Ex 1:

$\Rightarrow$ Assume that S is Jordan-measurable

S is bounded, so there exists a rectangle R s.t. $S \subset R$

We define $\chi_S : R \to \mathbb{R}$ by $\chi_S(x) = 0$ if $x \notin S$

$\chi_S(x) = 1$ if $x \in S$

Notice that the discontinuity set of χ_S is ∂S which has ZC by assumption, so χ_S is integrable on R

$\Leftarrow$: Proof 1 using Lebesgue criterion (easier but not part of MAT237)

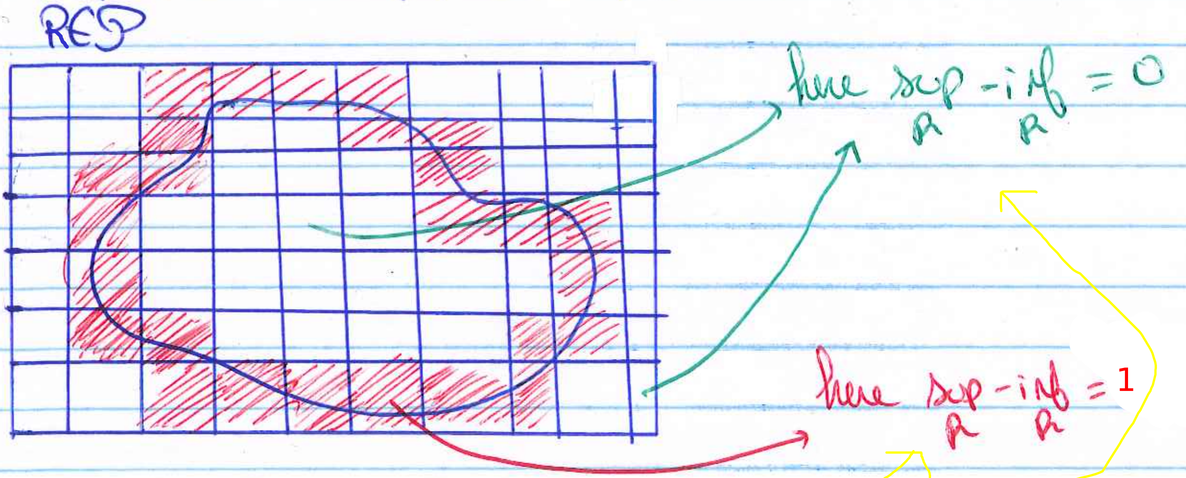
χ_S is integrable on R $\Rightarrow$ ∂S its discontinuity set has measure 0

$\Rightarrow$ ∂S has zero content since it is compact (Heine-Borel)

Proof 2: Let $\varepsilon > 0$ Let P a partition of R s.t. $U_P(\chi_S) - L_P(\chi_S) < \varepsilon$

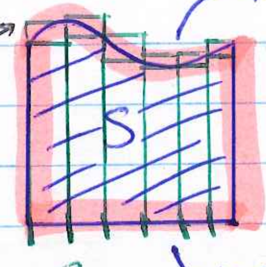
We denote by $\mathcal{P}$ the set of subrectangles of P intersecting ∂S then

$$\sum_{R \in \mathcal{P}}{}' \nu(R) = U_P(\chi_S) - L_P(\chi_S) < \varepsilon$$



Let me add an extra explanation: $\sum_{R \in \mathcal{P}} \nu(R) = \sum_R (\sup_R \chi_S - \inf_R \chi_S)\nu(R) = U_P(\chi_S) - L_P(\chi_S) < \varepsilon$

Exo 3.



the idea is that ∂S has zero-content

$\rightarrow$ the difference is a small as we want by definition of integrability

Let $\varepsilon > 0$. Let P be a partition of T s.t.

$$U_P(f) - L_P(f) < \varepsilon$$

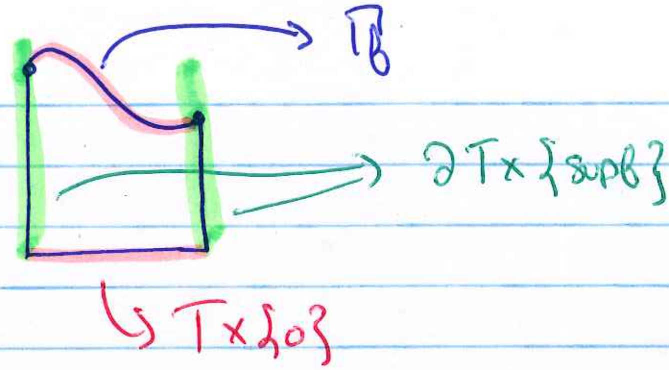
Then
$$\overline{\int}\chi_S - \underline{\int}\chi_S \leq \sum_{\substack{R \\ \text{subrectangle} \\ \text{of } P}} \nu(R\times[0,\sup_R f]) - \nu(R\times[0,\inf_R f])$$
$$= \sum_R \left(\sup_R f - \inf_R f\right)\nu(R)$$
$$< \varepsilon$$

So $\overline{\int}\chi_S = \underline{\int}\chi_S$ and χ_S is integrable thus S is Jordan measurable by Exo 1.

Notice also that
$$\overline{\int_T} f - \underline{\int}\chi_S \leq \sum_R \nu(R\times[0,\sup_R f]) - \nu(R\times[0,\inf_R f])$$
$$< \varepsilon$$

So $$\underline{\int_T} f = \overline{\int_T} f = \underline{\int}\chi_S = \overline{\int}\chi_S =: \int_S 1$$

Method 2

$\partial S =$ 

$$\partial S \subset (T \times \{0\}) \cup (\partial T \times \{\sup_T f\}) \cup \Gamma_f$$

- $T \times \{0\}$ has ZC:

  Let $\varepsilon > 0$, $T \times \{0\} \subset T \times \left[-\frac{\varepsilon}{3\vartheta(T)}, \frac{\varepsilon}{3\vartheta(T)}\right]$

  and $\vartheta\left(T \times \left[-\frac{\varepsilon}{3\vartheta(T)}, \frac{\varepsilon}{3\vartheta(T)}\right]\right) = \vartheta(T) \times \frac{2\varepsilon}{3\vartheta(T)} = \frac{2\varepsilon}{3} < \varepsilon$

- Let $\varepsilon > 0$,

  ∂T has zero content so $\exists R_i, i=1,\dots,r$ s.t. $\partial T \subset \bigcup R_i$
  and $\sum \vartheta(R_i) < \sqrt{\varepsilon}$

  then $\partial T \times \{\sup_T f\} \subset \bigcup \underbrace{R_i \times \left[\sup - \frac{\sqrt{\varepsilon}}{2}, \sup + \frac{\sqrt{\varepsilon}}{2}\right]}$

  and $\sum \vartheta(\ \ ) = \sum \vartheta(R_i)\sqrt{\varepsilon} < \sqrt{\varepsilon}\sqrt{\varepsilon} = \varepsilon$

- Γ_f has ZC as the graph of an integrable function

CCL: ∂S has ZC as the subset of a set having ZC (as the finite union...)

