The complex plane $\mathbb{C}$ – continuation

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Reviews from last lecture

- Website with the material used in this section: `http://uoft.me/MAT334-LEC0101`
- Informally, $\mathbb{C} = \{x + iy \ : \ x, y \in \mathbb{R}\}$ where $i^2 = -1$ with addition and multiplication which behave as you expect them to.
- $\mathbb{C}$ is a 2-dimension $\mathbb{R}$-vector space spanned by $\langle 1, i \rangle$.
- $\mathbb{C}$ is a field and for $z = x + iy \in \mathbb{C} \setminus \{0\}$, $z^{-1} = \dfrac{1}{z} = \dfrac{\bar{z}}{z\bar{z}} = \dfrac{x}{x^2 + y^2} - i\dfrac{y}{x^2 + y^2}$.
- *Real part of* $z = x + iy \in \mathbb{C}$*:* $\Re(z) := x$ (or $\mathrm{Re}(z) := x$).
  Imaginary part of $z = x + iy \in \mathbb{C}$*:* $\Im(z) := y$ (or $\mathrm{Im}(z) := y$).
- Given $z = x + iy \in \mathbb{C}$, we define the *(complex) conjugate of* z by $\bar{z} := x - iy$.
  - $\forall z \in \mathbb{C}$, $\bar{\bar{z}} = z$
  - $\forall z_1, z_2 \in \mathbb{C}$, $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$ and $\overline{z_1 z_2} = \overline{z_1} \cdot \overline{z_2}$
  - Let $z = x + iy \in \mathbb{C}$ then $z\bar{z} = x^2 + y^2$
  - $\forall z \in \mathbb{C}$, $\Re(z) = \dfrac{z + \bar{z}}{2}$ and $\Im(z) = \dfrac{z - \bar{z}}{2i}$

Polar representation: modulus – 1

Definition

The *modulus* (or *magnitude*, or *absolute value*) of $z = x + iy \in \mathbb{C}$ is defined by

$$|z| := \sqrt{x^2 + y^2}$$

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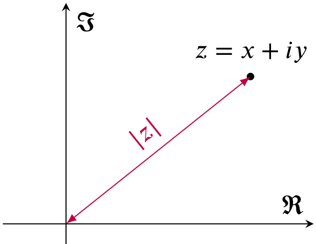
- The modulus of $x + iy$ is the Euclidean norm of (x, y), ie $|x + iy| = \|(x, y)\|$, it is the distance to the origin.



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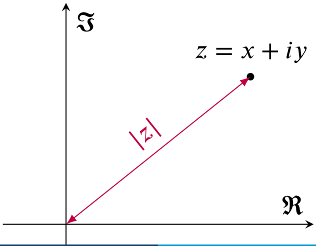
- For a real (i.e. $\Im(z) = 0$), it coincides with the usual absolute value: $|x + i0| = |x|$



Polar representation: modulus – 2

Properties of the modulus

- $\forall z \in \mathbb{C},\ z = 0 \Leftrightarrow |z| = 0$
- $\forall z \in \mathbb{C},\ |z|^2 = z\bar{z} \quad \left(\text{or } |z| = \sqrt{z\bar{z}}\right)$
- $\forall z_1, z_2 \in \mathbb{C},\ |z_1 + z_2| \leq |z_1| + |z_2|$ *(Triangle inequality)*
- $\forall z_1, z_2 \in \mathbb{C},\ \big||z_1| - |z_2|\big| \leq |z_1 - z_2|$ *(Reverse triangle inequality)*
- $\forall z_1, z_2 \in \mathbb{C},\ |z_1 z_2| = |z_1||z_2|$
- $\forall z_1 \in \mathbb{C},\ \forall z_2 \in \mathbb{C} \setminus \{0\},\ \left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$
- $\forall z \in \mathbb{C},\ \forall n \in \mathbb{Z},\ |z^n| = |z|^n$
- $\forall z \in \mathbb{C},\ |\bar{z}| = |z|$
- $\forall z \in \mathbb{C} \setminus \{0\},\ \frac{1}{z} = \frac{\bar{z}}{|z|^2}$

In general[1], $|z_1 + z_2| \neq |z_1| + |z_2|$. For example $|1+i| = \sqrt{2} \neq 2 = |1| + |i|$.

