Cardinality: infinite sets

March 30$^{\text{th}}$, 2021

Infinite sets – 1

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Definition: infinite set

We say that a set is *infinite* if it is not finite.

Infinite sets – 2

Theorem

$\mathbb{N}$ is infinite.

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Proof. Assume by contradiction that $\mathbb{N}$ is finite. Then $\mathbb{N} \setminus \{0\} \subset \mathbb{N}$ so $\mathbb{N} \setminus \{0\}$ is finite too.
We define $f : \mathbb{N} \to \mathbb{N} \setminus \{0\}$ by $f(n) = n + 1$.
Note that f is bijective with inverse $f^{-1} : \mathbb{N} \setminus \{0\} \to \mathbb{N}$ defined by $f^{-1}(n) = n - 1$.
Thus $|\mathbb{N}| = |\mathbb{N} \setminus \{0\}| = |\mathbb{N}| - |\{0\}| = |\mathbb{N}| - 1$, i.e. $0 = 1$.
Hence a contradiction. ∎

Definition: cardinality

We say that two sets E and F have same *cardinality*, denoted by $|E| = |F|$, if there exists a bijection $f : E \to F$.
We also say that E and F are *equinumerous* or *equipotent*.

Cardinality of infinite sets – 1

Definition: cardinality

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Remark

We proved that $|\mathbb{N} \setminus \{0\}| = |\mathbb{N}|$ although $\mathbb{N} \setminus \{0\} \subsetneq \mathbb{N}$.
That's a first quirkiness about infinite cardinals.

Cardinality of infinite sets – 2

Proposition

1. If E is a set then $|E| = |E|$.
2. Given two sets E and F, if $|E| = |F|$ then $|F| = |E|$.
3. Given three sets E, F and G, if $|E| = |F|$ and $|F| = |G|$ then $|E| = |G|$.

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Proof.

1. $id : E \to E$ is a bijection.
2. Assume that $|E| = |F|$, i.e. that there exists a bijection $f : E \to F$.
   Then $f^{-1} : F \to E$ is a bijection, so $|F| = |E|$.
3. Assume that $|E| = |F|$ and $|F| = |G|$, i.e. that there exist bijections $f : E \to F$ and $g : F \to G$.
   Then $g \circ f : E \to G$ is a bijection, thus $|E| = |G|$. ∎

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Remark

Equipotence seems to be an equivalence relation since it satisfies reflexivity, symmetry and transitivity. Nonetheless, recall that an equivalence relation is a binary relation on a set whereas the set of all sets doesn't exist (we will prove this fact later).

Cardinality of infinite sets – 3

Theorem

A set E is infinite if and only if for every $n \in \mathbb{N}$ there exists $S \subset E$ such that $|S| = n$.

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Proof.

$\Rightarrow$ Assume that E is infinite.

We are going to prove by induction that for every $n \in \mathbb{N}$ there exists $S \in \mathcal{P}(E)$ such that $|S| = n$.

- *Base case at $n = 0$:* $\varnothing \subset E$ satisfies $|\varnothing| = 0$.
- *Induction step.* Assume that for some $n \in \mathbb{N}$ there exists $T \subset E$ such that $|T| = n$.
  Note that $E \setminus T \neq \varnothing$ (otherwise $E = T$, which is impossible since E is infinite).
  Therefore there exists $x \in E \setminus T$. Define $S \coloneqq T \sqcup \{x\}$, then $S \subset E$ is finite and $|S| = |T| + 1 = n + 1$.
  Which ends the induction step.

$\Leftarrow$ Let E be a set such that for every $n \in \mathbb{N}$ there exists $S \subset E$ such that $|S| = n$.
Assume by contradiction that E is finite. Then there exists $k \in \mathbb{N}$ such that $|E| = k$.
Since $k + 1 \in \mathbb{N}$, there exists $S \subset E$ such that $|S| = k + 1$.
Since $S \subset E$, we get $k + 1 = |S| \leq |E| = k$. Hence a contradiction. ∎

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Corollary

A set E is infinite if and only if for every $n \in \mathbb{N}$ there exists an injective function $\{0, 1, 2, \ldots, n-1\} \to E$.