Hence S is Jordan measurable

Next: $\vartheta(S) = \int_S 1 = \int_T \int_{[0, \sup f]} \chi_{[0, f(x)]}(y)\, dy\, dx = \int_T \int_0^{f(x)} 1\, dy\, dx$

$= \int_T f(x)\, dx$

Exo 4:

① $$\int_S y \, dx\,dy = \int_{-1}^{1}\int_{0}^{\sqrt{1-x^2}} y \, dy \, dx = \int_{-1}^{1} \left[\frac{y^2}{2}\right]_0^{\sqrt{1-x^2}} dx$$

$$= \frac{1}{2}\int_{-1}^{1} (1-x^2)\, dx = 2/3$$

② $$\int_S y \, dx\,dy = \int_0^{\pi}\int_0^1 r^2 \sin\theta \, dr \, d\theta = \left[\frac{r^3}{3}\right]_0^1 \times \left[-\cos\theta\right]_0^{\pi}$$

$$= 2/3$$

Exo 5: Notice that $\sqrt{|y-x^2|} = \begin{cases} \sqrt{y-x^2} & \text{when } y \geq x^2 \\ \sqrt{x^2-y} & \text{otherwise} \end{cases}$

We have $[-1,1]\times[0,2] = S_1 \cup S_2$

where $S_1 = \{(x,y) : x \in [-1,1],\ 0 \leq y \leq x^2\}$

$S_2 = \{(x,y) : x \in [-1,1],\ x^2 \leq y \leq 2\}$

then $$\int_{[-1,1]\times[0,2]} \sqrt{|y-x^2|} = \int_{S_1} \sqrt{x^2-y} + \int_{S_2} \sqrt{y-x^2}$$

$$= \int_{-1}^{1}\int_0^{x^2} \sqrt{x^2-y}\, dy\,dx + \int_{-1}^{1}\int_{x^2}^{2} \sqrt{y-x^2}\, dy\,dx$$

$$= \frac{2}{3}\int_{-1}^{1} |x^3| \mathrm{d}x + \frac{2}{3}\int_{-1}^{1} (2-x^2)^{3/2}\, dx$$

$$= \frac{4}{3}\int_0^1 x^3 \mathrm{d}x + \frac{4}{3}\int_0^1 (2-x^2)^{3/2} dx = \frac{16}{3}\int_0^{\pi/4} \cos^4 t \, dt + \frac{1}{3}$$

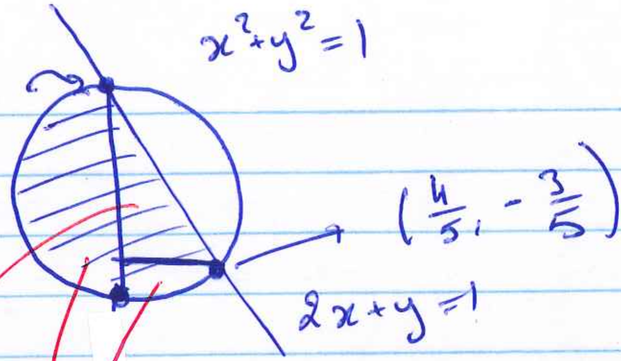
$x = \sqrt{2}\sin t$

$$= \frac{5}{3} + \frac{\pi}{2}$$

$$\cos^4 t = (\cos^2 t)^2 = \frac{1}{4}\left(\cos(2t)+1\right)^2 = \frac{1}{4}\underbrace{\cos^2(2t)}_{\text{again...}} + \frac{1}{2}\cos(2t) + \frac{1}{4}$$

Exo 6

Method 1 :



$$= \{(x,y) : x^2+y^2 \leq 1, x \leq 0\} \quad S_1$$

$$\cup \left\{(x,y) : -\frac{3}{5} \leq y \leq 1, 0 \leq x \leq \frac{1-y}{2}\right\} \quad S_2$$

$$\cup \left\{(x,y) : 0 \leq x \leq \frac{4}{5}, -\sqrt{1-x^2} \leq y \leq -\frac{3}{5}\right\} \quad S_3$$

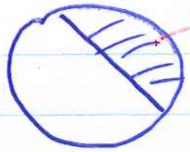
$$\int_{S_2} xy = \int_{-3/5}^{1} y \int_0^{\frac{1-y}{2}} x \, dx \, dy = \frac{1}{8} \int_{-3/5}^{1} y^3 - 2y^2 + y \, dy = -\frac{64}{1875}$$

$$\int_{S_3} xy = \int_0^{4/5} x \int_{-\sqrt{1-x^2}}^{-3/5} y \, dy \, dx = \frac{1}{2}\int_0^{4/5} x^3 - \frac{16}{25} x \, dx = -\frac{32}{625}$$

$$\int_{S_1} xy = \int_0^1 \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} r^3 \cos\theta \sin\theta \, d\theta \, dr = \frac{1}{8} \int_{\pi/2}^{\frac{3\pi}{2}} \sin(2\theta) \, d\theta = 0$$

$$\text{So } \int_A xy = -\frac{64}{1875} - \frac{32}{625} = -\frac{32}{375}$$

Method 2 :



T

- $\displaystyle\iint_{\overline{B}(0,1)} xy = \int_0^1\int_{-\pi}^{\pi} r^3\cos\theta\sin\theta \, d\theta\, dr = \frac{1}{8}\int_{-\pi}^{\pi} \sin(2\theta)\, d\theta = 0$

- $T = \left\{ (x,y) : -\frac{3}{5} \leq y \leq 1 ,\ \frac{1-y}{2} \leq x \leq \sqrt{1-y^2} \right\}$

So $\displaystyle\int_T xy = \int_{-3/5}^{1} y \int_{\frac{1-y}{2}}^{\sqrt{1-y^2}} x\, dx\, dy = \frac{1}{8}\int_{-3/5}^{1} -5y^3 + 2y^2 + 3y\, dy$

$$= \frac{32}{375}$$

Hence $\displaystyle\int_A xy = \int_{\overline{B}(0,1)} xy - \int_T xy = 0 - \frac{32}{375} = -\frac{32}{375}$

Ex7

① Using polar coordinates

Define $\psi(r,\theta) = \Phi(r\cos\theta, r\sin\theta)$

$= r^2(\cos(2\theta), \sin(2\theta))$

It is easy to check that

$\psi : \mathbb{R}_{>0} \times (-\frac{\pi}{2}, \frac{\pi}{2}) \longrightarrow \mathbb{R}^2 \setminus \{(x,0) : x \leq 0\}$

is a bijection.

Moreover $\det D\psi(r,\theta) = \begin{vmatrix} 2r\cos(2\theta) & -2r^2\sin(2\theta) \\ 2r\sin(2\theta) & 2r^2\cos(2\theta) \end{vmatrix}$

$= 4r^3 > 0$

Hence ψ and hence Φ is a C^1-diffeomorphism

Method 2: $\|\Phi(x,y)\| = x^2+y^2$

- $\Phi : \mathbb{R}_{>0} \times \mathbb{R} \longrightarrow \mathbb{R}^2 \setminus \{(x,0) : x \leq 0\}$ is well defined
- Let $(u,v) = \Phi(x,y) = (x^2-y^2, 2xy)$

  so that $u = x^2-y^2$, $v = 2xy$, $\sqrt{u^2+v^2} = x^2+y^2$

Then $\Theta : \mathbb{R}^2 \setminus \{(u,0) : u \leq 0\} \longrightarrow \mathbb{R}_{>0} \times \mathbb{R}$

$(u, v) \longmapsto \left(\sqrt{\frac{u+\sqrt{u^2+v^2}}{2}}, \frac{v}{\sqrt{2(u+\sqrt{u^2+v^2})}} \right)$

is well defined, C^1

and $\Theta = \Phi^{-1}$

(2) By symmetry $\int_S (y^2-x^2)^{xy}(x^2+y^2) = 2\int_{S\cap\{x>0\}} (y^2-x^2)^{xy}(x^2+y^2)$

And Φ maps $S\cap\{x>0\}$ to

$$\{(u,v) : -1 \le u \le 0,\ 2a \le v \le 2b\}$$

Notice that

$$\operatorname{Det} D\Phi(x,y) = \begin{vmatrix} 2x & -2y \\ 2y & 2x \end{vmatrix} = 4(x^2+y^2) > 0$$

so $\int_S (y^2-x^2)^{xy}(x^2+y^2)\,dx\,dy$

$$= 2\int_{S\cap\{x\ge 0\}} (y^2-x^2)^{xy}(x^2+y^2)\,dx\,dy$$

$$= \frac{1}{2}\int_{2a}^{2b}\int_{-1}^{0} (-u)^{\frac{v}{2}}\,du\,dv$$

$$= \frac{1}{2}\int_{2a}^{2b}\left[-\frac{(-u)^{v/2+1}}{v/2+1}\right]_{-1}^{0} dv$$

$$= \int_{2a}^{2b} \frac{1}{v+2}\,dv$$

$$= \ln\left(\frac{2b+2}{2a+2}\right)$$

$$= \ln\left(\frac{b+1}{a+1}\right)$$

Ex 8

$\Phi(x,y) = (x+y, y)$ defines a C^1-diffeo $\mathbb{R}^2 \to \mathbb{R}^2$

$$\det D\Phi(x,y) = \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} = 1 > 0$$

