Euclidean division

January 26th, 2021

Absolute value – 1

Definition: absolute value of an integer

For $n \in \mathbb{Z}$, we define the *absolute value of* n by $|n| := \begin{cases} n & \text{if } n \in \mathbb{N} \\ -n & \text{if } n \in (-\mathbb{N}) \end{cases}.$

Proposition

1. $\forall n \in \mathbb{Z},\ |n| \in \mathbb{N}$
2. $\forall n \in \mathbb{Z},\ n \leq |n|$
3. $\forall n \in \mathbb{Z},\ |n| = 0 \Leftrightarrow n = 0$
4. $\forall a, b \in \mathbb{Z},\ |ab| = |a||b|$
5. $\forall a, b \in \mathbb{Z},\ |a| \leq b \Leftrightarrow -b \leq a \leq b$

Proof.

1. If $n \in \mathbb{N}$ then $|n| = n \in \mathbb{N}$.
   If $n \in (-\mathbb{N})$ then $n = -m$ for some $m \in \mathbb{N}$ and $|n| = -n = -(-m) = m \in \mathbb{N}$.
2. *First case:* $n \in \mathbb{N}$. Then $n \leq n = |n|$.
   Second case: $n \in (-\mathbb{N})$. Then $n \leq 0 \leq |n|$.
3. Note that $|0| = 0$ and that if $n \neq 0$ then $|n| \neq 0$.
4. You have to study separately the four cases depending on the signs of a and b.
5. If $b < 0$ then $|a| \leq b$ and $-b \leq a \leq b$ are both false. So we may assume that $b \in \mathbb{N}$. Then
   First case: $a \in \mathbb{N}$. Then $|a| \leq b \Leftrightarrow a \leq b \Leftrightarrow -b \leq a \leq b$.
   Second case: $a \in (-\mathbb{N})$. Then $|a| \leq b \Leftrightarrow -a \leq b \Leftrightarrow -b \leq a \Leftrightarrow -b \leq a \leq b$. ∎

Euclidean division – 1

Theorem: Euclidean division

Given $a \in \mathbb{Z}$ and $b \in \mathbb{Z} \setminus \{0\}$, there exists a unique couple $(q, r) \in \mathbb{Z}^2$ such that

$$\begin{cases} a = bq + r \\ 0 \leq r < |b| \end{cases}$$

The integers q and r are respectively the *quotient* and the *remainder* of the division of a by b.

The proof doesn't appear in the handout: see either the slides or the lecture notes.

Euclidean division – 2

Examples

- Division of 22 by 5:
  $$22 = 5 \times 4 + 2$$
  The quotient is $q = 4$ and the remainder is $r = 2$.

- Division of -22 by 5:
  $$-22 = 5 \times (-5) + 3$$
  The quotient is $q = -5$ and the remainder is $r = 3$.

- Division of 22 by -5:
  $$22 = (-5) \times (-4) + 2$$
  The quotient is $q = -4$ and the remainder is $r = 2$.

- Division of -22 by -5:
  $$-22 = (-5) \times 5 + 3$$
  The quotient is $q = 5$ and the remainder is $r = 3$.

Euclidean division – 3

Proposition: parity of an integer

Given $n \in \mathbb{Z}$, exactly one of the followings occurs:

- either $n = 2k$ for some $k \in \mathbb{Z}$ *(then we say that n is even)*,
- or $n = 2k + 1$ for some $k \in \mathbb{Z}$ *(then we say that n is odd)*.

Proof. Let $n \in \mathbb{Z}$.
By Euclidean division by 2, there exist $k, r \in \mathbb{Z}$ such that $n = 2k + r$ and $0 \leq r < 2$.
Hence either $r = 0$ or $r = 1$.
And these cases are exclusive by the uniqueness of the Euclidean division. ∎