3 - Functions, Sequences & Series

__Definition:__ A function (or map) is the data of two sets A and B together with a "process" which associates to each element $x \in A$ a unique element $f(x) \in B$

$$f : \begin{array}{l} A \longrightarrow B \\ x \longmapsto f(x) \end{array}$$

domain

codomain

name

⚠ The domain and codomain are part of the definition of a function, that's not just a formula

Nonetheless, you can find question of the type : "What is the domain of $f(z) = \frac{1}{z+1}$?" implicitely, it is the largest set such that the formula makes sense : $\mathbb{C} \setminus \{-1\}$

__Definitions:__ Let $f : A \to B$ be a function.

- the image of $E \subset A$ by f is the set $f(E) := \{ f(x) : x \in E \} \subset B$
- the image of f (or range of f) is $\text{Range}(f) := f(A)$ (all the values reached by f)
- the preimage of $F \subset B$ by f is the set $f^{-1}(F) := \{ x \in A : f(x) \in F \} \subset A$
- the graph of f is the set $\Gamma_f := \{ (x,y) \in A \times B : y = f(x) \}$
- f is injective (or one-to-one) if $\forall x_1, x_2 \in A, \ x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$
  or equivalently by taking the contrapositive, $\forall x_1, x_2 \in A, \ f(x_1) = f(x_2) \Rightarrow x_1 = x_2$
- f is surjective (or onto) if $\forall y \in B, \ \exists x \in A, \ y = f(x)$
- f is bijective if it is injective and surjective, i.e.: $\forall y \in B, \exists ! x \in A, \ y = f(x)$

__Theorem:__ $f : A \to B$ is bijective iff $\exists g : B \longrightarrow A$ s.t. $\begin{cases} \forall x \in A, & g(f(x)) = x \\ \forall y \in B, & f(g(y)) = y \end{cases}$
then g is unique and we say that g is the inverse of f, denoted f^{-1}

In this course, we will focus on functions of the form $f : S \longrightarrow \mathbb{C}$ where $S \subset \mathbb{C}$.
Contrary to a function $f : \mathbb{R} \to \mathbb{R}$, the graph is difficult to visualize : it is a 2 dimensional object lying in a 4 dimensional space.

Limits of complex functions

<u>Definition</u>: Let $S \subset \mathbb{C}$. We say that $z_0 \in \mathbb{C}$ is a limit point of S if

$$\forall \delta > 0, \exists z \in S, \ 0 < |z - z_0| < \delta$$

or geometrically: $\forall \delta > 0, (D_\delta(z_0) \cap S) \setminus \{z_0\} \neq \emptyset$

<u>Theorem</u>: z_0 is a limit point of $S \Leftrightarrow z_0 \in \overline{S \setminus \{z_0\}}$

<u>Definition</u>: $S \subset \mathbb{C}$, $f : S \to \mathbb{C}$, z_0 a limit point of S, $\ell \in \mathbb{C}$

We say that ℓ is the limit of f at z_0 denoted $\lim_{z \to z_0} f(z) = \ell$ if

$$\forall \varepsilon > 0, \exists \delta > 0, \forall z \in S, \ 0 < |z - z_0| < \delta \Rightarrow |f(z) - \ell| < \varepsilon$$

<u>Eg</u>: $\lim_{z \to i} z^2 + 1 = 0$

Δ Let $\varepsilon > 0$, take $\delta = \min\left(1, \frac{\varepsilon}{3}\right)$ then, for $z \in \mathbb{C}$, $|z - i| < \delta \Rightarrow |z + i| < 3$

and $|z^2 + 1| = |z + i||z - i| \leq 3\delta < \varepsilon$ $\square$

<u>Definition</u>: $S \subset \mathbb{C}$ unbounded, $f : S \to \mathbb{C}$, $\ell \in \mathbb{C}$.

We say that f tends to ℓ at ∞ denoted $\lim_{z \to \infty} f(z) = \ell$ if

$$\forall \varepsilon > 0, \exists M > 0, \forall z \in S, |z| > M \Rightarrow |f(x) - \ell| < \varepsilon$$

<u>Eg</u>: $\lim_{z \to \infty} \frac{1}{z} = 0$

Δ Let $\varepsilon > 0$, take $M = \frac{1}{\varepsilon}$ then $|z| > \frac{1}{\varepsilon} \Rightarrow \left|\frac{1}{z}\right| = \frac{1}{|z|} < \varepsilon$ $\square$