Hence $$\int_0^{\pi}\int_0^{\pi} |\cos(x+y)|\,dx\,dy = \int_0^{\pi}\left(\int_v^{v+\pi} |\cos(u)|\,du\right) dv$$

$$= \int_0^{\pi}\left(\int_v^{\pi/2} |\cos(u)|\,du + \int_{\pi/2}^{\pi} |\cos(u)|\,du + \int_{\pi}^{v+\pi} |\cos(u)|\,du\right) dv$$

Here I used the CoV $s = u - \pi$ and that $\cos(s+\pi) = -\cos s$

$$= \int_0^{\pi}\left(\int_v^{\pi/2} |\cos u|\,du + \int_{\pi/2}^{\pi} |\cos u|\,du + \int_0^{v} |\cos u|\,du\right) dv$$

$$= \int_0^{\pi}\int_0^{\pi} |\cos u|\,du\,dv$$

$$= 2\pi \int_0^{\pi/2} \cos u\,du$$

$$= 2\pi$$

Method 2:

$$\int_0^{\pi}\left(\int_v^{v+\pi} |\cos u| \mathrm{d}u\right)\mathrm{d}v = \int_0^{\frac{\pi}{2}}\left(\int_v^{\frac{\pi}{2}} \cos u \mathrm{d}u - \int_{\frac{\pi}{2}}^{v+\pi} \cos u \mathrm{d}u\right)\mathrm{d}v + \int_{\frac{\pi}{2}}^{\pi}\left(-\int_v^{\frac{3\pi}{2}} \cos u \mathrm{d}u + \int_{\frac{3\pi}{2}}^{v+\pi} \cos u \mathrm{d}u\right)\mathrm{d}v$$

$$= \int_0^{\frac{\pi}{2}} (1 - \sin v - \sin(v+\pi) + 1)\mathrm{d}v + \int_{\frac{\pi}{2}}^{\pi} (1 + \sin v + \sin(\pi+v) + 1)\mathrm{d}v$$

$\sin v + \sin(\pi + v) = 0$

$$= \int_0^{\frac{\pi}{2}} 2\mathrm{d}v + \int_{\frac{\pi}{2}}^{\pi} 2\mathrm{d}v$$

$$= 2\pi$$

Ex 9

① $F(x) = \int_a^b f(x,y)\,dy$

FTC $\rightarrow = \int_a^b \left(\int_c^x \frac{\partial f}{\partial x}(s,y)\,ds + f(c,y) \right) dy$

$= \int_a^b \int_c^x \frac{\partial f}{\partial x}(s,y)\,ds\,dy + \int_a^b f(c,y)\,dy$

Fubini $\rightarrow = \int_c^x \int_a^b \frac{\partial f}{\partial x}(s,y)\,dy\,ds + \int_a^b f(c,y)\,dy$

So by the FTC: $F'(x) = \int_a^b \frac{\partial f}{\partial x}(x,y)\,dy$

($s \mapsto \int_a^b \frac{\partial f}{\partial x}(s,y)\,dy$ is C^0 by the C^0 thm)

② It is enough to study $G(x) = \int_a^{\varphi(x)} f(x,y)\,dy$

We define $F(x,t) = \int_a^t f(x,y)\,dy$

$\frac{\partial F}{\partial t}(x,t) \underset{\text{FTC}}{=} f(x,t)$ $\qquad$ $\frac{\partial F}{\partial x}(x,t) \underset{\text{by ①}}{=} \int_a^t \frac{\partial f}{\partial x}(x,y)\,dy$

Then $G(x) = F(x,\varphi(x))$

By the chain rule $G'(x) = \frac{\partial F}{\partial x}(x,\varphi(x)) + \varphi'(x)\frac{\partial F}{\partial t}(x,\varphi(x))$

$= \int_a^{\varphi(x)} \frac{\partial f}{\partial x}(x,y)\,dy + \varphi'(x) f(x,\varphi(x))$

Ex 10:

① Let $y_0 \in [0,1]$

- If $y_0 = 0$ or $y_0 = 1$ then $f_{y_0}(x) = 0$ and f_{y_0} is integrable
- If $y_0 \in (0,1)$, $f_{y_0}(x) = \begin{cases} 0 & \text{at } x = 0 \\ y_0^{-2} & \text{if } 0 < x < y_0 \\ 0 & \text{if } x = y_0 \\ -x^{-2} & \text{if } y_0 < x < 1 \\ 0 & \text{if } x = 1 \end{cases}$

Notice that the discontinuity set is finite and that f_{y_0} is bounded so f_{y_0} is integrable

② $$\int_0^1 \left(\int_0^1 f(x,y)\, dx \right) dy = \int_0^1 \left(\int_0^y y^{-2}\, dx + \int_y^1 -x^{-2}\, dx \right) dy$$
$$= \int_0^1 \frac{1}{y} + 1 - \frac{1}{y}\, dy = 1$$

$$\int_0^1 \left(\int_0^1 f(x,y)\, dy \right) dx = \int_0^1 \left(\int_0^x -x^{-2}\, dy + \int_x^1 y^{-2}\, dy \right) dx$$
$$= \int_0^1 -\frac{1}{x} - 1 + \frac{1}{x}\, dx = -1$$

③ f is not bounded / integrable

In class, we proved Fubini's theorem for the usual Darboux/Riemann integral. In particular f has to be integrable (hence bounded) to apply the result stated in class.
So we can't directly apply the theorem here.

Actually, Fubini's theorem is far more general. Here the issue is actually that the integral of f is not improperly convergent (i.e. absolutely convergent).

Exo 11

① Let $F(x,y) = \frac{\partial^2 f}{\partial x \partial y}(x,y) - \frac{\partial^2 f}{\partial y \partial x}(x,y)$

Let $(x_0,y_0) \in \mathcal{U}$.
Since $F(x_0,y_0) > 0$, by continuity of F, there exists a rectangle R containing (x_0,y_0) s.t.

$$\forall (x,y) \in R,\ F(x,y) > \frac{F(x_0,y_0)}{2}$$

then $\int_R F(x,y) > \frac{F(x_0,y_0)}{2} \, \mathcal{J}(R) > 0$

② Assume that $R = [a,b] \times [c,d]$ then

$$0 < \int_R \frac{\partial^2 f}{\partial x \partial y} - \frac{\partial^2 f}{\partial y \partial x} = \int_c^d \int_a^b \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) dx\, dy - \int_a^b \int_c^d \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) dy\, dx$$

$$= \int_c^d \frac{\partial f}{\partial y}(b,y) - \frac{\partial f}{\partial y}(a,y)\, dy - \int_a^b \frac{\partial f}{\partial x}(x,d) - \frac{\partial f}{\partial x}(x,c)\, dx$$

$$= f(b,d) - f(a,d) - f(b,c) + f(a,c) - f(b,d) + f(b,c) + f(a,d) - f(a,c)$$

$$= 0$$

Contradiction!

Ex 12:

We can apply the theorem $\int \leftrightarrow \frac{\partial}{\partial x}$ (check the assumptions)

so $F'(x) = \int_0^1 \frac{2x}{x^2+y^2}\,dy = \int_0^1 \frac{2}{1+(y/x)^2} \cdot \frac{dy}{x}$

$u = \frac{y}{x}$ $\quad = \int_0^{1/x} \frac{2}{1+u^2}\,du$

$= 2\arctan(1/x)$

Ex 13:

Oups, I've just realized that I forgot this exercise...

You have to use the generalized theorem to differentiate under the integral from Exercise 9!

