WHAT IS A SET?

UNIVERSITY OF TORONTO

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Naive set theory

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Sets are governed by two principles:

1. The *comprehension principle*: any predicate defines a set (i.e. we can define the set of all elements satisfying a given property).
2. The *extension principle*: two sets are equal if and only if they contain the same elements.

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Russell's paradox (Zermelo 1899, Russell 1901)

By the comprehension principle, the set $S = \{x : x \notin x\}$ is well-defined.

- If $S \in S$ then $S \notin S$.
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Axiomatic set theory

- Zermelo (1908): a more careful axiomatic set theory.
  Besides the comprehension principle is weakened to the *separation principle*: given a set, we can define its subset of elements satisfying a given predicate

$$\{x \in E \ : \ P(x)\}$$

- This theory has been subsequently refined by Fraenkel, Solem, von Neumann, and others, giving rise to Zermelo–Fraenkel (ZF) set theory.
  ZF is a *first order theory* with equality and membership: we extend propositional calculus by introducing quantified variables and the symbol $\in$.
  In such a theory, we don't define what is a set: they are the atomic objects over which we use quantifiers.

- ZF is not the only axiomatic set theory.
  For instance, there is von Neumann–Bernays–Gödel theory of classes (in this theory a set is a class contained in another class).

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- **Axiom of foundation:** $\forall x (x \neq \emptyset \Rightarrow \exists y \in x (x \cap y = \emptyset))$

Un morceau de choix – 1

We may extend ZF with the following axiom to obtain ZFC:

Axiom of choice: $\forall x((\emptyset \notin x \wedge \forall u, v \in x(u = v \vee u \cap v = \emptyset)) \implies \exists y \forall u \in x \exists w (u \cap y) = \{w\})$

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Equivalent statement

For $(X_i)_{i\in I}$ a family of sets indexed by a set I, we have

$$\left(\forall i \in I,\ X_i \neq \varnothing\right) \implies \prod_{i\in I} X_i \neq \varnothing$$

i.e. there exists $(x_i)_{i\in I}$ where $x_i \in X_i$.

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"The axiom of choice is obviously true, the well-ordering principle obviously false, and who can tell about Zorn's lemma?"

– Jerry L. Bona

Un morceau de choix – 2

Tarski proved that the following statement is equivalent to the axiom of choice:

Trichotomy principle for cardinality

Given two sets A and B, exactly one of the following occurs:

- $|A| < |B|$
- $|A| = |B|$
- $|A| > |B|$

When Tarski submitted to the *Comptes Rendus de l'Académie des Sciences* his proof that the trichotomy principle is equivalent to the axiom of choice, both Fréchet and Lebesgue refused it: Fréchet because *"an implication between two well known propositions is not a new result"*, and Lebesgue because *"an implication between two false propositions is of no interest"*.

Cantor's original proof of Cantor–Schröder–Bernstein theorem relied on the trichotomy principle.

Un morceau de choix – 3

We used the axiom of choices several times in Chapter 7:

- When we proved that if there exists a surjective function $g : F \to E$ then there exists an injective function $f : E \to F$. Indeed, we picked $(y_x)_{x\in E} \in \prod_{x\in E} g^{-1}(x)$.

  Actually the axiom of choice is equivalent to the fact that a function is surjective if and only if it admits a right inverse, i.e. $g : F \to E$ is surjective if and only if there exists $f : E \to F$ such that $g \circ f = id_E$.

- ”A countable union of countable sets is countable” is equivalent to the axiom of countable choice.
  We used the axiom of countable choice to pick simultaneously injective functions $f_i : E_i \to \mathbb{N}$.

- We used the axiom of countable choice to prove that a set is infinite if and only if it contains a subset with cardinality $\aleph_0$ (we used that a countable union of countable sets is countable).
  Without it, there exist models of ZF with infinite sets which doesn't contain subset with cardinaloty $\aleph_0$ (recall that the trichotomy principle doesn't hold: such an infinite set is not comparable with $\mathbb{N}$).