<u>Eg</u>: $\lim_{z \to 0} \frac{z}{\bar{z}}$ DNE, $\lim_{z \to \infty} \frac{z}{\bar{z}}$ DNE

$z = re^{i\theta}$, $\frac{z}{\bar{z}} = e^{2i\theta}$ so the "limit depends on the direction"

<u>Proposition</u>: $\lim_{z \to z_0} f(z) = L$ if and only if $\begin{cases} \lim_{z \to z_0} \operatorname{Re}(f) = \operatorname{Re}(L) \\ \lim_{z \to z_0} \operatorname{Im}(f) = \operatorname{Im}(L) \end{cases}$

$(z_0 \in \mathbb{C} \cup \{\infty\})$

<u>Proposition</u>: assume that $\lim_{z \to z_0} f(z) = L$, $\lim_{z \to z_0} g(z) = M$, $\alpha \in \mathbb{C}$, $\beta \in \mathbb{C}$ then

Here $z_0 \in \mathbb{C} \cup \{\infty\}$

- $\lim_{z \to z_0} (\alpha f + \beta g) = \alpha L + \beta M$
- $\lim_{z \to z_0} fg = LM$
- $\lim_{z \to z_0} \bar{f} = \bar{L}$
- $\lim_{z \to z_0} |f| = |L|$
- if $M \neq 0$, $\lim_{z \to z_0} \frac{f}{g} = \frac{L}{M}$

Continuity:

Definition: $S \subset \mathbb{C}$, $f: S \longrightarrow \mathbb{C}$, $z_0 \in S$

⚠ there is no "$0<$"

We say that f is continuous at z_0 if $\forall \varepsilon > 0, \exists \delta > 0, \forall z \in S, |z - z_0| < \delta \Rightarrow |f(z) - f(z_0)| < \varepsilon$

Note that if z_0 is a limit point of S then f is continuous at z_0 iff $\lim_{z \to z_0} f(z) = f(z_0)$

And we say that f is continuous if it is continuous at every $z_0 \in S$

Theorem: $S \subset \mathbb{C}$, $f: S \longrightarrow \mathbb{C}$

The following are equivalent:
1. f is continuous
2. $\forall \mathcal{U} \subset \mathbb{C}$ open, $\exists \mathcal{V} \subset \mathbb{C}$ open s.t. $f^{-1}(\mathcal{U}) = \mathcal{V} \cap S$
3. $\forall \mathcal{C} \subset \mathbb{C}$ closed, $\exists \mathcal{D} \subset \mathbb{C}$ closed s.t. $f^{-1}(\mathcal{C}) = \mathcal{D} \cap S$

You can skip this proof.

$\triangleright$ $1 \Rightarrow 2$: let $\mathcal{U} \subset \mathbb{C}$ open and $z_0 \in f^{-1}(\mathcal{U})$ then $f(z_0) \in \mathcal{U}$ open
hence $\exists \varepsilon > 0$, $D_\varepsilon(f(z_0)) \subset \mathcal{U}$
by continuity, $\exists \delta_{z_0} > 0$, $f(D_{\delta_{z_0}}(z_0)) \subset D_\varepsilon(f(z_0)) \subset \mathcal{U}$
We can take $\mathcal{V} = \bigcup_{z_0 \in f^{-1}(\mathcal{U})} D_{\delta_{z_0}}(z_0)$

$2 \Rightarrow 1$: let $z_0 \in S$, let $\varepsilon > 0$, then $D_\varepsilon(f(z_0))$ is open, so by assumption
$\exists \mathcal{V} \subset \mathbb{C}$ s.t. $f^{-1}(D_\varepsilon(f(z_0))) = \mathcal{V} \cap S$.
Since $\mathcal{V}$ is open, and $z_0 \in \mathcal{V}$, $\exists \delta > 0$ s.t. $D_\delta(z_0) \subset \mathcal{V}$
then, for $z \in S$, $|z - z_0| < \delta \Rightarrow z \in D_\delta(z_0) \subset \mathcal{V}$

$$\Rightarrow z \in \mathcal{V} \cap S = f^{-1}(D_\varepsilon(f(z_0)))$$
$$\Rightarrow f(z) \in D_\varepsilon(f(z_0))$$
$$\Rightarrow |f(z) - f(z_0)| < \varepsilon$$

so f is continuous at z_0

$2 \Leftrightarrow 3$: $f^{-1}(\mathbb{C} \setminus \mathcal{U}) = S \setminus f^{-1}(\mathcal{U}) = S \cap (f^{-1}(\mathcal{U}))^c = S \cap (\mathcal{V}^c)$ $\square$

Proposition: f is continuous at z_0 if and only if $\Re(f)$ and $\Im(f)$ are.