Sorry for that!

ADDENDUM (March 25): the full solution is next page!

$$v(x) = \int_0^x (x-y)\, e^{x-y} f(y)\, dy$$

- $F(x,y) = (x-y)\, e^{x-y} f(y)$ is C^0 on $\mathbb{R}^2$
- $\frac{\partial F}{\partial x}(x,y) = (x-y+1)\, e^{x-y} f(y)$ is C^0 on $\mathbb{R}^2$

So we can apply the formula from Ex 9, 2: v is C^1 and

$$v'(x) = \int_0^{x \to \varphi_2(x)} \frac{\partial F}{\partial x}(x,y)\, dy + \underbrace{1}_{\varphi_2'(x)} \cdot F(x, \underset{\varphi_2(x)}{x})$$

$$= \int_0^x (x-y+1)\, e^{x-y} f(y)\, dy + 0$$

$$= \int_0^x (x-y+1)\, e^{x-y} f(y)\, dy$$

- $G(x,y) = (x-y+1)\, e^{x-y} f(y)$ and $\frac{\partial G}{\partial x}(x,y) = (x-y+2)\, e^{x-y} f(y)$ are C^0 on $\mathbb{R}^2$

So (again by Ex 9 2)

$$v''(x) = \int_0^x \frac{\partial G}{\partial x}(x,y)\, dy + 1 \cdot G(x,x)$$

$$= \int_0^x (x-y+2)\, e^{x-y} f(y)\, dy + (x-x+1)\, e^{x-x} f(x)$$

$$= \int_0^x (x-y+2)\, e^{x-y} f(y)\, dy + f(x)$$

Then $v''(x) - 2v'(x) + v(x) = \int_0^x (x-y+2)\, e^{x-y} f(y)\, dy + f(x)$

$$- 2\int_0^x (x-y+1)\, e^{x-y} f(y)\, dy + \int_0^x (x-y)\, e^{x-y} f(y)\, dy$$

$$= f(x) + \int_0^x e^{x-y} f(y)\, \underbrace{(x-y+2-2x+2y-2+x-y)}_{=0}\, dy$$

$$= f(x)$$

Ex 15

① $\displaystyle\int_1^\infty \left(\int_1^\infty \frac{y-x}{(x+y)^3}\,dx \right) dy$

$$= \int_1^\infty \left[\frac{x}{(x+y)^2} \right]_1^\infty dy$$

$$= \int_1^\infty - \frac{1}{(1+y)^2}$$

$$= \left[\frac{1}{1+y} \right]_1^\infty$$

$$= -1/2$$

② $\displaystyle\int_1^\infty \left(\int_1^\infty \frac{y-x}{(x+y)^3}\,dy \right) dx = \int_1^{+\infty} \frac{1}{(1+x)^2}\,dx = \left[-\frac{1}{1+x} \right]_1^\infty = 1/2$

③ $\displaystyle\int_{[1,\infty]\times[1,\infty]} \frac{y-x}{(x+y)^3}$ is not improperly convergent

(ie absolutely !)

Ex 16

Since $f(x) = e^{-\alpha \|x\|^2}$ is positive, we have:

$$\int_{\mathbb{R}^m} e^{-\alpha \|x\|^2} = \int_{\mathbb{R}^m} e^{-\alpha (\Sigma x_i^2)}$$

$$= \int_{\mathbb{R}^m} e^{-\Sigma \alpha x_i^2}$$

$$= \prod_{i=1}^{m} \int_{\mathbb{R}} e^{-\alpha x_i^2} dx_i$$

$$= \left(\int_{-\infty}^{+\infty} e^{-\alpha x^2} dx \right)^m$$ (eventually $+\infty$ at this point)

then, since $e^{-\alpha x^2} > 0$, we may take:

$$\int_{-\infty}^{+\infty} e^{-\alpha x^2} dx = \lim_{m \to +\infty} \int_{-m}^{m} e^{-\alpha x^2} dx = \lim_{m \to +\infty} \int_{-\sqrt{\alpha} m}^{\sqrt{\alpha} m} e^{-u^2} \frac{du}{\sqrt{\alpha}}$$

the value (possibly $+\infty$) doesn't depend on the exhaustion since ≥ 0

$$= \frac{1}{\sqrt{\alpha}} \int_{-\infty}^{+\infty} e^{-u^2} du$$

$$= \sqrt{\frac{\pi}{\alpha}}$$

So $$\int_{\mathbb{R}^m} e^{-\alpha \|x\|^2} = \left(\frac{\pi}{\alpha}\right)^{m/2}$$

Ex 17

Since the integrand is positive, the value of the integral (possibly $+\infty$) doesn't depend on the exhaustion:

$$\int_{\{x^2+y^2<1\}} \frac{dx\,dy}{(1-x^2-y^2)^\alpha} = \lim_{m\to+\infty} \int_{(x^2+y^2)^{1/2}\leq 1-\frac{1}{m}} \frac{dx\,dy}{(1-x^2-y^2)^\alpha}$$

for the exhaustion, we take disks bigger and bigger

$$= \lim_{m\to+\infty} \int_{-\pi}^{\pi}\int_0^{1-\frac{1}{m}} \frac{r\,dr\,d\theta}{(1-r^2)^\alpha}$$

CoV : polar coordinates

$$= \lim_{m\to+\infty} 2\pi \int_0^{1-\frac{1}{m}} \frac{r}{(1-r^2)^\alpha}\,dr$$

CoV: $u = 1-r^2$

$$= \lim_{m\to+\infty} -\pi \int_1^{1-(1-\frac{1}{m})^2} \frac{1}{u^\alpha}\,du$$

$$= \lim_{m\to+\infty} \pi \int_{1-(1-\frac{1}{m})^2}^{1} \frac{1}{u^\alpha}\,du$$

which is CV iff $\alpha < 1$
(Riemann improper integrals)

If you don't remember:

$$\int_N^1 \frac{1}{u^\alpha}\,du = \begin{cases} \left[\frac{u^{1-\alpha}}{1-\alpha}\right]_N^1 & \text{if } \alpha \neq 1 \\ \left[\ln u\right]_N^1 & \text{if } \alpha = 1 \end{cases}$$

$$= \begin{cases} \frac{1}{1-\alpha} - \frac{N^{1-\alpha}}{1-\alpha} & \text{if } \alpha\neq 1 \\ -\ln N \xrightarrow{N\to 0^+} +\infty & \text{if } \alpha = 1 \end{cases} \xrightarrow{N\to 0^+} \begin{cases} \frac{1}{1-\alpha} & \text{if } \alpha<1 \\ +\infty & \text{if } \alpha>1 \end{cases}$$

Ex 18:

(1) $$\int_0^1\int_0^1\int_0^\infty \frac{1}{(1+x^2z^2)(1+y^2z^2)}\,dz\,dx\,dy$$ ← the integrand is ≥ 0 so no need to be too cautious (but be careful still)

$$= \int_0^1\int_0^1 \frac{1}{x^2-y^2}\left(\int_0^\infty \frac{x^2}{1+x^2z^2} - \frac{y^2}{1+y^2z^2}\,dz\right)dx\,dy$$

$$= \int_0^1\int_0^1 \frac{1}{x^2-y^2}\Big[x\arctan(xz) - y\arctan(yz)\Big]_0^\infty dx\,dy$$

$$= \frac{\pi}{2}\int_0^1\int_0^1 \frac{x-y}{x^2-y^2}\,dx\,dy$$

$$= \frac{\pi}{2}\int_0^1\int_0^1 \frac{1}{x+y}\,dx\,dy$$ Be careful, here it is improper at (0,0), but the integrand is positive.

