CONFORMAL MAPPINGS

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Definition: conformal mapping

Let $U \subset \mathbb{C}$ be open and $f : U \to \mathbb{C}$. Let $z_0 \in U$. Assume that there exists $r > 0$ such that $D_r(z_0) \subset U$ and $\forall z \in D_r(z_0) \setminus \{z_0\}$, $f(z) \neq f(z_0)$.
We say that f is **conformal at** z_0 if f preserves angles at z_0, i.e.

$$\lim_{\varepsilon \to 0^+} e^{-i\theta} \frac{f(z_0 + \varepsilon e^{i\theta}) - f(z_0)}{|f(z_0 + \varepsilon e^{i\theta}) - f(z_0)|}$$

exists and doesn't depend on $\theta \in \mathbb{R}$.

We say that f is conformal if it is conformal at every $z \in U$.

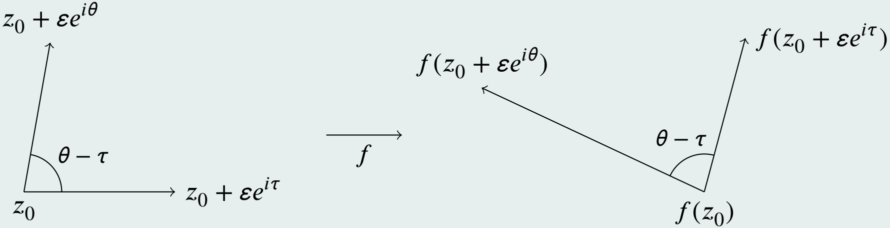
Conformal mappings – 2

Intuition

In the previous definition,

$$\frac{f(z_0+\varepsilon e^{i\theta})-f(z_0)}{|f(z_0+\varepsilon e^{i\theta})-f(z_0)|}\in S^1$$

is the direction from $f(z_0)$ to $f(z_0+\varepsilon e^{i\theta})$.
Hence the the definition means that the oriented angle of the images of two rays originated from z_0 is the same as the oriented angle of these two rays.



Conformal mappings – 3

Example

Set $f(z) = az$ where $a \in \mathbb{C} \setminus \{0\}$. Then f is conformal at every $z_0 \in \mathbb{C}$, indeed

$$\lim_{\varepsilon \to 0^+} e^{-i\theta} \frac{f(z_0 + \varepsilon e^{i\theta}) - f(z_0)}{|f(z_0 + \varepsilon e^{i\theta}) - f(z_0)|} = \frac{a}{|a|}$$

exists and doesn't depend on $\theta \in \mathbb{R}$.

Example

Set $f(z) = z^2$. Then f is conformal at every $z_0 \in \mathbb{C} \setminus \{0\}$, indeed

$$\lim_{\varepsilon \to 0^+} e^{-i\theta} \frac{f(z_0 + \varepsilon e^{i\theta}) - f(z_0)}{|f(z_0 + \varepsilon e^{i\theta}) - f(z_0)|} = \frac{z_0}{|z_0|}$$

exists and doesn't depend on $\theta \in \mathbb{R}$.

However f is not conformal at 0 since $\lim_{\varepsilon \to 0^+} e^{-i\theta} \frac{f(\varepsilon e^{i\theta}) - f(0)}{|f(\varepsilon e^{i\theta}) - f(0)|} = e^{i\theta}$ depends on $\theta \in \mathbb{R}$.

Connection with holomorphicity/analiticity – 1

Theorem

Let $U \subset \mathbb{C}$ be open and $f : U \to \mathbb{C}$.

1. If f is holomorphic/analytic at z_0 and if $f'(z_0) \neq 0$ then f is conformal at z_0.
2. Conversely, if $\tilde{f} : \tilde{U} \to \mathbb{R}^2$ is $\mathbb{R}$-differentiable at (x_0, y_0) and $\operatorname{Jac}_{\tilde{f}}(x_0, y_0) \neq \mathbf{0}$ and f is conformal at $z_0 = x_0 + iy_0$ then f is holomorphic at z_0 and $f'(z_0) \neq 0$.

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Proof.

1. $$\lim_{\varepsilon \to 0^+} e^{-i\theta} \frac{f(z_0 + \varepsilon e^{i\theta}) - f(z_0)}{|f(z_0 + \varepsilon e^{i\theta}) - f(z_0)|} = \lim_{\varepsilon \to 0^+} \frac{f(z_0 + \varepsilon e^{i\theta}) - f(z_0)}{\varepsilon e^{i\theta}} \frac{|\varepsilon e^{i\theta}|}{|f(z_0 + \varepsilon e^{i\theta}) - f(z_0)|} = \frac{f'(z_0)}{|f'(z_0)|}$$
   exists and doesn't depend on $\theta \in \mathbb{R}$.

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Proof.

