1 - The Complex Plane

→ Construction of $\mathbb{C}$ (You can safely skip the beginning)

We define $\mathbb{C}$, the field of complex numbers, as $\mathbb{R}^2$ together with the following operations:

$$\circledast \begin{cases} \bullet \text{ Addition: } (x,y)+(s,t) = (x+s, y+t) \\ \bullet \text{ Multiplication by a real: } \lambda(x,y) = (\lambda x + \lambda y) \text{ where } \lambda \in \mathbb{R} \end{cases}$$

- Multiplication: $(x,y)\cdot(s,t) = (xs - yt, xt + ys)$

→ $\mathbb{C}$ is a 2-dimensional $\mathbb{R}$-vector space for $\circledast$

→ We see $\mathbb{R}$ as a subset of $\mathbb{C}$ by $\begin{array}{ccc} \mathbb{R} & \hookrightarrow & \mathbb{C} \\ x & \longmapsto & (x,0) \end{array}$

For instance 1 stands for $(1,0)$ and π for $(\pi,0)$

→ We set $i = (0,1)$, then note that

- $i^2 = (0,1)\cdot(0,1) = (-1,0) = -1$
- $\langle 1, i \rangle$ is a basis of $\mathbb{C}$ (seen as a $\mathbb{R}$-vector space)

→ Notation: therefore we write $x+iy$ for (x,y)

and this notation is compatible with the usual distributive laws:

$$(x+iy)\cdot(s+it) = xs + ixt + iys + i^2yt = (xs - yt) + i(xt + ys)$$

→ For $z = x+iy \in \mathbb{C}\setminus\{0\}$, set $w = \dfrac{x}{x^2+y^2} - i\dfrac{y}{x^2+y^2}$ then $zw = wz = 1$.

How to remember this formula: $\dfrac{1}{x+iy} = \dfrac{x-iy}{(x+iy)(x-iy)} = \dfrac{x}{x^2+y^2} - i\dfrac{y}{x^2+y^2}$

<u>Theorem</u>: $\mathbb{C}$ is a field, meaning that:

- $\forall z_1, z_2, z_3 \in \mathbb{C}$, $z_1 + (z_2+z_3) = (z_1+z_2) + z_3$
- $\forall z \in \mathbb{C}$, $z + 0 = 0 + z = z$
- $\forall z \in \mathbb{C}$, $z + (-z) = (-z) + z = 0$ where $-(x+iy) = (-x) + i(-y)$
- $\forall z_1, z_2 \in \mathbb{C}$, $z_1 + z_2 = z_2 + z_1$
- $\forall z_1, z_2, z_3 \in \mathbb{C}$, $z_1(z_2 z_3) = (z_1 z_2) z_3$
- $\forall z_1, z_2, z_3 \in \mathbb{C}$, $z_1(z_2+z_3) = z_1 z_2 + z_1 z_3$ and $(z_1+z_2)z_3 = z_1 z_3 + z_2 z_3$
- $\forall z \in \mathbb{C}$, $1\cdot z = z \cdot 1 = z$
- $\forall z \in \mathbb{C}\setminus\{0\}$, $z\cdot z^{-1} = z^{-1}\cdot z = 1$ where $(x+iy)^{-1} = \dfrac{x}{x^2+y^2} - i\dfrac{y}{x^2+y^2}$
- $\forall z_1, z_2 \in \mathbb{C}$, $z_1 z_2 = z_2 z_1$

What should you remember from this page:

$\mathbb{C} = \{x + iy : x, y \in \mathbb{R}\}$ where $i^2 = -1$ and the operations behave as expected

⚠ In the litterature you may see other constructions such as $\mathbb{C} = \mathbb{R}[X]/_{X^2+1}$ or $\mathbb{C} = \left\{ \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \right\}$, they are equivalent but not useful for us.

⚠ The order on $\mathbb{R}$ doesn't extend to an order on $\mathbb{C}$ compatible with $+$ and $\times$:

$\left.\begin{array}{l} \bullet \text{ assume } i > 0 \text{ then } i^2 > 0 \text{, ie } -1 > 0 \\ \bullet \text{ assume } i < 0 \text{ then } i^2 > 0 \text{, ie } -1 > 0 \end{array}\right\}$ contradiction

You should NOT write $z_1 < z_2$ for complex numbers or say that $z \in \mathbb{C}$ is positive.