$$= \frac{\pi}{2}\int_0^1 \Big[\ln(x+y)\Big]_0^1 dy$$

$$= \frac{\pi}{2}\int_0^1 \ln(1+y) - \ln(y)\,dy$$

$$= \frac{\pi}{2}\Big[(1+y)\ln(1+y) - (1+y) - y\ln y + y\Big]_0^1$$

Here I went very fast, that's an improper integral at 0, for the lower bound, I took the limit when y goes to 0 (I didn't evaluate at 0): $\lim_{y\to 0^+} y\ln y = 0$

$$= \pi\ln(2)$$

If you forget the antiderivative of ln:

$$\int_1^x \ln(t)\,dt \underset{\text{parts}}{=} \Big[t\ln t\Big]_1^x - \int_1^x dt = x\ln x - x + 1$$

$u = \ln t$, $v' = 1$

$u' = \frac{1}{t}$, $v = t$

So $\boxed{\int \ln x = x\ln x - x}$

Ex 18

(2) $$\int_0^\infty \int_0^1 \int_0^1 \frac{1}{(1+x^2z^2)(1+y^2z^2)}\, dx\, dy\, dz \qquad = \pi \ln 2$$

$$= \int_0^\infty \left(\int_0^1 \frac{1}{1+x^2z^2}\, dx \right)^2 dz$$

$$= \int_0^\infty \left(\frac{\arctan z}{z} \right)^2 dz$$

Ex 19 :

① $\sigma(t) = (e^t \cos t, e^t \sin t)$

$\sigma'(t) = (e^t(\cos t - \sin t), e^t(\sin(t) + \cos t))$

$\|\sigma'(t)\|^2 = e^{2t}((\cos t - \sin t)^2 + (\cos t + \sin t)^2)$

$= e^{2t}(\cos^2 t + \sin^2 t - 2\cos t \sin t + \cos^2 t + \sin^2 t + 2\cos t \sin t)$

$= 2e^{2t}$

So $\|\sigma'(t)\| = \sqrt{2}\, e^t$

$\ell(C) = \int_0^{\pi/2} \sqrt{2}\, e^t = \sqrt{2}(e^{\frac{\pi}{2}} - 1)$

② $\sigma(t) = (\ln t, 2t, t^2)$

$\sigma'(t) = (1/t, 2, 2t)$

$\|\sigma'(t)\|^2 = \frac{1}{t^2} + 4 + 4t^2 = \frac{1 + 4t^2 + 4t^4}{t^2} = \frac{(1+2t^2)^2}{t^2}$

So $\|\sigma'(t)\| = \frac{1+2t^2}{t}$ ($t \in [1, e]$ so > 0)

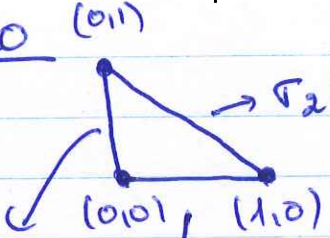
and $\ell(C) = \int_1^e \frac{1}{t} + 2t = [\ln t + t^2]_1^e$

$= 1 + e^2 - 1 = e^2$

In Exercise 20, we are computing line integrals of SCALAR fields (not vector fields), so the answer doesn't depend on the orientation (that's why I didn't precise any orientation in the question)!

Ex 20

(1)



$\sigma_2(t) = (1-t, t)$, $t \in [0,1]$

$\sigma_3 = (0, 1-t)$, $t \in [0,1]$

$\sigma_1(t) = (t, 0)$, $t \in [0,1]$

So $$\int_C x+y = \int_{C_1} x+y + \int_{C_2} x+y + \int_{C_3} x+y$$

$$= \int_0^1 t + \int_0^1 \sqrt{2} + \int_0^1 (1-t)$$

$$= \frac{1}{2} + \sqrt{2} + \frac{1}{2} = 1 + \sqrt{2}$$

(2) $\sigma(t) = (t\cos t, t\sin t, t)$, $\sigma'(t) = (\cos t - t\sin t, \sin t + t\cos t, 1)$

$$\|\sigma'(t)\|^2 = (\cos t - t\sin t)^2 + (\sin t + t\cos t)^2 + 1$$

$$= \cos^2 t - 2t\cos t\sin t + t^2\sin^2 t$$
$$+ \sin^2 t + 2t\cos t\sin t + t^2\cos^2 t + 1$$

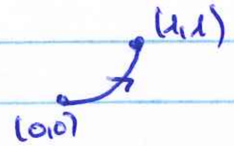
$$= 2 + t^2$$

$$\int_C z = \int_0^a t\sqrt{2+t^2}\, dt \underset{u=t^2}{=} \frac{1}{2}\int_0^{a^2} \sqrt{2+u}\, du = \left[\frac{(2+u)^{3/2}}{3}\right]_0^{a^2}$$

$$= \frac{(2+a^2)^{3/2} - 2^{3/2}}{3}$$

Ex 21

① $\sigma(t) = (t, t^2)$, $t \in [0,1]$



$\sigma'(t) = (1, 2t)$

$$\int_C x e^{-y} dx + \sin(\pi x) dy = \int_0^1 t e^{-t^2} + 2t \sin(\pi t)\, dt$$

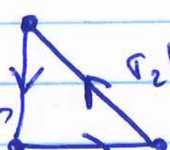
parts: $u = t$, $v' = \sin(\pi t)$; $u' = 1$, $v = -\frac{\cos(\pi t)}{\pi}$

$$= \left[-\frac{1}{2} e^{-t^2}\right]_0^1 + 2\left(\left[-\frac{t\cos(\pi t)}{\pi}\right]_0^1 + \frac{1}{\pi}\int_0^1 \cos \pi t\, dt\right)$$

$$= \frac{1}{2} - \frac{1}{2e} + \frac{2}{\pi} + \frac{2}{\pi^2}\left[\sin \pi t\right]_0^1$$

$$= \frac{1}{2} - \frac{1}{2e} + \frac{2}{\pi}$$

②



$\sigma_2(t) = (1-t, t)$, $t \in [0,1]$

$\sigma_3(t) = (0, 1-t)$, $t \in [0,1]$

$\sigma_1(t) = (t, 0)$, $t \in [0,1]$

Oups, you'll notice that I am solving for the counter-clockwise orientation, whereas I asked in the question to use the clockwise orientation...
Sorry for that, I was wearing my trigonometric watch!

That's easy to fix: we will multiply by -1 at the end!

$$\int_{-C} y^2 dx - 2x dy = \int_0^1 0 + \int_0^1 t^2 - 2(1-t)\, dt + \int_0^1 0$$

Here I use "-C" to say that I work with the opposite orientation!

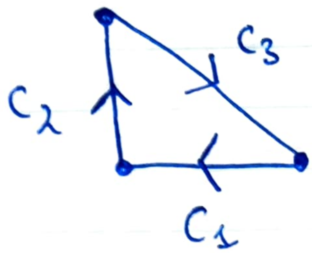
$$= \int_0^1 -t^2 + 2t - 2\, dt$$

$$= \left[-\frac{t^3}{3} + t^2 - 2t\right]_0^1$$

$$= -\frac{1}{3} + 1 - 2 = -\frac{4}{3}$$

Then $\displaystyle\int_C y^2 \mathrm{d}x - 2x\mathrm{d}y = -\int_{-C} y^2 \mathrm{d}x - 2x\mathrm{d}y = \frac{4}{3}$

② Directly with the good orientation



$C_1 : \sigma_1(t) = (1-t, 0)\ ,\ t \in [0,1]$

$C_2 : \sigma_2(t) = (0, t)\ ,\ t \in [0,1]$

$C_3 : \sigma_3(t) = (t, 1-t)\ ,\ t \in [0,1]$

So $\displaystyle\int_C y^2 dx - 2x dy$

$$= \int_{C_1} y^2 dx - 2x dy + \int_{C_2} y^2 dx - 2x dy + \int_{C_3} y^2 dx - 2x dy$$

$$= \int_0^1 0 \, dt + \int_0^1 0 \, dt + \int_0^1 t^2 \times 1 - 2t \times (-1) \, dt$$

$$= \int_0^1 t^2 + 2t \, dt$$

$$= \left[\frac{t^3}{3} + t^2 \right]_0^1$$

$$= \frac{1}{3} + 1$$

$$= \frac{4}{3}$$

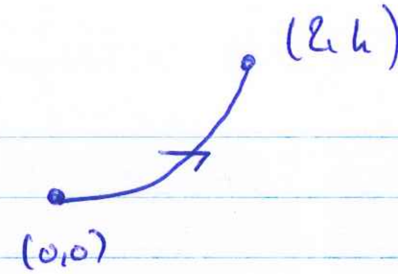
Comment: Here, it was also possible to use Green's theorem

$$\int_C y^2 dx - 2x dy = -\iint_S -2 - 2y = 2\iint_S 1+y = 2\int_0^1 \int_0^{1-x} 1+y \, dy \, dx = \frac{4}{3}$$