Proposition: If f and g are continuous at z_0 and $\alpha, \beta \in \mathbb{C}$ then

- $\alpha f + \beta g$
- fg
- $\bar{f}$
- $|f|$
- $\frac{f}{g}$ (assuming $g(z_0) \neq 0$)

are continuous at z_0

Sequences:

Definition: A sequence in $\mathbb{C}$ is a function $\{k\in\mathbb{N} : k\geq k_0\} \longrightarrow \mathbb{C}$, $k \longmapsto z_k$

It is denoted by $(z_k)_{k\geq k_0}$

Definition: We say that $(z_k)_{k\geq k_0}$ converges to $L\in\mathbb{C}$, denoted by $\lim\limits_{k\to+\infty} z_k = L$

if $\forall \varepsilon>0, \exists K\in\mathbb{N}, \forall k\in\mathbb{N}_{\geq k_0}, \ k\geq K \Rightarrow |z_k - L| < \varepsilon$

Proposition: $\lim\limits_{k\to+\infty} z_k = L \Leftrightarrow \lim\limits_{k\to k_0} \underbrace{|z_k - L|}_{\text{real-valued sequence}} = 0$

Proposition: $\lim\limits_{k\to+\infty} z_k = L \Leftrightarrow \begin{cases} \lim\limits_{k\to+\infty} \mathrm{Re}(z_k) = \mathrm{Re}(L) \\ \lim\limits_{k\to+\infty} \mathrm{Im}(z_k) = \mathrm{Im}(L) \end{cases}$

$\Delta \ \Rightarrow \ |\mathrm{Re}(z_k) - \mathrm{Re}(L)| = |\mathrm{Re}(z_k - L)|$

$$\leq \left(\mathrm{Re}(z_k-L)^2 + \mathrm{Im}(z_k-L)^2\right)^{1/2} = |z_k - L| \xrightarrow[k\to+\infty]{} 0$$

Hence $\lim\limits_{k\to+\infty} \mathrm{Re}(z_k) = \mathrm{Re}(L)$ and similarly for $\lim\limits_{k\to+\infty} \mathrm{Im}(z_k) = \mathrm{Im}(L)$

$\Leftarrow \ |z_m - L| = |(\mathrm{Re}(z_k) - \mathrm{Re}(L)) + i(\mathrm{Im}(z_k) - \mathrm{Im}(L))|$

$$\leq |\mathrm{Re}(z_k) - \mathrm{Re}(L)| + |\mathrm{Im}(z_k) - \mathrm{Im}(L)| \xrightarrow[k\to+\infty]{} 0$$

$\square$

Proposition: If $\lim\limits_{k\to+\infty} z_k = L \neq 0$ then $\exists K\in\mathbb{N}$ s.t. $\forall k\geq K$, $z_k\neq 0$

Δ Homework $\square$

Proposition: Assume that $\lim\limits_{k\to+\infty} z_k = L$ and $\lim\limits_{k\to+\infty} w_k = M$ then

$\lim\limits_{k\to+\infty} (z_k + w_k) = L+M$

$\lim\limits_{k\to+\infty} z_k w_k = LM$

Assuming $M\neq 0$, $\lim\limits_{k\to+\infty} \dfrac{z_k}{w_k} = \dfrac{L}{M}$

$\lim\limits_{k\to+\infty} \overline{z_k} = \overline{L}$

$\lim\limits_{k\to+\infty} |z_k| = |L|$

Δ Homework $\square$

Series:

Definition: let $(z_k)_k$ be a sequence

The n-th partial sum of the series associated to $(z_k)_k$ is $S_n = \sum_{k=0}^{n} z_k$

We say that the series $\sum_{k=0}^{\infty} z_k$ is convergent if $\lim_{n\to+\infty} S_n$ exists and then we denote: $\sum_{k=0}^{+\infty} z_k = \lim_{n\to+\infty} \sum_{k=0}^{n} z_k$

Otherwise we say that the series $\sum_{k=0}^{+\infty} z_k$ is divergent

⚠ "Convergence" doesn't depend on the starting term: $\sum_{k=0}^{\infty} z_k$ CV $\Leftrightarrow \sum_{k=k_0}^{+\infty} z_k$ CV

indeed $\sum_{k=0}^{n} z_k = \sum_{k=0}^{k_0-1} z_k + \sum_{k=k_0}^{n} z_k$

Proposition: $\sum_{k=0}^{+\infty} z_k$ CV $\Rightarrow \lim_{k\to+\infty} z_k = 0$

