Limits of multivariable functions

In class activity: start with the definition of $\ell = \lim_{x\to a} f(x)$ for $f: I \to \mathbb{R}$ a one variable function defined on an interval I containing a.
Generalize the above definition and check that we need to restrict to limit points

Def: let $S \subset \mathbb{R}^n$. We say that $a \in \mathbb{R}^n$ is a limit point of S if

$$\forall \delta > 0, \exists x \in S, \ 0 < \|x - a\| < \delta$$

or geometrically: $\forall \delta > 0, \ (B(a,\delta) \cap S) \setminus \{a\} \neq \emptyset$

Theorem: a is a limit point of $S \Leftrightarrow a \in \overline{S \setminus \{a\}}$

Δ Notice that $(B(a,\delta) \cap S) \setminus \{a\} = B(a,\delta) \cap (S \setminus \{a\})$ $\square$

Ex: ① 0 is a limit point of $\{\frac{1}{n} : n \in \mathbb{N}_{>0}\} \subset \mathbb{R}$

② 0 is a limit point of $[0,3)$ or of $(0,3)$

③ 0 is NOT a limit point of $\{0\} \cup [1,2)$

Intuition: a limit point is a closure point which is not isolated

Remark: if $a \in \mathring{S}$ then a is a limit point of S

Def: let $S \subset \mathbb{R}^n$, a be a limit point of S, $f: S \to \mathbb{R}^k$ and $L \in \mathbb{R}^k$
We say that L is the limit of f at a, denoted $\lim_{x\to a} f(x) = L$ if

$$\forall \varepsilon > 0, \exists \delta > 0, \forall x \in S, \ 0 < \|x - a\| < \delta \Rightarrow \|f(x) - L\| < \varepsilon$$

Proposition: let $f,g: S \to \mathbb{R}$ (Δ $k=1$) and a be a limit point of S then

$$\lim_{x\to a} f(x) = L, \ \lim g(x) = M \Rightarrow \begin{cases} \lim (f+g) = M+L \\ \lim fg = ML \end{cases}$$

Proposition: $f,g,h: S \to \mathbb{R}$ (Δ $k=1$) and a limit point of S

$$\begin{cases} f \leq g \leq h \\ \lim_a f = \lim_a h = L \end{cases} \Rightarrow \lim_{x\to a} g = L$$

Theorem: let $f = (f_1, \dots, f_k) : S \longrightarrow \mathbb{R}^k$, a be a limit point of S and $L = (L_1, \dots, L_k) \in \mathbb{R}^k$.

$$\lim_{x \to a} f(x) = L \iff \forall i = 1, \dots, k, \quad \lim_{x \to a} f_i(x) = L_i$$

Comment: hence it's enough to understand the real-valued case

Δ First notice that $\lim_a f = L \iff \lim_{x \to a} \|f(x) - L\| = 0$

$\Rightarrow$: $|f_i(x) - L_i| \leqslant \left(|f_1(x) - L_1|^2 + \dots + |f_k(x) - L_k|^2\right)^{1/2} = \|f(x) - L\|$

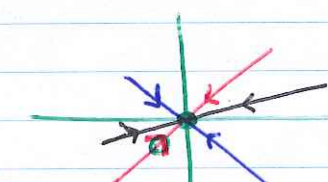
so $\lim_{x \to a} f(x) = L \Rightarrow \lim_{x \to a} f_i(x) = L_i$

$\Leftarrow$: $\|f(x) - L\| = \left(|f_1(x) - L_1|^2 + \dots + |f_k(x) - L_k|^2\right)^{1/2}$

so $\left(\forall i, \ |f_i - L_i| \longrightarrow 0\right) \Rightarrow \lim_{x \to a} f(x) = L$ $\square$

Rem: in the one variable case it's enough to check that the limits from the right and from the left coincide.
In $\mathbb{R}^n$ the situation is more subtle since we have more "freedom" to approach $a \in \mathbb{R}^n$