I don't use the positive orientation so $-$

(3) **Method 1:** $\sigma(t) = (t, t^2),\ t \in [0,2]$

$\sigma'(t) = (1, 2t)$



$$\int_C y\,dx + x\,dy = \int_0^2 t^2 + 2t^2\,dt = 3\left[\frac{t^3}{3}\right]_0^2$$

$$= \frac{3\ 2^3}{3} = 8$$

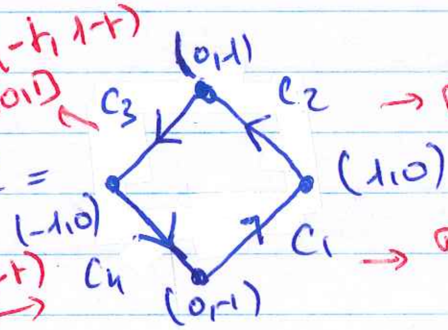
Method 2: $F(x,y) = (y,x)$

$F = \nabla f$ where $f(x,y) = xy$

so $\displaystyle\int_C y\,dx + x\,dy = \int_C \vec{\nabla f} \cdot d\vec{x} \underset{\uparrow}{=} f(2,4) - f(0,0) = 8$

Gradient theorem

(4) $C =$



$\sigma_3(t) = (-t, 1-t)$, $t \in [0,1]$

$\to \sigma_2(t) = (1-t, t)$, $t \in [0,1]$

$\to \sigma_1(t) = (t, t-1)$, $t \in [0,1]$

$\sigma_4(t) = (t-1, -t)$, $t \in [0,1]$

$$\int_C y|y|\,dx + x|x|\,dy = \int_0^1 -(t-1)^2 + t^2\,dt + \int_0^1 -t^2 + (1-t)^2\,dt$$

$$+ \int_0^1 -(1-t)^2 + t^2\,dt + \int_0^1 -t^2 + (t-1)^2\,dt$$

$$= 0$$

Ex 22:

① Let $f(x,y) = x^3y + \frac{1}{2}x^2y^2$

then $\nabla f(x,y) = (3xy + xy^2, x^3 + x^2y) = F(x,y)$

② By the Gradient theorem:

$$\int_C \vec{F} \cdot d\vec{x} = f(7,8) - f(0,0)$$

$$= 7^3 \cdot 8 + \frac{1}{2} 7^2 8^2$$

$$= 4312$$

Ex 23: Let $\sigma : [a,b] \to C$ be a parametrization of C
(I don't care about the orientation because of the absolute value: if I don't use the good one, the $|\cdot|$ will kill the $-$)

$$\left| \int_C \vec{F} \cdot d\vec{x} \right| = \left| \int_a^b F(\sigma(t)) \cdot \sigma'(t)\, dt \right|$$

$$\leqslant \int_a^b |F(\sigma(t)) \cdot \sigma'(t)|\, dt$$

Cauchy-Schwarz $\to$ $$\leqslant \int_a^b \|F(\sigma(t))\|\, \|\sigma'(t)\|\, dt$$

$$\leqslant \max_C \|F\| \int_a^b \|\sigma'(t)\| dt$$

$$= \max_C \|F\|\, \mathcal{L}(C)$$

Ex24: 1 Method 1

$$S = \{\sigma(\theta,\varphi) : \theta \in [0,2\pi],\ \varphi \in [0,\pi]\}$$

$$\sigma(\theta,\varphi) = (\cos\theta \sin\varphi, \sin\theta \sin\varphi, \cos\varphi)$$

$$\|\partial_\theta \sigma \times \partial_\varphi \sigma\| = \left\| \begin{pmatrix} -\sin\theta \sin\varphi \\ \cos\theta \sin\varphi \\ 0 \end{pmatrix} \times \begin{pmatrix} \cos\theta \cos\varphi \\ \sin\theta \cos\varphi \\ -\sin\varphi \end{pmatrix} \right\|$$

$$= \left\| \begin{pmatrix} -\cos\theta \sin^2\varphi \\ -\sin\theta \sin^2\varphi \\ -\sin\varphi \cos\varphi \end{pmatrix} \right\|$$

$$= \left(\cos^2\theta \sin^4\varphi + \sin^2\theta \sin^4\varphi + \sin^2\varphi \cos^2\varphi \right)^{1/2}$$

$$= \left(\sin^4\varphi + \sin^2\varphi \cos^2\varphi \right)^{1/2} = \sin\varphi$$

So $$\iint_S z^2 = \int_0^\pi \int_0^{2\pi} \sin\varphi \cos^2\varphi \, d\theta \, d\varphi$$

$$= 2\pi \int_0^\pi \sin\varphi \cos^2\varphi \, d\varphi$$

$u = \cos\varphi$
$du = -\sin\varphi \, d\varphi$
$$= -2\pi \int_1^{-1} u^2 \, du$$

$$= \frac{2}{3}\pi \left[u^3\right]_{-1}^{+1}$$

$$= \frac{4\pi}{3}$$

Method 2 :

By symmetry $\iint_S x^2 = \iint_S y^2 = \iint_S z^2$

So $\iint_S x^2 = \frac{1}{3} \iint_S x^2+y^2+z^2$

$$= \frac{1}{3} \iint_S 1$$

$$= \frac{1}{3} \text{Area}(S)$$

$$= \frac{4\pi}{3}$$

(2) $\sigma(r,\theta) = (r\cos\theta,\ r\sin\theta,\ \theta)$

$$\|\partial_r\sigma \times \partial_\theta\sigma\| = \left\| \begin{pmatrix} \cos\theta \\ \sin\theta \\ 0 \end{pmatrix} \wedge \begin{pmatrix} -r\sin\theta \\ r\cos\theta \\ 1 \end{pmatrix} \right\|$$

$$= \left\| \begin{pmatrix} \sin\theta \\ -\cos\theta \\ r \end{pmatrix} \right\| = \sqrt{1+r^2}$$

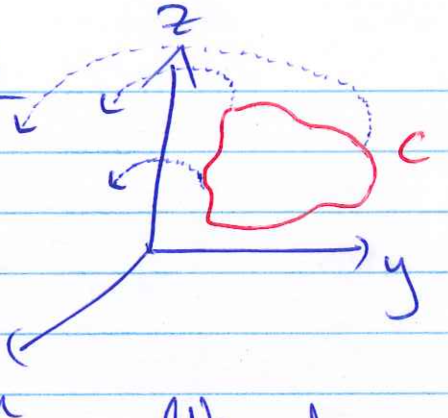
$$\iint_S \sqrt{1+x^2+y^2} = \int_0^{2\pi}\int_0^1 \sqrt{1+r^2}\sqrt{1+r^2}\, dr\, d\theta$$

$$= 2\pi \int_0^1 1+r^2\, dr$$

$$= 2\pi\left(1+\frac{1}{3}\right) = \frac{8\pi}{3}$$

Ex 25

(1)



We apply a rotation of angle θ around z:

$$\begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ y(t) \\ z(t) \end{pmatrix} = \begin{pmatrix} -\sin\theta\, y(t) \\ \cos\theta\ y(t) \\ z(t) \end{pmatrix}$$

