal{N}$ is invertible in A.*
5. *$A \models \forall x,\ f(x) \neq 0$, for every irreducible $f \in \mathbf{Z}[X]$ whose leading coefficient has a prime factor not in $\mathcal{N}$.*
6. *$K = \mathrm{Frac}(A) \equiv \mathbf{Q}_\mathcal{F}$.*
7. *$L(A) \equiv L(B)$.*

Moreover if the above conditions 1 to 6 are fullfilled then $\mathrm{Alg}(A, \leq_A) \simeq (B, \leq_B)$.
Conditions 1, 2, 3 and 7 are recursively axiomatisable, so are 4 and 5 if $\mathcal{N}$ *is recursive, and so is 6 if* $\mathbf{Q}_{\mathcal{F}}$ *is decidable.*

Proof: A structure is decidable if and only if its complete theory is recursively axiomatisable. The recursive axiomatisability of conditions 1 to 3 is immediate, as well as 4 and 5 if $\mathcal{N}$ is recursive. For condition 7 one can either use remark 7.1 which guaranties that $L(B)$ has a decidable complete theory whatever it can be, or check that $L(B)$ is dense and derive from this a recursive (actually a finite) axiomatisation of its complete theory.

Since all the above properties are elementary and satisfied by B, any domain elementary equivalent to B must share them.

Conversely let A be a domain satisfying conditions 1 to 6. Since $K \equiv \mathbf{Q}_{\mathcal{F}}$ we have $\mathrm{Frac}(\mathrm{Alg}(A)) = \mathrm{Alg}(K) \simeq \mathbf{Q}_{\mathcal{F}}$. After identifying $\mathrm{Alg}(K)$ with $\mathbf{Q}_{\mathcal{F}}$ we have $B \subseteq \mathrm{Alg}(A)$ because A is integrally closed and contains $1/n$ for every $n \in \mathcal{N}$, and $\mathrm{Alg}(A) = \mathrm{Alg}(A) \cap \mathbf{Q}_{\mathcal{F}} = B$ by condition 5. It remains to prove that the radical relation induced on B by $(A, \leq_A)$ is the radical relation $\leq_B$.

Let $a, b \in B$ be such that $a \leq_B b$. If $a \neq 0$ then for every $\mathbf{m} \in \mathrm{Specmax}(A)$ containing a we have $\mathbf{m} \cap B \in \mathrm{Specmax}(B)$ (because B has dimension 1) hence $b \in \mathbf{m}$ by definition of $\leq_B$, which proves that $a \leq_A b$. If $a = 0$ then either $J(B) = (0)$ and then $b = 0$ hence $a \leq_A b$, or $J(B) \neq (0)$ and then there is some nonzero integer N such that $0 \leq_B N$ hence $0 \leq_A N$ by condition 2. Then $N \leq_B b$ and by the above argument $N \leq_A b$ hence $0 = a \leq_A b$.

Conversely suppose $a \not\leq_B b$, that is $M_B(a) \setminus M_B(b) \neq \emptyset$. If $a = 0$ then either $J(B) = (0)$ and then $J(A) = (0)$ by condition 1 and $b \neq 0$ hence $a \not\leq_A b$, or $J(A) \neq (0)$ and then replacing a by some nonzero integer N such that $0 \leq_B N$ we reduce to the case $a \neq 0$. If $a \neq 0$ let $c \in B$ be such that $M_B(c) = M_B(a) \setminus M_B(b)$ (see remark 6.6, c can be taken nonzero). For any maximal ideal $\mathbf{m}$ of A containing c we have $\mathbf{m} \cap B \in M_B(c)$ hence $a \in \mathbf{m}$ and $b \notin \mathbf{m}$, which contradicts $a \leq_A b$. It follows that $c \in A^*$ hence $c \in B^* = A^* \cap \mathbf{Q}_{\mathcal{F}}$. In particular $M_B(c)$ is empty, that is $M_B(a) \subseteq M_B(b)$ hence $a \leq_B b$.
∎

Theorem 7.3 *Suppose that* $\mathcal{F}$ *is a finite number of fields among* $\mathbf{R}$ *and the* $\mathbf{Q}_p$*'s.*