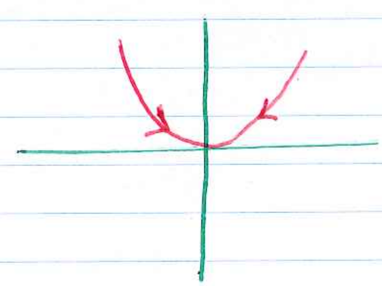
Ex ① let $f(x, y) = \dfrac{xy}{x^2 + y^2}$



then $f(x, cx) = \dfrac{c}{1 + c^2}$ so $\lim_{(x,y) \to (0,0)} f(x,y)$ DNE

② let $f(x, y) = \dfrac{x^2 y}{x^4 + y^2}$

then $f(x, cx) = \dfrac{cx^3}{x^4 + c^2 x^2} \xrightarrow[x \to 0]{} 0$



but $f(x, x^2) = \dfrac{x^4}{x^4 + x^4} = \dfrac{1}{2} \longrightarrow \dfrac{1}{2} \neq 0$

so $\lim_{(x,y) \to (0,0)} f(x, y)$ DNE

Ccl: it's not enough to look along lines (or even parabolas, what of )

Ex : $f(x,y) = \frac{x^2y^2}{x^2+y^2}$ at $(0,0)$

$$|f(x,y)| = \frac{|xy|}{x^2+y^2} \cdot |xy| \leqslant \frac{1}{2}|xy| \xrightarrow[(x,y)\to(0,0)]{} 0$$

so $\lim_{(x,y)\to(0,0)} f(x,y) = 0$

Here, I used the following very useful inequality

$$\boxed{\forall (x,y) \in \mathbb{R}^2 \setminus \{(0,0)\}, \quad \frac{|xy|}{x^2+y^2} \leqslant \frac{1}{2}}$$

Δ let $(x,y) \in \mathbb{R}^2 \setminus \{(0,0)\}$ then

$$0 \leqslant (|x|-|y|)^2 = x^2+y^2-2|x||y|$$

$$\Rightarrow 2|xy| \leqslant x^2+y^2$$

$$\Rightarrow \frac{|xy|}{x^2+y^2} \leqslant \frac{1}{2}$$

$\square$

Ex : $\frac{|x|y^2}{x^2+y^4} \leqslant \frac{1}{2}$

Continuity of multivariable functions

Def: let $S \subset \mathbb{R}^m$, $f: S \to \mathbb{R}^k$ and $a \in S$.
We say that f is continuous at a if:

$$\forall \varepsilon > 0, \exists \delta > 0, \forall x \in S, \|x-a\| < \delta \Rightarrow \|f(x)-f(a)\| < \varepsilon$$

Remarks. ① We don't require a to be a limit point: if a is isolated of S then f is continuous at a.

② However, if a is a limit point of S, then
f is continuous at $a \Leftrightarrow \lim_{x \to a} f(x) = f(a)$

Theorem: $f = (f_1, \dots, f_k) : S \to \mathbb{R}^k$ is continuous at a if and only if each component $f_i : S \to \mathbb{R}$ is continuous at a.

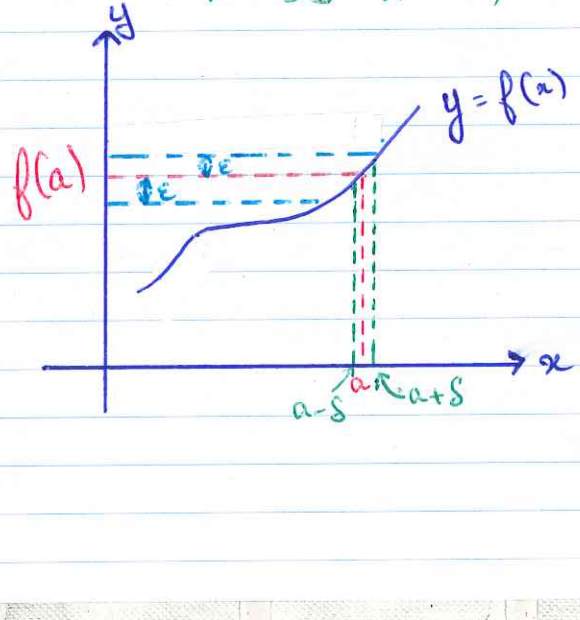
Again, it is enough to understand well the real valued case ($k=1$)

Remark: in the real valued case ($k=1$) the usual "limit laws" remain true so we can build continuous functions using the elementary functions.

