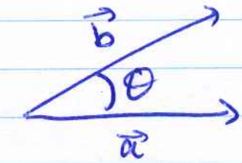
Geometric interpretation of the dot product

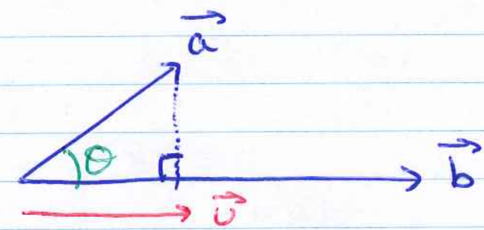
$$\vec{a} \cdot \vec{b} = \|\vec{a}\| \cdot \|\vec{b}\| \cdot \cos\theta$$



Def: We say that $a, b \in \mathbb{R}^n$ are orthogonal when $a \cdot b = 0$

Consequences: (of the geometric definition)

①



$\|\vec{u}\| = \|\vec{a}\| \cos\theta$

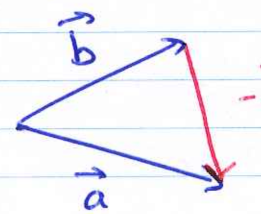
so $\vec{u} = \|\vec{a}\| \cos\theta \cdot \dfrac{\vec{b}}{\|\vec{b}\|}$

$= \dfrac{\|\vec{a}\| \cdot \|\vec{b}\| \cdot \cos\theta}{\|\vec{b}\|^2} \vec{b}$

$= \dfrac{(\vec{a} \cdot \vec{b})}{\|\vec{b}\|^2} \vec{b}$

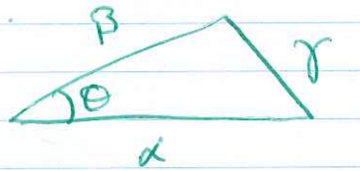
Conclusion: $\left(\dfrac{\vec{a} \cdot \vec{b}}{\|\vec{b}\|^2}\right) \vec{b}$ is the orthogonal projection of $\vec{a}$ on the line spanned by $\vec{b}$

② Law of cosines: $\|a - b\|^2 = \|a\|^2 + \|b\|^2 - 2(a \cdot b)$



$-\vec{b} + \vec{a} = \vec{a} - \vec{b}$

So in a triangle:



$\gamma^2 = \alpha^2 + \beta^2 - 2\alpha\beta\cos\theta$

Homework. Find the angles of the triangle whose vertices are $A(-1, 0)$, $B\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$, $C\left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$

Hint: compute $\overrightarrow{AB} \cdot \overrightarrow{AC}$ using both the geometric and algebraic definition

Rem: if $a \cdot b = 0$ then $\|a + b\|^2 = \|a\|^2 + \|b\|^2$ (Pythagorean thm)

Cross product (⚠ only in $\mathbb{R}^3$, $n=3$)

Def: (Cross product) for $a, b \in \mathbb{R}^3$

$$a \times b = (a_2b_3 - a_3b_2,\ a_3b_1 - a_1b_3,\ a_1b_2 - a_2b_1) \in \mathbb{R}^3$$

(takes 2 vectors, gives 1 vector)

Mnemonic devices:

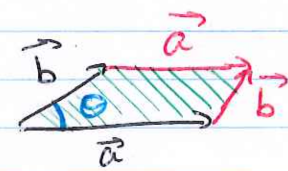
① $\begin{array}{cc} \cancel{a_1} & \cancel{b_1} \\ a_2 & b_2 \\ a_3 & b_3 \\ a_1 & b_1 \\ a_2 & b_2 \end{array}$ $\qquad \begin{array}{l} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{array}$

② Compute the following determinant w.r.t the last column:

$$\begin{vmatrix} a_1 & b_1 & \vec{i} \\ a_2 & b_2 & \vec{j} \\ a_3 & b_3 & \vec{k} \end{vmatrix} = \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix} \vec{i} - \begin{vmatrix} a_1 & b_1 \\ a_3 & b_3 \end{vmatrix} \vec{j} + \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \vec{k}$$

⚠

Prop: for $a, b, c \in \mathbb{R}^3$, $\lambda \in \mathbb{R}$



① $b \times a = -(a \times b)$

② $(\lambda a + b) \times c = \lambda(a \times c) + b \times c$

③ $a \times a = 0$

④ $\|a \times b\|^2 + (a \cdot b)^2 = \|a\|^2 \cdot \|b\|^2$

⑤ $\|a \times b\| = \|a\| \cdot \|b\| \cdot |\sin\theta|$ ⟶ area of the parallelogram defined by $\vec{a}$ and $\vec{b}$

⑥ $a \cdot (a \times b) = 0$, $b \cdot (a \times b) = 0$

⑦ $\vec{i} \times \vec{j} = \vec{k}$, $\vec{j} \times \vec{k} = \vec{i}$, $\vec{k} \times \vec{i} = \vec{j}$

⚠ $(a \times b) \times c \neq a \times (b \times c)$: example $(\vec{i} \times \vec{i}) \times j = \vec{0}$ $\quad \vec{i} \times (\vec{i} \times \vec{j}) = -\vec{j}$

But: ⑧ $a \times (b \times c) = (a \cdot c)b - (a \cdot b)c$
$(a \times b) \times c = (a \cdot c)b - (b \cdot c)a$

⑨ $a \times (b \times c) + b \times (c \times a) + c \times (a \times b) = 0$ (Jacobi identity)

(10) $(a \times b) \cdot c = \det \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}$

Homework: let $\vec{a} = (2,-3,1)$, $\vec{b} = (3,-5,2)$, $\vec{c} = (4,-5,1)$

(i) Compute $\vec{a} \times \vec{b}$. Are $\vec{a}$ and $\vec{b}$ collinear?
(ii) Compute $(\vec{a} \times \vec{b}) \cdot \vec{c}$. Are $\vec{a}, \vec{b}, \vec{c}$ coplanar?