1. *If* $\mathbf{R} \notin \mathcal{F}$ *then* $\mathbf{Z}_{\mathcal{F}}$ *is decidable if and only if* $\mathbf{Q}_{\mathcal{F}}$ *is decidable.*
2. *For any non empty* $\mathcal{N} \subseteq \mathbf{N} \setminus \{0,1\}$ *the ring* $\mathbf{Z}_{\mathcal{F}}[1/n]_{n \in \mathcal{N}}$ *is decidable if and only if* $\mathbf{Q}_{\mathcal{F}}$ *is decidable and* $\mathcal{N}$ *is recursive.*

Proof: The two cases can be treated together, by proving that for an arbitrary subset $\mathcal{N}$ of $\mathbf{N} \setminus \{0,1\}$ if $B = \mathbf{Z}_{\mathcal{F}}[1/n]_{n \in \mathcal{N}}$ satisfies LGPH* (which happens in both cases by the fundamental theorem) then B is decidable if and only if $\mathbf{Q}_{\mathcal{F}}$ is decidable and $\mathcal{N}$ is recursive.

One direction is trivial: if B is decidable then so is its fraction field because it is interpretable in B, and $\mathcal{N}$ is recursive because it is the set of integers in $\mathbf{N} \setminus \{0,1\}$ which are invertible in B.

Suppose now that $\mathbf{Q}_{\mathcal{F}}$ is decidable and $\mathcal{N}$ is recursive. Let $\mathcal{P}$ be the set of primes p such that $\mathbf{Q}_p \in \mathcal{F}$ and let $m = \prod_{p \in \mathcal{P}} p$. Let $\mathcal{N}_1 = \mathcal{N} \cup \mathcal{P}$, m_1 be the product of all elements of $\mathcal{N} \cap \mathcal{P}$ and $B_1 = \mathbf{Z}_{\mathcal{F}}[1/n]_{n \in \mathcal{N}_1} = B[1/m]$. Since $\mathbf{Q}_{\mathcal{F}}$ is regularly closed with respect to a family of p-adically closed fields of finitely many different kind its holomorphy ring $R_{\mathcal{F}}$ (the intersection of all its p-adic valuation rings) is definable without parameters in $\mathbf{Q}_{\mathcal{F}}$. Moreover $B = (R_{\mathcal{F}} \cap B[1/m])[1/m_1]$ hence B is definable in $B[1/m]$ and we have only to prove that $B[1/m]$ is decidable.

B is a PAC* domain by the fundamental theorem, hence by 6.7 and 7.2 it is axiomatised by the theory of PAC* domains and the list of conditions given in 7.2 with $\mathcal{N}$ replaced by $\mathcal{N}_1$. Since $\mathbf{Q}_{\mathcal{F}}$ is decidable and $\mathcal{N}_1$ is recursive (as the union of a recursive and a finite set) this provides a recursive axiomatisation B.

■

Remark 7.4 In [2] we studied the integral closure of $\mathbf{Z}$ inside one p-adic closure of $\mathbf{Q}_p$, which we may call the ring of p-adic integers. In this paper the case under consideration is typically the integral closure of $\mathbf{Z}$ inside $\mathbf{Q}_{\{\mathbf{Q}_p\}}$ that is the ring of absolutely (or totally) p-adic integers, which we proved to be decidable if and only if its fraction field is decidable. Experts agree that since the structure of the absolute Galois group of $\mathbf{Q}_{\{\mathbf{Q}_p\}}$ has been elucidated by F. Pop, this field is known to have a decidable theory. More generally $\mathbf{Q}_{\mathcal{F}}$ is decidable when $\mathcal{F}$ is a finite set of $\mathbf{Q}_p$'s, see [8] theorem 3. It then follows from 7.3 that $\mathbf{Z}_{\mathcal{F}}$ is decidable.

In general it is not possible to reduce the decidability of a domain to the decidability of its fraction field: in [9] it was proved that the field $\mathbf{Q}_{\{\mathbf{R}\}}$ of absolutely real numbers was decidable, although $\mathbf{Z}_{\{\mathbf{R}\}}$ is known to be undecidable by a result of J. Robinson. However this is a somewhat surprising result, and even more surprising is the fact that for example $\mathbf{Z}_{\{\mathbf{R}\}}[1/n]$ is decidable by 7.3 for any $n \in \mathbf{N} \setminus \{0, 1\}$. We turn now to some cases of positive characteristic.