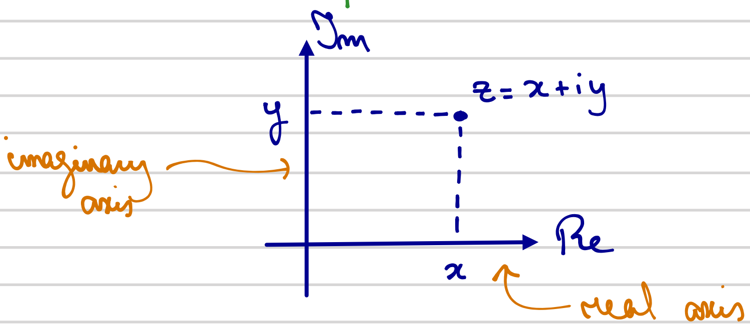
Definition: Given $z = x + iy \in \mathbb{C}$, we define:

$\mathrm{Re}(z) := x$ "the real part of z"

$\mathrm{Im}(z) := y$ "the imaginary part of z"

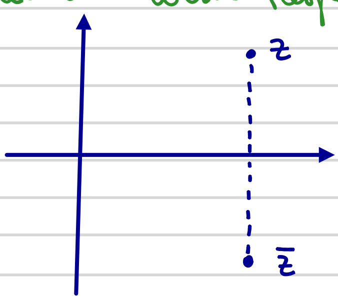
⚠ $\mathrm{Re}(z), \mathrm{Im}(z) \in \mathbb{R}$

→ It may be convenient to geometrically identify $z = x + iy$ with coordinates (x,y) of a point in the Euclidean plane.



Definition: Let $z = x + iy \in \mathbb{C}$. Then the (complex) conjugate of z is $\bar{z} := x - iy$

Geometrically, it is the reflection with respect to the real axis



Proposition: $\bar{\bar{z}} = z$

Proposition: Let $z_1, z_2 \in \mathbb{C}$ then
- $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$
- $\overline{z_1 z_2} = \bar{z}_1 \bar{z}_2$

Proposition: Let $z = x + iy \in \mathbb{C}$ then $z\bar{z} = x^2 + y^2$

Note that $z\bar{z} \in \mathbb{R}$, that's very useful to write a fraction in its canonical form: $\frac{1}{z} = \frac{\bar{z}}{z\bar{z}}$, $\frac{z_1}{z_2} = \frac{z_1 \bar{z}_2}{z_2 \bar{z}_2}$, eg: $\frac{3+4i}{1+i} = \frac{(3+4i)(1-i)}{(1+i)(1-i)} = \frac{7+i}{2} = \frac{7}{2} + i\frac{1}{2}$

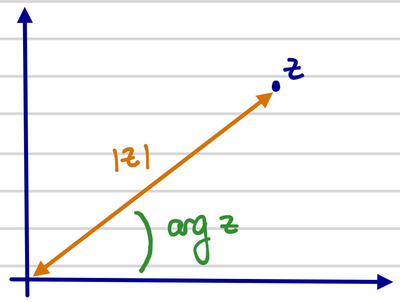
Polar representation:

Definition: For $z = x+iy \in \mathbb{C}$, we define its modulus (or magnitude, or absolute value) by : $|z| := \sqrt{x^2+y^2}$

Note that $|z| \in \mathbb{R}_{\geq 0}$: it coincides with the norm $\|(x,y)\|$.

Proposition:
- $\forall z \in \mathbb{C}$, $|z|^2 = z\bar{z}$
- $\forall z_1, z_2 \in \mathbb{C}$, $|z_1+z_2| \leq |z_1| + |z_2|$ (Triangle inequality)
- $\forall z_1, z_2 \in \mathbb{C}$, $||z_1| - |z_2|| \leq |z_1 - z_2|$
- $\forall z_1, z_2 \in \mathbb{C}$, $|z_1 z_2| = |z_1||z_2|$
- $\forall z_1 \in \mathbb{C}$, $\forall z_2 \in \mathbb{C}\setminus\{0\}$, $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$
- $\forall z \in \mathbb{C}\setminus\{0\}$, $\frac{1}{z} = \frac{\bar{z}}{|z|^2}$
- $\forall z \in \mathbb{C}$, $|\bar{z}| = |z|$
- $\forall z \in \mathbb{C}$, $|z^n| = |z|^n$
- $\forall z \in \mathbb{C}$: $z = 0 \iff |z| = 0$

Theorem: For $z \in \mathbb{C}\setminus\{0\}$, there exists a unique $\theta \in [0, 2\pi)$ such that
$$z = |z|(\cos\theta + i\sin\theta)$$
It is called the principal argument of z and denoted by $\text{Arg}(z) := \theta$.