[1] There is equality if and only if there exists $\lambda \in \mathbb{R}_{\geq 0}$ such that $z_1 = \lambda z_2$ or $z_2 = \lambda z_1$ (Homework).

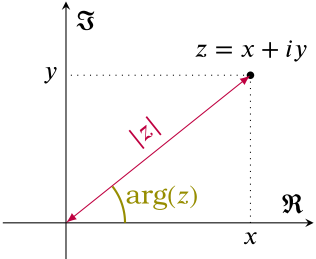
Polar representation: argument – 1

Theorem

For $z \in \mathbb{C} \setminus \{0\}$, there exists a unique $\theta \in [0, 2\pi)$ such that $z = |z|(\cos\theta + i\sin\theta)$.

It is called the *principal argument of* z and denoted by $\mathrm{Arg}(z)$.

Beware: the argument is only defined for $z \neq 0$.
The choice of the interval $[0, 2\pi)$ is not that important, we could have picked $[-\pi, \pi)$ or any other half-open interval of length 2π. In practice, it is common to pick the interval simplifying the computations.



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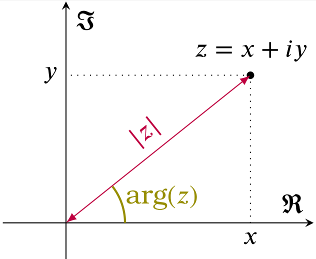
Beware – the cap is important!

If we allow $\theta \in \mathbb{R}$, then it is only defined modulo 2π and we say that θ is **an** argument of z.
Then we use the notation $\arg(z)$ (only defined up to 2π, i.e. $\arg(z) = \theta + 2\pi n$ for some $n \in \mathbb{Z}$).



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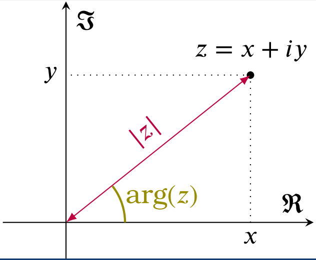

Some useful formulae:

$$\cos\theta = \frac{x}{|z|}$$

$$\sin\theta = \frac{y}{|z|}$$

$$\tan\theta = \frac{y}{x}$$

Polar representation: argument – 2

Properties of the argument

- $\forall z \in \mathbb{C} \setminus \{0\}, \arg\left(\overline{z}\right) \equiv -\arg(z) \mod 2\pi$
- $\forall z_1, z_2 \in \mathbb{C} \setminus \{0\}, \arg(z_1 z_2) \equiv \arg(z_1) + \arg(z_2) \mod 2\pi$
- $\forall z_1, z_2 \in \mathbb{C} \setminus \{0\}, \arg\left(\frac{z_1}{z_2}\right) \equiv \arg(z_1) - \arg(z_2) \mod 2\pi$
- $\forall z \in \mathbb{C} \setminus \{0\}, \forall n \in \mathbb{Z}, \arg\left(z^n\right) \equiv n \arg(z) \mod 2\pi$

Again, the cap and (more especially) the modulo are important

In general $\operatorname{Arg}(z_1 z_2) \neq \operatorname{Arg}(z_1) + \operatorname{Arg}(z_2)$.
Indeed, for $z_1 = z_2 = -1$, $\operatorname{Arg}(z_1 z_2) = \operatorname{Arg}(1) = 0 \neq 2\pi = \operatorname{Arg}(z_1) + \operatorname{Arg}(z_1)$.

Homework

Write in the form $x + iy$: the complex number of modulus 3 and argument $\frac{\pi}{3}$.