Theorem 7.5 *A PAC domain of positive characteristic is decidable if and only if its fraction field is decidable.*
In particular any integrally closed ring whose fraction field is separably closed and algebraic over $\mathbf{F}_p(t)$ is decidable.

Proof: Let A be a PAC domain of positive characteristic. By 6.8 its complete theory is axiomatised by:

- The recursively axiomatisable theory of PAC domains.
- The formula saying if $J(A) = (0)$ or not.
- The axioms describing the complete theory of its fraction field and residual lattice.

Since the residual lattice is relatively complemented it has a decidable complete theory, and so does the fraction field by hypothesis, hence the above list of axioms is recursively enumerable and A is then decidable.

To derive the second assertion observe that by the fundamental theorem any integrally closed ring whose fraction field is separably closed and algebraic over $\mathbf{F}_p(t)$ is a PAC domain. Moreover its fraction field is known to be recursively axiomatised by the axioms of separably closed fields of characteristic p and fixed imperfect degree (0 or 1). The conclusion follows.

■

Our last application is purely algebraic. It does not actually use our extensive study of PAC domains but only the particular case of PAC domains with algebraically closed fraction field and a dense residual lattice studied by Prestel and Schmid in [15]. On the other hand it works in arbitrary characteristic, and it is interesting to notice that contrary to theorem 5.3 of [15] no simpler algebraic proof is known by now.

Theorem 7.6 *Let A be a PAC domain with an algebraically closed fraction fields and a dense residual lattice. Then for every $f_1, \ldots, f_m \in A[X_1, \ldots, X_n, Y]$ the following assertions are equivalent:*

1. *For every $\alpha \in A^n$ the $f_i(\alpha, Y)$'s have a common root in A.*
2. *There exists a non-constant $f \in A[X_1, \ldots, X_n, Y]$ monic in Y which divides simultaneously $f_1, \ldots, f_m$.*

Proof: If a non-constant $f \in A[X_1, \ldots, X_n, Y]$ monic in Y divides simultaneously $f_1, \ldots, f_m$ then for every $\alpha \in A^n$ any root of $f(\alpha, Y)$ will be a common root the $f_i(\alpha, Y)$'s. Since A is integrally closed with algebraically closed fraction field and $f(\alpha, Y)$ is non-constant and monic such a root exists and must belong to A.

Conversely let $\varphi(x, y)$ be the formula with $n+1$ variables and parameters the coefficients of the f_i's which writes $\bigwedge_{i \leq n} f_i(x, y) = 0$ and suppose that $A \models \forall x \exists y \varphi(x, y)$. Let B be the integral closure of $A[X_1, \ldots, X_n]$ in an algebraically closed field. Equip B with the radical relation defined by its maximal spectrum. The r-domain $(B, \leq_B)$ is an extension of $(A, \leq_A)$ and $J(B) = (0)$ if and only if $J(A) = (0)$. The construction of [15] 3.3. apply here: it proves that $(B, \leq_B)$ embeds in a r-domain $(C, \leq)$ which is PAC, has an algebraically closed fraction field, a dense residual lattice, a radical relation defined by its maximal spectrum, a Jacobson radical which is zero if and only if so is $J(B)$ and such that every maximal ideal of B can be lifted to a maximal ideal of C. Since the theory of algebraically closed field and the theory of dense relatively complemented lattices are model-complete, we conclude by theorem 4.3 of [15] (or more generally by 4.8) that $(A, \leq_A) \preceq (C, \leq)$.

In particular $(C, \leq) \models \forall x \exists y \varphi(x, y)$. Apply this to $(X_1, \ldots, X_n) \in C^n$: there exists $\theta \in C$ such that $C \models \varphi(X_1, \ldots, X_n, \theta)$. Such a θ is algebraic over $A[X_1, \ldots, X_n]$ hence $\theta \in \mathrm{Frac}(B)$. Since B is integrally closed and every maximal ideal of B lifts to C we have $C \cap \mathrm{Frac}(B) = B$ hence $\theta \in B$. Let $f \in A[X_1, \ldots, X_n, Y]$ be the minimal polynomial of θ over $A[X_1, \ldots, X_n]$. It is monic in Y because θ is integral over $A[X_1, \ldots, X_n]$, and it divides every $f_i(X_1, \ldots, X_n, Y)$ since all of them vanish on θ and the conclusion follows. ■

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