Homework: read theorem 5 of section 1.2 (online notes)

⚠ f is continuous at a iff: $\forall \varepsilon > 0, \exists \delta > 0, f(B(a,\delta) \cap S) \subset B(f(a), \varepsilon)$

(it's where topology appears)



$\forall \varepsilon > 0, \exists \delta > 0,$

$f((a-\delta, a+\delta)) \subset (f(a)-\varepsilon, f(a)+\varepsilon)$

Theorem: let $S \subset \mathbb{R}^m$ and $f : S \to \mathbb{R}^k$

TFAE ① f is continuous

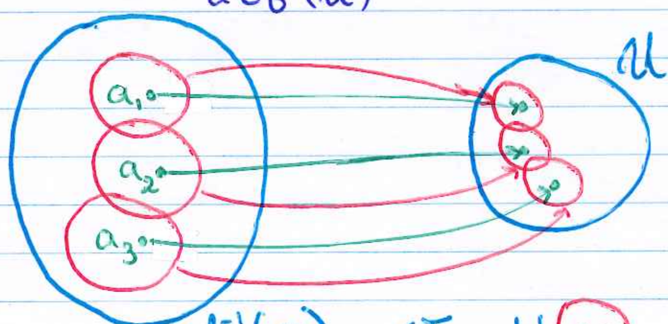
② $\forall \mathcal{U} \subset \mathbb{R}^k$ open set, $\exists \mathcal{V} \subset \mathbb{R}^m$ open set, s.t. $f^{-1}(\mathcal{U}) = \mathcal{V} \cap S$

③ $\forall C \subset \mathbb{R}^k$ closed set, $\exists D \subset \mathbb{R}^m$ closed set, s.t. $f^{-1}(C) = D \cap S$

if the domain is $\mathbb{R}^m$: f is continuous $\Leftrightarrow$ the inverse image of an open is open
$\Leftrightarrow$ ——— a closed is closed

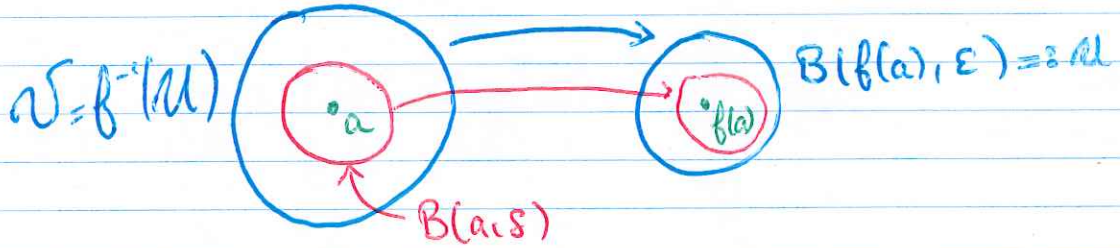
Δ 1⇒2: let $a \in f^{-1}(\mathcal{U})$ then $f(a) \in \mathcal{U}$ open so $\exists \varepsilon > 0$ s.t. $B(f(a), \varepsilon) \subset \mathcal{U}$
then, by continuity of f, $\exists \delta_a > 0$ s.t. $f(B(a, \delta_a) \cap S) \subset B(f(a), \varepsilon) \subset \mathcal{U}$

we can take $\mathcal{V} = \bigcup_{a \in f^{-1}(\mathcal{U})} B(a, \delta_a)$



$f^{-1}(\mathcal{U}) = \mathcal{V} = \bigcup_{a \in f^{-1}(\mathcal{U})} (\circ a)$

2⇒1. let $a \in S$, let $\varepsilon > 0$, then $B(f(a), \varepsilon)$ is open as an open ball.
by assumption $\exists \mathcal{V} \subset \mathbb{R}^m$ open s.t. $f^{-1}(B(f(a), \varepsilon)) = S \cap \mathcal{V}$
since $a \in \mathcal{V}$ open, $\exists \delta > 0$, $B(a, \delta) \subset \mathcal{V}$
then $f(B(a, \delta) \cap S) \subset f(\mathcal{V} \cap S) \subset B(f(a), \varepsilon)$



2⇔3. $f^{-1}(\mathbb{R}^k \setminus \mathcal{U}) = S \setminus f^{-1}(\mathcal{U}) = S \cap ((f^{-1}(\mathcal{U}))^c) = S \cap (\mathcal{V}^c)$

$\square$

Ex: ① $S = \{(x,y) \in \mathbb{R}^2 : |x-y| = 1\}$ is closed

indeed $S = f^{-1}(\{1\})$ where $f(x,y) = |x-y|$ is continous and $\{1\} \subset \mathbb{R}$ is closed

② $T = \{(x,y) \in \mathbb{R}^2 : |x-y| > 1\}$ is open

indeed $T = f^{-1}((1, +\infty))$ where $(1, +\infty)$ is open

Homework: Q from S.1.2 of the lecture notes