⚠ The converse is false: $\frac{1}{k} \to 0$ but $\sum_{k=1}^{\infty} \frac{1}{k}$ DV

Δ $z_n = \sum_{k=0}^{n} z_k - \sum_{k=0}^{n-1} z_k \xrightarrow[n\infty]{} \sum_{k=0}^{+\infty} z_k - \sum_{k=0}^{+\infty} z_k = 0$ $\square$

The contrapositive can be very useful: $\lim_{k\to+\infty} z_k \neq 0 \Rightarrow \sum_{k=0}^{+\infty} z_k$ is divergent.

Proposition: Assume that $\sum_{k=0}^{\infty} z_k$ and $\sum_{k=0}^{\infty} w_k$ are CV and $\alpha \in \mathbb{C}$ then $\sum (\alpha z_k + w_k)$ is CV and $\sum_{k=0}^{\infty} (\alpha z_k + w_k) = \alpha \sum_{k=0}^{\infty} z_k + \sum_{k=0}^{\infty} w_k$

Δ Homework $\square$

Definition: We say that $\sum_{k=0}^{\infty} z_k$ is absolutely convergent if $\sum_{k=0}^{\infty} |z_k|$ is CV

Prop: $\sum z_k$ ACV $\Rightarrow \sum z_k$ CV

⚠ the converse is false: $\sum_{k=1}^{\infty} \frac{(-1)^k}{k}$ is CV but not ACV

You can skip the proof

Proof 1: Let $\varepsilon > 0$. Since $\sum_{k=0}^{n} |z_k|$ is convergent then it is a Cauchy sequence

so $\exists K > 0$ s.t. $q \geq p \geq K \Rightarrow \sum_{k=p+1}^{q} |z_k| = \sum_{k=0}^{q} |z_k| - \sum_{k=0}^{p} |z_k| < \varepsilon$

then for $q \geq p \geq K$: $\left| \sum_{k=0}^{q} z_k - \sum_{k=0}^{p} z_k \right| = \left| \sum_{k=p+1}^{q} z_k \right| \leq \sum_{k=p+1}^{q} |z_k| < \varepsilon$

hence $S_n = \sum_{k=0}^{n} z_k$ is a Cauchy sequence and hence CV

Proof 2: $\sum |\mathrm{Re}(z_k)| \leq \sum |z_k|$

hence $\mathrm{Re}(z_k)$ is ACV and hence CV, similarly for $\mathrm{Im}(z_k)$

thus $z_k = \mathrm{Re}(z_k) + i\,\mathrm{Im}(z_k)$ is CV $\square$

Homework: study $\sum_{k=k_0}^{+\infty} z^k$ (geometric series)

D'Alembert's ratio test:

We consider a serie $\sum_{k=0}^{+\infty} z_k$ and we assume that $\ell = \lim_{k\to+\infty} \left|\frac{z_{k+1}}{z_k}\right|$ exists.

- If $\ell < 1$ then $\sum z_k$ is absolutely convergent
- If $\ell > 1$ then $\sum z_k$ is divergent

Δ • $\ell < 1$: we take r s.t. $\ell < r < 1$ then $\exists K \in \mathbb{N}$ s.t. $k \geqslant K \Rightarrow \left|\frac{z_{k+1}}{z_k}\right| < r$

then $\sum_{k=K}^{m} |z_k| \leqslant \sum_{k=K}^{m} |z_K| r^{k-K} = \frac{|z_K|}{r^K} \sum_{k=K}^{m} r^k$ CV (geometric series with $|r| < 1$)

• $\ell > 1$: $\exists K \in \mathbb{N}$, $k \geqslant K \Rightarrow \left|\frac{z_{k+1}}{z_k}\right| > 1$ then $|z_{k+1}| > |z_k|$

then $|z_k|$ is an eventually increasing sequence, so $|z_k| \not\longrightarrow 0$

hence $\sum z_k$ is divergent and $z_k \not\longrightarrow 0$

$\square$

Remark: We can't conclude when $\ell = 1$:

$\sum_{k=1}^{+\infty} 1$ is divergent

$\sum_{k=1}^{+\infty} \frac{(-1)^k}{k^2}$ is absolutely convergent

$\sum_{k=1}^{+\infty} \frac{(-1)^k}{k}$ is convergent but not absolutely convergent.