Some useful identities to find $\theta = \text{Arg}(z)$:

$$\cos\theta = \frac{x}{|z|}$$
$$\sin\theta = \frac{y}{|z|}$$
$$\tan\theta = \frac{y}{x}$$

Notation: if we allow $\theta \in \mathbb{R}$ then it is only defined modulo 2π and we say that $\arg(z) := \theta$ is an argument of z.

⚠ $\text{Arg}(z)$ ⤳ uniquely defined in $[0, 2\pi)$
$\arg(z)$ ⤳ defined only modulo 2π, ie up to some $2n\pi$, $n \in \mathbb{Z}$ (multivalued)

Proposition: $\forall z \in \mathbb{C}\setminus\{0\}$, $\arg(\bar{z}) \equiv -\arg(z) \mod 2\pi$

$\forall z_1, z_2 \in \mathbb{C}\setminus\{0\}$, $\arg(z_1 z_2) \equiv \arg(z_1) + \arg(z_2) \mod 2\pi$

$\forall z_1, z_2 \in \mathbb{C}\setminus\{0\}$, $\arg\left(\frac{z_1}{z_2}\right) \equiv \arg(z_1) - \arg(z_2) \mod 2\pi$

$\forall z \in \mathbb{C}\setminus\{0\}$, $\forall n \in \mathbb{Z}$, $\arg(z^n) \equiv n\arg(z) \mod 2\pi$

⚠ $\text{Arg}(z_1 z_2) \neq \text{Arg}(z_1) + \text{Arg}(z_2)$ in general :

Eg : $z_1 = z_2 = -1$ then $\text{Arg}((-1)(-1)) = \text{Arg}(1) = 0 \neq 2\pi = \text{Arg}(-1) + \text{Arg}(-1)$

Δ Proof of:

• $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$:

Assume that $z_k = r_k(\cos\theta_k + i\sin\theta_k)$ then

$$\begin{aligned} z_1 z_2 &= r_1(\cos\theta_1 + i\sin\theta_1)\, r_2(\cos\theta_2 + i\sin\theta_2) \\ &= r_1 r_2 (\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2 + i\cos\theta_1\sin\theta_2 + i\sin\theta_1\cos\theta_2) \\ &= r_1 r_2 (\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)) \end{aligned}$$

• $\arg(z_1/z_2) = \arg(z_1) - \arg(z_2)$:

$$\arg(z_1) = \arg\left(\frac{z_1}{z_2} z_2\right) = \arg\left(\frac{z_1}{z_2}\right) + \arg(z_2)$$

so $\arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2)$ □

De Moivre's formula

__Theorem__: (De Moivre's formula)

$$\forall \theta \in \mathbb{R}, \forall m \in \mathbb{Z}, \quad (\cos\theta + i\sin\theta)^m = \cos(m\theta) + i\sin(m\theta)$$

Δ • For $m \geq 0$ by induction:

⇝ Base case: $m = 0$

$$(\cos\theta + i\sin\theta)^0 = 1 \overset{\shortparallel}{} \cos(0) + i\sin(0) = 1$$

⇝ Induction step: assume that the formula holds for some $m \geq 0$

$$\begin{aligned} (\cos\theta + i\sin\theta)^{m+1} &= (\cos\theta + i\sin\theta)^m (\cos\theta + i\sin\theta) \\ &= (\cos(m\theta) + i\sin(m\theta))(\cos\theta + i\sin\theta) \\ &= \cos(m\theta)\cos\theta - \sin(m\theta)\sin(\theta) \\ &\quad + i[\sin(m\theta)\cos\theta + \cos(m\theta)\sin\theta] \\ &= \cos((m+1)\theta) + i\sin((m+1)\theta) \end{aligned}$$

• Assume that $m < 0$: $(\cos(\theta) + i\sin(\theta))^m = \frac{1}{(\cos\theta + i\sin\theta)^{-m}} = \frac{1}{\cos(-m\theta) + i\sin(-m\theta)}$

$-m > 0$ so we may apply the above.

complex conjugate $\rightarrow\ = \frac{\cos(-m\theta) - i\sin(-m\theta)}{1} = \cos(m\theta) + i\sin(m\theta)$ □