Write in polar representation (what are the modulus and argument?): $\frac{\sqrt{6} - i\sqrt{2}}{2}$

De Moivre's formula

Theorem: De Moivre's formula

$$\forall \theta \in \mathbb{R},\ \forall n \in \mathbb{Z},\ (\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta)$$

Homework

Find formulae respectively for $\cos(3t)$ and $\sin(3t)$ in terms of $\cos(t)$ and $\sin(t)$.

Exponential representation – 1

Definition

For $\theta \in \mathbb{R}$, we set $e^{i\theta} := \cos\theta + i\sin\theta$.

Then we may lighten the notations for the polar representation and De Moivre's formula:

$$z = |z|e^{i\arg(z)} \qquad \text{and} \qquad \left(e^{i\theta}\right)^n = e^{in\theta}$$

Homework

Prove that $\forall \theta_1, \theta_2 \in \mathbb{R}$, $e^{i(\theta_1+\theta_2)} = e^{i\theta_1}e^{i\theta_2}$

Definition

For $x, y \in \mathbb{R}$, we set $e^{x+iy} := e^x e^{iy}$.

Homework

Prove that $\forall z_1, z_2 \in \mathbb{C}$, $e^{z_1+z_2} = e^{z_1}e^{z_2}$.

Exponential representation – 2

Proposition: Euler's formulae

$$\Re\left(e^{i\theta}\right) = \cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2} \qquad \Im\left(e^{i\theta}\right) = \sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i}$$

Homework

Linearize $\cos^3 t$ (i.e. find an expression with no power of trigonometric functions).

n-th roots

Definition: n-th root

Let $z \in \mathbb{C}$ and $n \in \mathbb{N}_{>0}$. We say that $w \in \mathbb{C}$ is a *n-th root of z* if $w^n = z$.

Theorem

Let $z \in \mathbb{C} \setminus \{0\}$. Then z admits exactly n n-th roots.
More precisely, if $z = \rho e^{i\theta}$, $\rho > 0$, then the n-th roots of z are exactly

$$\rho^{1/n} e^{i\left(\frac{\theta}{n} + \frac{2k\pi}{n}\right)}, \ k = 0, \dots, n-1$$

Homework

Find the square roots of: 1, -1, i, $1+i$.
Find the cubic roots of: 1, $2-2i$.

Homework

Study the r-th roots of $z \in \mathbb{C} \setminus \{0\}$ where $r \in \mathbb{Q}$. *(Hint: write $r = p/q$ where $\gcd(p,q) = 1$)*

Square roots – 1

How to compute the square roots of $z = a + ib \neq 0$ without using exponential representation?
Let $w = x + iy$ then

$$w^2 = z \Leftrightarrow \begin{cases} w^2 = z \\ |w|^2 = |z| \end{cases} \Leftrightarrow \begin{cases} x^2 - y^2 + 2ixy = a + ib \\ x^2 + y^2 = \sqrt{a^2 + b^2} \end{cases} \Leftrightarrow \begin{cases} x^2 - y^2 = a \\ 2xy = b \\ x^2 + y^2 = \sqrt{a^2 + b^2} \end{cases}$$

The last system is easy to solve: the first and last equations give 4 possible couples (x, y) and the second one allows to restrict to the expected 2 (using the sign).

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For $z = 8 - 6i$, we get $\begin{cases} x^2 - y^2 = 8 \\ 2xy = -6 \\ x^2 + y^2 = 10 \end{cases} \Leftrightarrow \begin{cases} x^2 = 9 \\ y^2 = 1 \\ 2xy = -6 \end{cases} \Leftrightarrow \begin{cases} x = \pm 3 \\ y = \pm 1 \\ 2xy = -6 \end{cases}$

So the solutions are $(3, -1)$ and $(-3, 1)$ since x and y have opposite signs thanks to $xy < 0$.
Hence the square roots of $8 - 6i$ are $-3 + i$ and $3 - i$.

Square roots – 2

Homework

1. Compute the square roots of $\frac{1+i}{\sqrt{2}}$.
2. Deduce the values of $\cos\frac{\pi}{8}$ and $\sin\frac{\pi}{8}$.