University of Toronto – MAT334H1-F – LEC0101

Complex Variables

19 – The Schwarz–Christoffel formula

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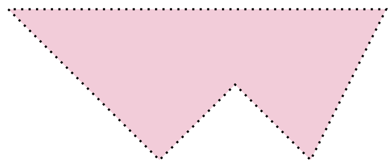
1 Informal introduction

According to the Riemann mapping theorem, given $U \subsetneq \mathbb{C}$ a simply connected open subset which is not $\mathbb{C}$, there exists a biholomorphism mapping $f : \mathbb{H} \to U$ where $\mathbb{H} = \{z \in \mathbb{C} : \Im(z) > 0\}$.
However the statement doesn't give an explicit expression for f (which, in general, can be quite complicated: for instance U can be the interior of the Koch snowflake or the set defined in the last example of the previous chapter, in these cases the behavior of f around the boundary of U should be quite complicated).

In this chapter we are going to focus on the special case where U is the interior of a polygon.



The Schwarz–Christoffel formula is a differential equation satisfied by such a f.
Although this equation doesn't admit a closed solution in general, it can be used in some special cases to obtain an explicit f.

2 A preliminary remark

Let $U \subset \mathbb{C}$ be open, $f : U \to \mathbb{C}$ be holomorphic/analytic and $\gamma : [a, b] \to \mathbb{C}$ be a smooth curve entirely included in U such that $\forall t \in (a, b),\ \gamma'(t) \neq 0$.

Then

$$\arg((f \circ \gamma)'(t)) = \arg(\gamma'(t)) + \arg(f'(\gamma(t)))$$

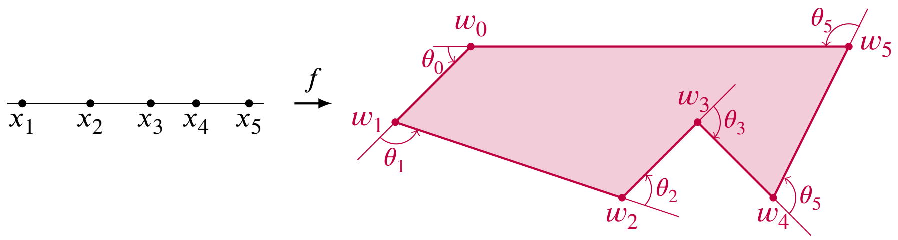
If γ runs through the real axis from the left to the right then $\gamma'(t) \in (0, \infty)$ so that

$$\arg((f \circ \gamma)'(t)) = \arg(f'(\gamma(t)))$$

Particularly, if $\arg(f'(z))$ is constant then $f \circ \gamma$ is also a segment line.

3 The intuition behind the statement

Let's fix a simple polygon P with consecutive vertices $w_0, \ldots, w_n$ (i.e. P has $n+1$ sides) and let $\theta_0, \ldots, \theta_n \in (-\pi, \pi)$ be the external angles at the vertices.



Then $\theta_1 + \theta_2 + \cdots + \theta_n = 2\pi$.

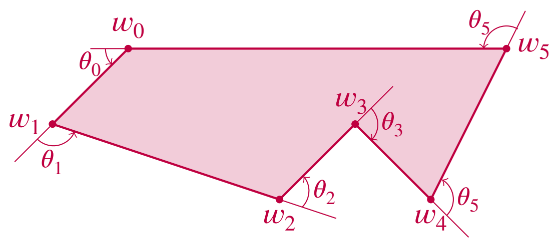
The idea is to place $x_1 < \cdots < x_n \in \mathbb{R}$ such that x_i will be mapped to w_i, i.e. $w_i = f(x_i)$.
We set $x_0 = \infty \in \mathbb{R} \cup \{\infty\}$, i.e. $w_0 = \lim_{x \to \pm\infty} f(x)$.
Obviously, $\arg(f'(x))$ should *jump* each time x passes through a x_i.

We set $\alpha_i = -\frac{\theta_i}{\pi}$ then $\alpha_i \in (-1, 1)$ and $\alpha_1 + \alpha_2 + \cdots + \alpha_n = -2$.
Let f be a function such that $f(x_i) = w_i$ and $f'(z) = A(z - x_1)^{\alpha_1} \cdots (z - x_n)^{\alpha_n}$. Then

$$\begin{aligned}
x_n < x &\implies \arg(f'(x)) = \arg A \\
x_{n-1} < x < x_n &\implies \arg(f'(x)) = \arg A + \pi\alpha_n \\
x_{n-2} < x < x_{n-1} &\implies \arg(f'(x)) = \arg A + \pi\alpha_{n-1} + \pi\alpha_n \\
&\vdots \\
x < x_1 &\implies \arg(f'(x)) = \arg A + \pi\alpha_1 + \pi\alpha_2 + \cdots + \pi\alpha_n
\end{aligned}$$

4 The statement

Theorem 1 (Schwarz–Christoffel)**.** *Let P be a simple polygon with consecutive vertices $w_0, \ldots, w_n$ (i.e. P has $n+1$ sides) and let $\theta_0, \ldots, \theta_n \in (-\pi, \pi)$ be the external angles at the vertices.*



We set $\alpha_i = -\frac{\theta_i}{\pi}$ so that $\alpha_i \in (-1, 1)$ and $\alpha_1 + \alpha_2 + \cdots + \alpha_n = -2$.
Then there exists an injective holomorphic function f from the Poincaré half-plane $\mathbb{H} := \{z \in \mathbb{C} \ : \ \mathfrak{I}(z) > 0\}$ onto the interior of P satisfying

$$f'(z) = A(z - x_1)^{\alpha_1} \cdots (z - x_n)^{\alpha_n}$$

where $x_1 < x_2 < \cdots < x_n$ are some real numbers and $A \in \mathbb{C} \setminus \{0\}$.