__Ex__: find formulae for $\cos(3t)$ and $\sin(3t)$ in terms of $\cos t$ and $\sin t$

$$\cos(3t) + i\sin(3t) = (\cos t + i\sin t)^3 = \cos^3 t + 3i\cos^2 t\sin t - 3\cos t\sin^2 t - 3i\sin^3 t$$

so, by identifying the real/imaginary parts:

$$\cos(3t) = \cos^3 t - 3\cos t\sin^2 t$$

$$\sin(3t) = -3\sin^3 t + 3\cos^2 t\sin t$$

Exponential representation

Definition: For $\theta \in \mathbb{R}$ we define: $e^{i\theta} := \cos\theta + i\sin\theta$

Proposition: (Euler's formulae)

$$\mathrm{Re}(e^{i\theta}) = \cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2}$$

$$\mathrm{Im}(e^{i\theta}) = \sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i}$$

Δ Homework $\square$

Notation: then we may lighten the polar representation: $z = |z|\, e^{i\arg(z)}$

Proposition: $\forall \theta_1, \theta_2 \in \mathbb{R}$, $e^{i(\theta_1+\theta_2)} = e^{i\theta_1} e^{i\theta_2}$

Δ Homework $\square$

Remark: De Moivre's formula can now be written as

$$(e^{i\theta})^n = e^{in\theta}$$

Notation: if $z = x+iy \in \mathbb{C}$ then we set $e^z = e^x e^{iy}$

Proposition: $e^{z+z'} = e^z e^{z'}$

Δ Homework $\square$

Ex: linearize $\cos^3 t$

$$\cos^3 t = \left(\frac{e^{it}+e^{-it}}{2}\right)^3 = \frac{1}{8}\left(e^{3it} + 3e^{2it}e^{-it} + 3e^{it}e^{-2it} + e^{-3it}\right)$$

$$= \frac{1}{4}\left(\frac{e^{3it}+e^{-3it}}{2} + 3\left(e^{it}+e^{-it}\right)\right)$$

$$= \frac{1}{4}\left(\cos(3t) + 3\cos(t)\right)$$

Remark: We have more generally that

$$\mathrm{Re}(z) = \frac{z+\bar{z}}{2}$$

$$\mathrm{Im}(z) = \frac{z-\bar{z}}{2i}$$

m-th roots

Definition: Let $z \in \mathbb{C}$. We say that $w \in \mathbb{C}$ is a m-th root of z if $w^m = z$ (here $m \in \mathbb{N}_{>0}$)

Theorem: Let $z \in \mathbb{C} \setminus \{0\}$. Then z admits m m-th roots.

More precisely, if $z = \rho e^{i\theta}$, $\rho > 0$, then the m-th roots of z are

$$\rho^{1/m} e^{i\left(\frac{\theta}{m} + \frac{2k\pi}{m}\right)}, \quad k = 0, \dots, m-1$$

Δ Let $w = re^{it}$ be an m-th root then

$$\rho e^{i\theta} = z = w^m = r^m e^{imt}$$

By identifying the modulus and the argument, we get:

$$\begin{cases} \rho = r^m \\ \theta \equiv mt \mod 2\pi \end{cases}$$

ie
$$\begin{cases} r = \rho^{1/m} \\ \theta = mt + 2k\pi \quad \text{for some } k \in \mathbb{Z} \end{cases}$$

ie
$$\begin{cases} r = \rho^{1/m} \\ t = \frac{\theta}{m} + \frac{2k\pi}{m}, \quad k \in \mathbb{Z} \end{cases}$$

$\square$

Method: how to compute the square roots of $z = a + ib$ without using the polar representation?

Write $w = x + iy$ then

$$w^2 = z \iff \begin{cases} w^2 = z \\ |w|^2 = |z| \end{cases}$$

$$\iff \begin{cases} (x+iy)^2 = a + ib \\ x^2 + y^2 = \sqrt{a^2+b^2} \end{cases}$$

$$\iff \begin{cases} x^2 - y^2 + 2ixy = a + ib \\ x^2 + y^2 = \sqrt{a^2+b^2} \end{cases}$$

$$\iff \begin{cases} x^2 - y^2 = a \\ 2xy = b \\ x^2 + y^2 = \sqrt{a^2+b^2} \end{cases}$$

Homework: Compute the square roots of $8 - 6i$

Homework: ① Find the square roots of $\frac{1+i}{\sqrt{2}}$

② Deduce the values of $\cos(\pi/8)$ and $\sin(\pi/8)$