So $S = \{\sigma(t,\theta) : t \in [a,b],\ \theta \in [0, 2\pi]\}$

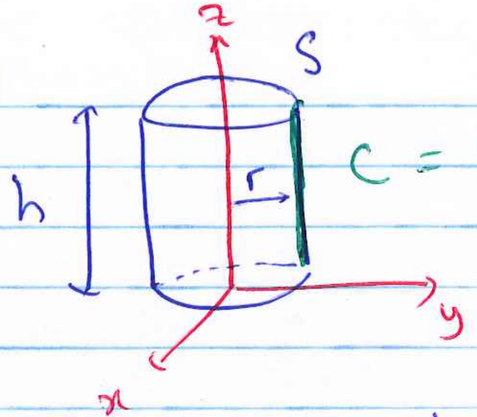
where $\sigma(t,\theta) = (-y(t)\sin\theta\ , y(t)\cos\theta,\ z(t))$

(2) $$\|\partial_t\sigma \times \partial_\theta\sigma\| = \left\| \begin{pmatrix} -y'(t)\sin\theta \\ y'(t)\cos\theta \\ z'(t) \end{pmatrix} \times \begin{pmatrix} -y(t)\cos\theta \\ -y(t)\sin\theta \\ 0 \end{pmatrix} \right\|$$

$$= \left\| \begin{matrix} y(t)z'(t)\sin\theta \\ -y(t)z'(t)\cos\theta \\ y'(t)y(t) \end{matrix} \right\| = |y(t)| \sqrt{y'(t)^2 + z'(t)^2}$$

So $$\iint_S f = \int_0^{2\pi}\int_a^b f(-y(t)\sin\theta, y(t)\cos\theta, z(t))\, |y(t)| \sqrt{y'(t)^2 + z'(t)^2}\, dt\, d\theta$$

(3)

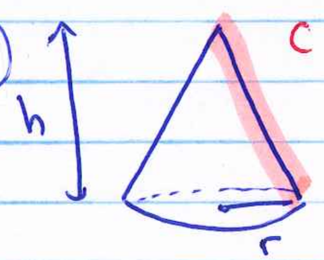


$C = \{(0, r, t) : t \in [0, h]\}$

So $\displaystyle\iint_S 1 = \int_0^{2\pi}\int_0^h 1 \cdot |r| \cdot \sqrt{0^2 + 1^2}\, dt\, d\theta$

$= 2\pi r h$

(4)



C is the line segment from $(0,0,h)$ to $(0,r,0)$

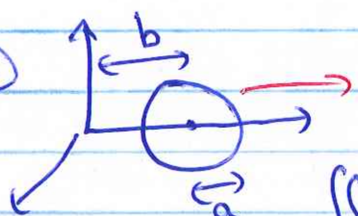
$r(t) = (0, tr, (1-t)h), \quad t \in [0,1]$

$\displaystyle\iint_S 1 = \int_0^{2\pi}\int_0^1 1 \cdot |tr| \cdot \sqrt{r^2+h^2}\, dt\, d\theta$

$\displaystyle = 2\pi r\sqrt{r^2+h^2} \int_0^1 t\, dt$

$= \pi r \sqrt{r^2+h^2}$

(5)



$C = \{(0, b + a\cos t, a\sin t) : t \in [0, 2\pi]\}$

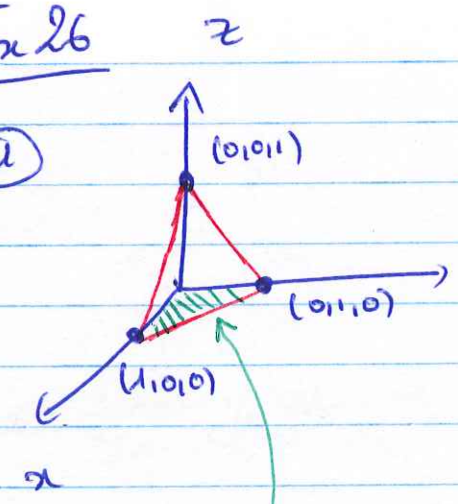
$\displaystyle\iint_S 1 = \int_0^{2\pi}\int_0^{2\pi} 1 \cdot |b + a\cos t|\, \sqrt{a^2}\, dt\, d\theta$

$\displaystyle = 2\pi \int_0^{2\pi} ab + a^2\cos t\, dt$

$= 4\pi^2 ab$

Ex 26

(1)



We know that $\vec{n}$ is orthogonal to $1x+1y+1z=1$

So $\vec{n} = \lambda(1,1,1)$

So $\vec{n} = \frac{1}{\sqrt{3}}(1,1,1)$ everywhere on S

($\vec{n}$ is pointing to you when you look at the drawing)

Let $T = \{(x,y) : x \geq 0, y \geq 0, x+y \leq 1\}$

then $z = 1-x-y$, so $\sigma(x,y) = (x,y,1-x-y)$

So $S = \{(x,y,1-x-y) : (x,y) \in T\}$

$$\partial_x\sigma \times \partial_y\sigma = \begin{pmatrix}1\\0\\-1\end{pmatrix} \times \begin{pmatrix}0\\1\\-1\end{pmatrix} = \begin{pmatrix}1\\1\\1\end{pmatrix} = \underset{\underset{=\sqrt{3}}{\lambda>0}}{\lambda} \vec{n}$$ so we have the good orientation !!! :-)

$$\iint_S \vec{F}\cdot\vec{n} = \iint_T \begin{pmatrix}x\\y^2\\1-x-y\end{pmatrix} \cdot \begin{pmatrix}1\\1\\1\end{pmatrix} = \int_0^1\int_0^{1-y} y^2-y+1 \, dx\, dy$$

$$= \int_0^1 (1-y)(y^2-y+1)\, dy$$

$$= \int_0^1 -y^3+2y^2-2y+1\, dy$$

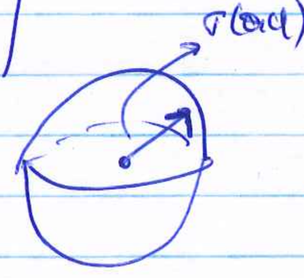
$$= -\frac{1}{4}+\frac{2}{3}-1+1$$

$$= \frac{5}{12}$$

② Using spherical coordinates

$$\sigma(\theta,\varphi) = (\cos\theta\sin\varphi,\ \sin\theta\sin\varphi,\ \cos\varphi) \quad \theta \in [0,2\pi],\ \varphi \in [0,\pi]$$

$$\partial_\theta\sigma \times \partial_\varphi\sigma = \begin{pmatrix} -\cos\theta\sin^2\varphi \\ -\sin\theta\sin^2\varphi \\ -\sin\varphi\cos\varphi \end{pmatrix} = -\sin\varphi \begin{pmatrix} \cos\theta\sin\varphi \\ \sin\theta\sin\varphi \\ \cos\varphi \end{pmatrix}$$



$$= \underbrace{-\sin\varphi}_{<0}\ \sigma(\theta,\varphi)$$

points inward

Oups...:... We don't have the good orientation :'c...