1. $\displaystyle \lim_{\varepsilon \to 0^+} e^{-i\theta} \frac{f(z_0 + \varepsilon e^{i\theta}) - f(z_0)}{|f(z_0 + \varepsilon e^{i\theta}) - f(z_0)|} = \lim_{\varepsilon \to 0^+} \frac{f(z_0 + \varepsilon e^{i\theta}) - f(z_0)}{\varepsilon e^{i\theta}} \frac{|\varepsilon e^{i\theta}|}{|f(z_0 + \varepsilon e^{i\theta}) - f(z_0)|} = \frac{f'(z_0)}{|f'(z_0)|}$
   exists and doesn't depend on $\theta \in \mathbb{R}$.

2. Since $\operatorname{Jac}_{\tilde{f}}(x_0, y_0) \neq \mathbf{0}$ we know that $f(z_0 + h) = f(z_0) + \alpha h + \beta \bar{h} + o(|h|)$ where $(\alpha, \beta) \neq (0, 0)$.
   Since $\displaystyle \lim_{\varepsilon \to 0^+} e^{-i\theta} \frac{f(z_0 + \varepsilon e^{i\theta}) - f(z_0)}{|f(z_0 + \varepsilon e^{i\theta}) - f(z_0)|} = \frac{\alpha + \beta e^{-2i\theta}}{|\alpha + \beta e^{-2i\theta}|}$ doesn't depend on $\theta \in \mathbb{R}$, we obtain that $\beta = 0$ so that $f(z_0 + h) = f(z_0) + \alpha h + o(|h|)$, i.e. f is holomorphic at z_0 and $f'(z_0) = \alpha \neq 0$.

Connection with holomorphicity/analiticity – 2

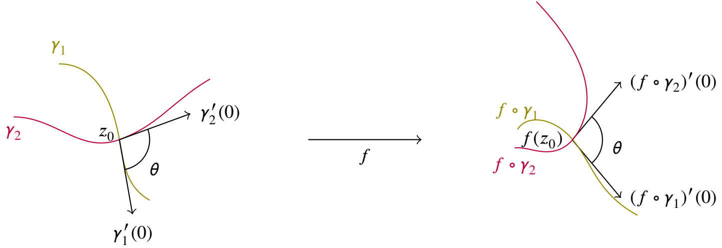
Corollary

Let $U \subset \mathbb{C}$ be open and $z_0 \in U$. Let $f : U \to \mathbb{C}$ be holomorphic/analytic.
Then f is conformal at z_0 if and only if $f'(z_0) \neq 0$.

Here is another geometric interpretation of conformality for f holomorphic at z_0 with $f'(z_0) \neq 0$.
Let $\gamma_1, \gamma_2 : (-1, 1) \to \mathbb{C}$ be two smooth curves such that $z_0 = \gamma_1(0) = \gamma_2(0)$ and $\gamma_i'(0) \neq 0$.
Then f being conformal at z_0 means that for any such curves, the oriented angles $(\gamma_1'(0), \gamma_2'(0))$ and $((f \circ \gamma_1)'(0), (f \circ \gamma_2)'(0))$ coincide,

$$\text{i.e} \quad \arg(\gamma_2'(0)) - \arg(\gamma_1'(0)) \equiv \arg((f \circ \gamma_2)'(0)) - \arg((f \circ \gamma_1)'(0)) \mod 2\pi$$

Indeed, by the chain rule $\arg((f \circ \gamma_i)'(0)) \equiv \arg(f'(z_0)) + \arg(\gamma_i'(0)) \mod 2\pi$ if $f'(z_0) \neq 0$.



A global result – 1

Theorem

Let $U \subset \mathbb{C}$ be open. Let $f : U \to \mathbb{C}$ be holomorphic/analytic.
If f is injective on U then $\forall z \in U,\ f'(z) \neq 0$.

A global result – 1

Theorem

Let $U \subset \mathbb{C}$ be open. Let $f : U \to \mathbb{C}$ be holomorphic/analytic.
If f is injective on U then $\forall z \in U,\ f'(z) \neq 0$.

Proof. We are going to prove the contrapositive.
Assume that there exists $z_0 \in U$ such that $f'(z_0) = 0$.

Then in a neighborhood of z_0 we have that $f(z) - f(z_0) = a_m(z - z_0)^m + \sum_{n=m+1}^{+\infty} a_n(z - z_0)^n$ where $a_m \neq 0$ and $m > 1$ (we know that at least $a_1 = f'(z_0) = 0$).

Hence, for $r > 0$ small enough, we have that $\forall z \in D_r(z_0)$, $\left| \sum_{n=m+1}^{+\infty} a_n(z - z_0)^n \right| < |a_m(z - z_0)^m|$.

Then, by Rouché's theorem, the functions $f(z) - f(z_0)$ and $a_m(z - z_0)^m$ have the same number of zeroes on $D_r(z_0)$ (counted with multiplicies), namely m. Hence there exists $z \in D_r(z_0)$ such that $f(z) - f(z_0) = 0$. ■

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Remark

Once again, this result is false for $\mathbb{R}$-differentiability:
Indeed $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^3$ is $\mathbb{R}$-differentiable and injective, but $f'(0) = 0$.