Geometric interpretation of the cross-product.

- if $\vec{a}$ and $\vec{b}$ are collinear then $\vec{a} \times \vec{b} = \vec{0}$
- otherwise $\vec{a} \times \vec{b}$ is the unique vector $\vec{c}$ such that
  (i) $\|\vec{c}\| = \|\vec{a}\| \cdot \|\vec{b}\| \, |\sin\theta|$
  (ii) $\vec{c}$ is orthogonal to $\vec{a}$ and $\vec{b}$
  (iii) $\det \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix} > 0$

  ↳ Right-hand rule: $\left\{ \begin{array}{l} \text{thumb} = \vec{a} \\ \text{index} = \vec{b} \\ \text{middle} = \vec{a} \times \vec{b} \end{array} \right.$
  ↳ ⚠ not left!

Δ • if $\vec{a}$ and $\vec{b}$ are collinear then $\|\vec{a} \times \vec{b}\| = \|\vec{a}\| \cdot \|\vec{b}\| \, |\sin\theta| = 0$ by (5)

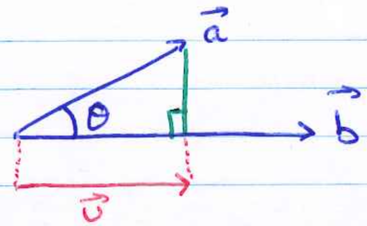
• otherwise: (i) $\|\vec{a} \times \vec{b}\| = \|\vec{a}\| \cdot \|\vec{b}\| \cdot |\sin\theta|$ by (5)
  (ii) $\vec{a} \cdot (\vec{a} \times \vec{b}) = 0$, $\vec{b} \cdot (\vec{a} \times \vec{b}) = 0$ by (6)
  (iii) $\det \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix} = (\vec{a} \times \vec{b}) \cdot (\vec{a} \times \vec{b})$ by (10) if $\vec{c} = \vec{a} \times \vec{b}$
  $= \|\vec{a} \times \vec{b}\|^2 > 0$ since $\vec{a} \times \vec{b} \neq \vec{0}$ by (i)

then $\vec{a} \times \vec{b}$ is uniquely determined since we know its direction and length
Δ

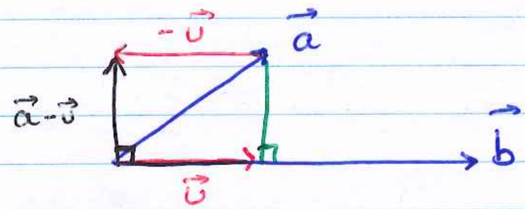
Extra definition: (via triple product)
$\vec{a} \times \vec{b}$ is the unique vector of $\mathbb{R}^3$ s.t.

$$\forall \vec{c} \in \mathbb{R}^3, \ \det \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix} = (\vec{a} \times \vec{b}) \cdot \vec{c}$$

Extra: since two of you asked me for a geometric proof of the Cauchy-Schwarz inequality, here it is:



$\vec{v} = \dfrac{(\vec{a}\cdot\vec{b})}{\|\vec{b}\|^2}\vec{b}$ is the orthogonal projection of $\vec{a}$ on the line spanned by $\vec{b}$



we see that $\vec{a}-\vec{v}$ is orthogonal to $\vec{v}$, ie $(\vec{a}-\vec{v})\cdot\vec{v} = 0$

(check that $(\vec{a}-\vec{v})\cdot\vec{v} = 0$ algebraically)

hence
$$\begin{aligned}
\|\vec{a}\|^2 &= \|(\vec{a}-\vec{v})+\vec{v}\|^2 \\
&= \|\vec{a}-\vec{v}\|^2 + \|\vec{v}\|^2 \quad \text{since } (\vec{a}-\vec{v})\cdot\vec{v} = 0 \\
&\geqslant \|\vec{v}\|^2 \quad \text{since } \|\vec{a}-\vec{v}\| \geqslant 0 \\
&= \frac{(\vec{a}\cdot\vec{b})^2}{\|\vec{b}\|^4}\cdot\|\vec{b}\|^2 \quad \text{by definition of } \vec{v} \\
&= \frac{(\vec{a}\cdot\vec{b})^2}{\|\vec{b}\|^2}
\end{aligned}$$

Hence $(\vec{a}\cdot\vec{b})^2 \leqslant \|\vec{a}\|^2\|\vec{b}\|^2$

and $|\vec{a}\cdot\vec{b}| \leqslant \|\vec{a}\|\,\|\vec{b}\|$

QED

$\square$