University of Toronto – MAT237Y1 – LEC5201

Multivariable calculus!

Interior, closure and boundary of $\mathbb{Q}$

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Proposition 1. $\forall x, y \in \mathbb{R},\ (x < y \implies \exists q \in \mathbb{Q},\ x < q < y)$
i.e. between two real numbers there is always a rational number.

Proof. Let $x, y \in \mathbb{R}$ satisfying $x < y$. Set $\varepsilon = y - x > 0$.
Since $\mathbb{R}$ is archimedean [⋆], there exists $n \in \mathbb{N}_{>0}$ such that $n\varepsilon > 1$, i.e. $\frac{1}{n} < \varepsilon$.
Set $m = \lfloor nx \rfloor + 1$, then $nx < m \leq nx + 1 \implies x < \frac{m}{n} \leq x + \frac{1}{n} < x + \varepsilon = y$.
Furthermore, $q = \frac{m}{n} \in \mathbb{Q}$ satisfies $x < q < y$. ∎

Proposition 2. *If $I \subset \mathbb{R}$ is an interval which is non-empty and not reduced to a singleton then $I \cap \mathbb{Q} \neq \varnothing$.*

Proof. Since I is non-empty and not reduced to a singleton, there exist $x, y \in I$ with $x < y$.
Then, by Proposition 1, there exists $q \in \mathbb{Q}$ such that $x < q < y$.
Since I is an interval, $q \in I$. Hence $q \in I \cap \mathbb{Q} \neq \varnothing$. ∎

Proposition 3 ($\mathbb{Q}$ is dense in $\mathbb{R}$)**.** $\overline{\mathbb{Q}} = \mathbb{R}$

Proof. Since $\overline{\mathbb{Q}} \subset \mathbb{R}$, it is enough to show the other inclusion.
Let $x \in \mathbb{R}$. Let $\varepsilon > 0$. Then $B(x, \varepsilon) = (x - \varepsilon, x + \varepsilon)$ is an interval which is non-empty and not reduced to a singleton. Hence, by Proposition 2, $B(x, \varepsilon) \cap \mathbb{Q} \neq \varnothing$.
Hence $\mathbb{R} \subset \overline{\mathbb{Q}}$. ∎

Proposition 4. $\forall x, y \in \mathbb{R},\ (x < y \implies \exists s \in \mathbb{R} \setminus \mathbb{Q},\ x < s < y)$
i.e. between two real numbers there is always an irrational number.

Proof. By Proposition 1, there exists $q \in \mathbb{Q}$ such that $x < q < y$. Still by Proposition 2, there exists $p \in \mathbb{Q}$ such that $x < p < q$.
Hence we obtained $p, q \in \mathbb{Q}$ such that $x < p < q < y$.
Set $s = p + \frac{\sqrt{2}}{2}(q - p)$. Then $s \in \mathbb{R} \setminus \mathbb{Q}$ (otherwise, by contradiction, $\sqrt{2}$ would be in $\mathbb{Q}$) and $p < s < q$ (notice that $0 < \frac{\sqrt{2}}{2} < 1$ so s is a number between p and q).
We obtained $s \in \mathbb{R} \setminus \mathbb{Q}$ such that $x < s < y$. ∎

Proposition 5. *If $I \subset \mathbb{R}$ is an interval which is non-empty and not reduced to a singleton then $I \cap (\mathbb{R} \setminus \mathbb{Q}) \neq \varnothing$.*

Proof. Follow the proof of Proposition 2 but using Proposition 4. ∎

[⋆] see the Slide 3 from Tuesday, September 24.

Proposition 6. $\partial\mathbb{Q} = \mathbb{R}$

Proof. Let $x \in \mathbb{R}$. Let $\varepsilon > 0$.
Then, by Proposition 2, $B(x,\varepsilon) \cap \mathbb{R} \cap \mathbb{Q} \neq \varnothing$ and, by Proposition 5, $B(x,\varepsilon) \cap \mathbb{R} \cap \mathbb{Q}^c \neq \varnothing$.
Hence $\mathbb{R} \subset \partial\mathbb{Q}$. The other inclusion is obvious. ■

Proposition 7. $\mathring{\mathbb{Q}} = \varnothing$

Proof. $\mathring{\mathbb{Q}} = \mathbb{R} \cap \mathring{\mathbb{Q}} = \partial\mathbb{Q} \cap \mathring{\mathbb{Q}} = \varnothing$ ■

Proposition 8. $\overline{\mathbb{R} \setminus \mathbb{Q}} = \mathbb{R}$, $\overbrace{\mathbb{R} \setminus \mathbb{Q}}^{\circ} = \varnothing$, $\partial(\mathbb{R} \setminus \mathbb{Q}) = \mathbb{R}$

Proof. Recall that $\partial(\mathbb{R} \setminus \mathbb{Q}) = \partial\mathbb{Q} = \mathbb{R}$.

Then $\overline{\mathbb{R} \setminus \mathbb{Q}} \supset \partial(\mathbb{R} \setminus \mathbb{Q}) = \mathbb{R}$ and $\overbrace{\mathbb{R} \setminus \mathbb{Q}}^{\circ} = \varnothing$ as above. ■

A useful consequence of the density of $\mathbb{Q}$ in $\mathbb{R}$ is that any real number can be approximated by rationals:

Proposition 9. *For any $x \in \mathbb{R}$, there exists a sequence (a_k) of rationals converging to x, i.e. such that $\left(\forall k,\ a_k \in \mathbb{Q}\right)$ and $\lim\limits_{k\to+\infty} a_k = x$.*

Proof. Let $k \in \mathbb{N}_{>0}$. Then $\left(x - \frac{1}{k}, x + \frac{1}{k}\right)$ is an interval which is non-empty and not reduced to a singleton. Hence, by Proposition 2, there exists $a_k \in \left(x - \frac{1}{k}, x + \frac{1}{k}\right) \cap \mathbb{Q}$.
We constructed a sequence such that $a_k \in \mathbb{Q}$ and $\left|x - a_k\right| < \frac{1}{k} \xrightarrow[k\to+\infty]{} 0$ ■

Example 10. The set $S = \left\{(x,y) \in \mathbb{R}^2\ :\ x \in \mathbb{Q} \text{ and } y \in \mathbb{Q}\right\} \subset \mathbb{R}^2$ is not closed.
Indeed, by Proposition 9, there exists (u_k) a sequence of rationals whose limit is $\sqrt{2}$.
Then $\forall k,\ (u_k, 0) \in S$.
Hence $(\sqrt{2}, 0) = \lim\limits_{k\to+\infty} (u_k, 0) \in \overline{S}$ but $(\sqrt{2}, 0) \notin S$.
Furthermore $\overline{S} \neq S$ and S is not closed.

Notice also that S is not open.
Indeed, let $\varepsilon > 0$, then there exists $n \in \mathbb{N}_{>0}$ big enough such that $0 < \frac{\sqrt{2}}{n} < \varepsilon$.
So $\left(\frac{\sqrt{2}}{n}, 0\right) \in B(\mathbf{0}, \varepsilon) \cap S^c$.
Furthermore $\mathbf{0} \in S$ but $\forall \varepsilon > 0,\ B(\mathbf{0}, \varepsilon) \not\subset S$.
Hence S is not open.

We may conclude that S is neither closed nor open without explicitely computing its interior or closure.