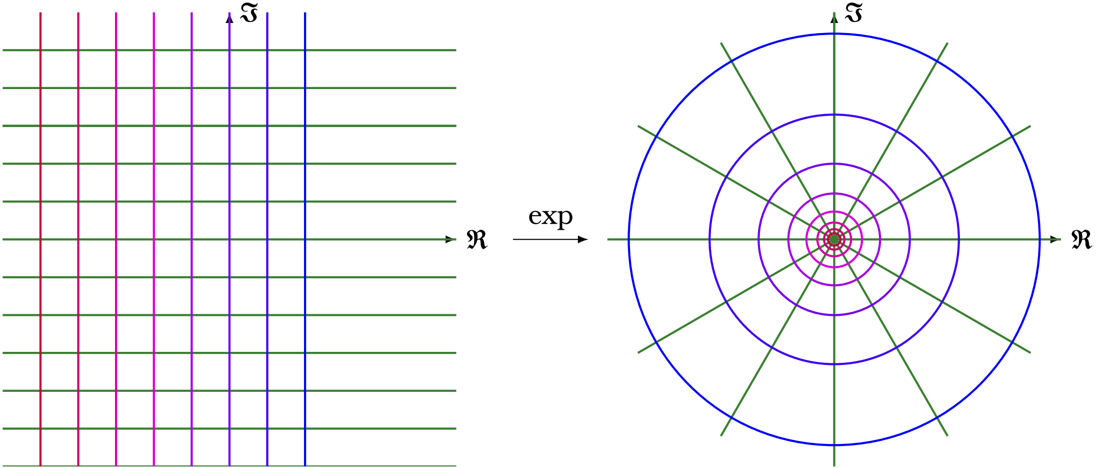
A global result – 2

Corollary

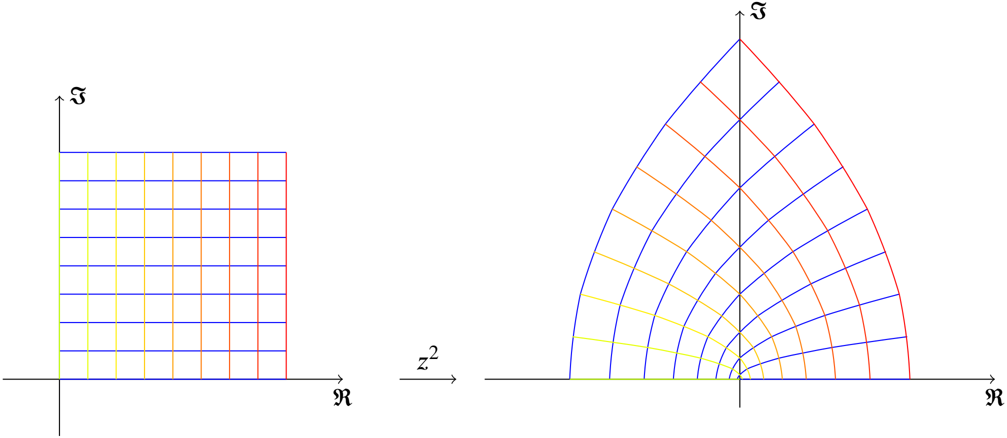
Let $U \subset \mathbb{C}$ be open. Let $f : U \to \mathbb{C}$ be holomorphic/analytic.
If f is injective on U then f is conformal on U.

Proof.
Let $z_0 \in U$. Since f is holomorphic and injective, we know from the previous that $f'(z_0) \neq 0$.
Hence f is conformal at z_0. ∎

Example 1

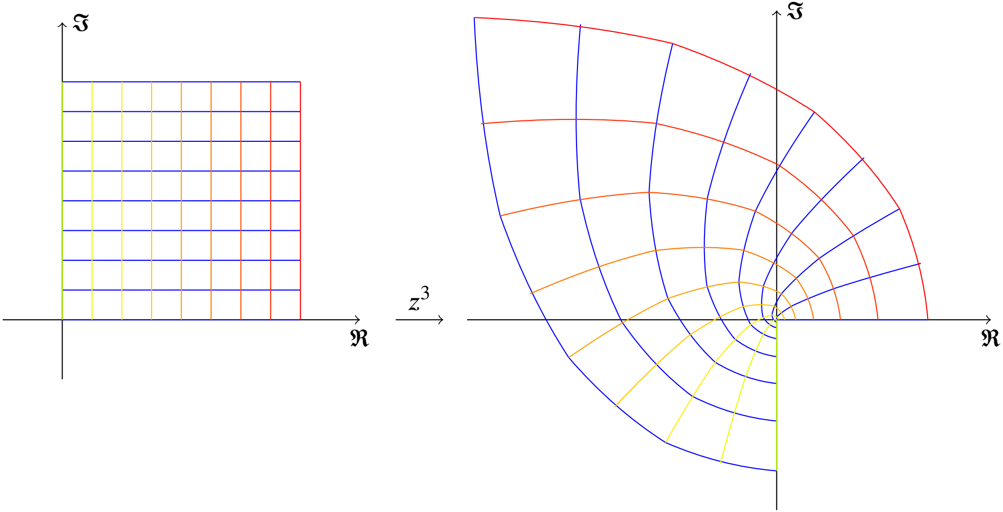


Example 2



Note that $z \mapsto z^2$ is not conformal at 0.

Example 3



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Example 4