Concepts in Abstract Mathematics

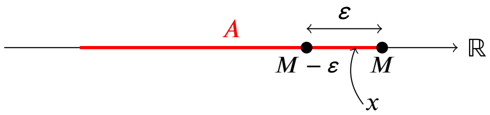
Real numbers – 2

March 16^{th}, 2021

Characterization of the supremum

Proposition

Let $A \subset \mathbb{R}$ and $M \in \mathbb{R}$. Then $M = \sup(A) \Leftrightarrow \begin{cases} \forall x \in A,\ x \leq M \\ \forall \varepsilon > 0,\ \exists x \in A,\ M - \varepsilon < x \end{cases}$



Proof.
$\Rightarrow$ Assume that $M = \sup(A)$. Then M is an upper bound of A so $\forall x \in A,\ x \leq S$.
We know that if T is an other upper bound of A then $M \leq T$ (since M is the least upper bound).
So, by taking the contrapositive, if $T < M$ then T isn't an upper bound of A.
Let $\varepsilon > 0$. Since $M - \varepsilon < M$, we know that $M - \varepsilon$ is not an upper bound of A, meaning that there exists $x \in A$ such that $M - \varepsilon < x$.

$\Leftarrow$ Assume that $\begin{cases} \forall x \in A,\ x \leq M \\ \forall \varepsilon > 0,\ \exists x \in A,\ M - \varepsilon < x \end{cases}$
Then, by the first condition, M is an upper bound of A. Let's prove it is the least one.
We will show the contrapositive: if $T < M$ then T isn't an upper bound of A.
Let $T \in \mathbb{R}$. Assume that $T < M$. Set $\varepsilon = M - T > 0$. Then there exists $x \in A$ such that $M - \varepsilon < x$, i.e. $T < x$.

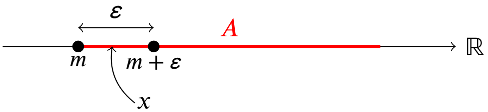
Hence T isn't an upper bound of A. ∎

Characterization of the infimum

Proposition

Let $A \subset \mathbb{R}$ and $m \in \mathbb{R}$. Then

$$m = \inf(A) \Leftrightarrow \begin{cases} \forall x \in A,\ m \leq x \\ \forall \varepsilon > 0,\ \exists x \in A,\ x < m + \varepsilon \end{cases}$$



$\mathbb{R}$ is archimedean

Theorem: $\mathbb{R}$ is archimedean

$\forall \varepsilon > 0, \forall A > 0, \exists n \in \mathbb{N}, n\varepsilon > A$

Proof.
Let $\varepsilon > 0$ and $A > 0$.
Assume by contradiction that $\forall n \in \mathbb{N}, n\varepsilon \leq A$.
Then $E = \{n\varepsilon \; : \; n \in \mathbb{N}\}$ is non-empty and bounded from above so it admits a supremum $M = \sup E$ by the least upper bound principle.
Since $M - \varepsilon < M$, $M - \varepsilon$ is not an upper bound of E, so there exists $n \in \mathbb{N}$ such that $n\varepsilon > M - \varepsilon$.
Therefore $(n+1)\varepsilon > M$, hence a contradiction. ∎

The above theorem means that $\lim_{n \to +\infty} \frac{1}{n} = 0$, or equivalently that $\mathbb{R}$ doesn't contain infinitesimal elements (i.e. there is not infinitely large or infinitely small elements).

Floor function

Proposition: floor function

For every $x \in \mathbb{R}$, there exists a unique $n \in \mathbb{Z}$ such that $n \leq x < n+1$.
We say that n is the integer part (or the floor function value) of x and we denote it by $\lfloor x \rfloor$.

Proof. Let $x \in \mathbb{R}$.
Existence.

- *First case: if $x \geq 0$.*
  We set $E = \{n \in \mathbb{N} \ : \ x < n\}$.
  By the archimedean property (with $\varepsilon = 1$), there exists $m \in \mathbb{N}$ such that $m > x$. Hence $E \neq \varnothing$.
  By the well-ordering principle, E admits a least element p.
  We have that $x < p$ since $p \in E$ and that $p - 1 \leq x$ since $p - 1 \notin E$.
  Therefore $n = p - 1$ satisfies $n \leq x < n + 1$.
- *Second case: if $x < 0$.* Then we apply the first case to $-x$.

Uniqueness. Assume that $n, n' \in \mathbb{Z}$ are two suitable integers. Then $(1)\ \ n \leq x < n+1$ and $n' \leq x < n' + 1$.
We deduce from the last inequality that $(2)\quad -n' - 1 < -x \leq -n'$.
Summing (1) and (2), we get that $n - n' - 1 < 0 < n - n' + 1$.
Hence $n - n' < 1$, i.e. $n - n' \leq 0$, and $-1 < n - n'$ i.e. $0 \leq n - n'$.
Therefore $n = n'$. ∎