Method 1: $\partial_\varphi\sigma \times \partial_\theta\sigma = -\partial_\theta\sigma \times \partial_\varphi\sigma$ so you swap the variables $\sigma(\varphi,\theta) = \cdots$ and you have a parametrization compatible with S

Method 2: Since the question doesn't ask me explicitely to find a parametrization compatible with the orientation, just to compute $\iint_S \vec{F}\cdot\vec{n}$. You can be lazy and simply multiply by -1.

$$\iint_S \vec{F}\cdot\vec{n} = \int_0^{\pi}\int_0^{2\pi} \begin{pmatrix} \cos^2\theta\sin^2\varphi \\ \cos\varphi \\ -\sin\theta\sin\varphi \end{pmatrix} \begin{pmatrix} \cos\theta\sin^2\varphi \\ \sin\theta\sin^2\varphi \\ \sin\varphi\cos\varphi \end{pmatrix} d\theta\, d\varphi$$

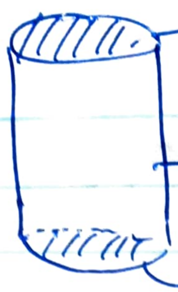
$$= \int_0^{\pi}\int_0^{2\pi} \cos^3\theta\sin^4\varphi + \cancel{\cos\varphi\sin\theta\sin^2\varphi} - \cancel{\sin\theta\sin^2\varphi\cos\varphi}\, d\theta\, d\varphi$$

$$= \int_0^{\pi}\int_0^{2\pi} \cos^3\theta\sin^4\varphi\, d\theta\, d\varphi$$

$= \dots$ I am lazy... you just have to linearize $\cos^3\theta$ and $\sin^4\varphi$ using $\cos^2 x = \frac{\cos(2x)+1}{2}$

$\sin^2 x = \frac{1-\cos 2x}{2}$

(3)

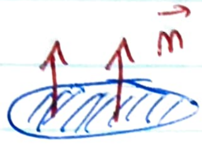


$\to S_1$ upper part

$\to S_2$ cylinder

$\to S_3$ bottom part

<u>S_1</u> :



$r(r,\theta) = (r\cos\theta, r\sin\theta, 1)$, $r \in [0,1]$, $\theta \in [0,2\pi]$

$$\partial_r r \times \partial_\theta r = \begin{pmatrix} \cos\theta \\ \sin\theta \\ 0 \end{pmatrix} \times \begin{pmatrix} -r\sin\theta \\ r\cos\theta \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ r \end{pmatrix}$$

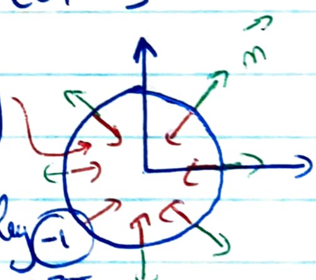
$r > 0$ so we have the good orientation (except at the origin, so that's OK, that's one point)

$$\iint_{S_1} \vec{F}\cdot\vec{m} = \int_0^{2\pi}\int_0^1 \begin{pmatrix} r\cos\theta \\ r\sin\theta \\ 1 \end{pmatrix}\cdot\begin{pmatrix} 0 \\ 0 \\ r \end{pmatrix} dr\,d\theta = 2\pi\int_0^1 r\,dr = \pi$$

<u>S_2</u>: when z is fixed, we have a circle, so it is easy to find a parametrization:

$r(z,\theta) = (\cos\theta, \sin\theta, z)$, $z \in [0,1]$, $\theta \in [0,2\pi]$

$$\partial_z r \times \partial_\theta r = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \times \begin{pmatrix} -\sin\theta \\ \cos\theta \\ 0 \end{pmatrix} = \begin{pmatrix} -\cos\theta \\ -\sin\theta \\ 0 \end{pmatrix}$$



So that's the bad orientation, so I multiply by -1

$$\iint_{S_2} \vec{F}\cdot\vec{m} = \int_0^1\int_0^{2\pi} \begin{pmatrix} \cos\theta \\ \sin\theta \\ z^2 \end{pmatrix}\cdot\begin{pmatrix} \cos\theta \\ \sin\theta \\ 0 \end{pmatrix} d\theta\,dz = \int_0^1\int_0^{2\pi} 1\,d\theta\,dz = 2\pi$$

<u>S_3</u>:

$r(r,\theta) = (r\cos\theta, r\sin\theta, 0)$, $r \in [0,1]$, $\theta \in [0,2\pi)$

$$\partial_r r \times \partial_\theta r = \begin{pmatrix} \cos\theta \\ \sin\theta \\ 0 \end{pmatrix} \times \begin{pmatrix} -r\sin\theta \\ r\cos\theta \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ r \end{pmatrix} \text{ bad orientation}$$

$$\text{So } \iint_{S_3} \vec{F}\cdot\vec{m} = \int_0^{2\pi}\int_0^1 \begin{pmatrix} r\cos\theta \\ r\sin\theta \\ 0 \end{pmatrix}\cdot\begin{pmatrix} 0 \\ 0 \\ -r \end{pmatrix} dr\,d\theta = \iint 0\,dr\,d\theta = 0$$

<u>CCL</u> $$\iint_S \vec{F}\cdot\vec{m} = \iint_{S_1} \vec{F}\cdot\vec{m} + \iint_{S_2} \vec{F}\cdot\vec{m} + \iint_{S_3} \vec{F}\cdot\vec{m} = \pi + 2\pi + 0 = 3\pi$$

Ex 27

(1) $C = \{(\cos\theta, \sin\theta) : \theta \in [0, 2\pi]\}$

$$\int_C \frac{-y}{x^2+4y^2}\,dx + \frac{x}{x^2+4y^2}\,dy$$

$$= \int_0^{2\pi} +\frac{\sin\theta}{\cos^2\theta+4\sin^2\theta}\cdot\sin\theta + \frac{\cos\theta}{\cos^2\theta+4\sin^2\theta}\cdot\cos\theta\,d\theta$$

$$= \int_0^{2\pi} \frac{1}{\cos^2\theta+4\sin^2\theta}\,d\theta$$

$$= 2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1/\cos^2\theta}{1+4\tan^2\theta}\,d\theta \qquad u = \tan\theta, \quad du = \frac{1}{\cos^2\theta}\,d\theta$$

By symmetry to stay in the domain of tan

$$= 2\int_{-\infty}^{+\infty} \frac{du}{1+4u^2}$$

$$= \Big[\arctan 2u\Big]_{-\infty}^{+\infty}$$

$$= \frac{\pi}{2} + \frac{\pi}{2} = \pi$$

(2) Just a computation ——— $C =$  $S =$ 

(3) By Green's theorem, we should obtain $\int_C \vec{F}\cdot\vec{m} = \iint_S 0 = 0 \neq \pi$

It seems to be a contradiction... We may have reached the limit of mathematics and should stop...

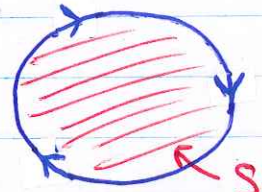
Or... Maybe... one assumption of Green's theorem is not satisfied...

I was scared for a short time... :-P

⚠ F has to be defined and C^1 on $\mathcal{U}$ open s.t. $S \subset \mathcal{U}$... But here $\vec{0} \in S = \overline{B}(\vec{0}, 1)$ and F is not defined at $\vec{0}$...

Ex 28: I have just written the last 5 solutions in a row and it is 2:30 AM... don't believe the following computations

(1)

C 

We don't have the good orientation to apply Green's theorem

$-C$ 

Now yes (by "$-C$" I mean C with the opposite orientation)

$$\int_C (1-x^2)y\,dx + (1+y^2)x\,dy$$

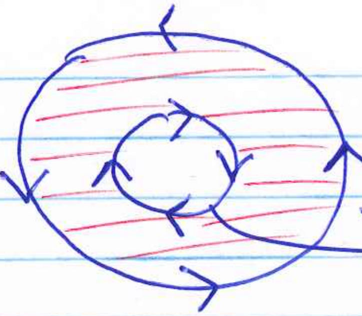
$$= -\int_{-C} (1-x^2)y\,dx + (1+y^2)x\,dy$$

$$= -\iint_S \cancel{1}+y^2 - \cancel{1}+x^2$$

$$= -\int_{-\pi}^{\pi}\int_0^a r^2 r\,dr\,d\theta = -2\pi\left[\frac{r^4}{4}\right]_0^a$$

$$= -\frac{\pi a^4}{2}$$

2



$C = C_1 \cup C_2$ is positively oriented by assumption

$$\int_C (-x^2y)\,dx + (xy^2)\,dy$$

Green's theorem

$$= \iint_S y^2 + x^2$$

$$= \int_0^{2\pi}\int_1^2 r^2\, r\, dr\, d\theta$$

$$= 2\pi \int_1^2 r^3\, dr$$

$$= 2\pi \left[\frac{r^4}{4}\right]_1^2$$

$$= \frac{\pi}{2}\left(2^4 - 1\right)$$

Good night

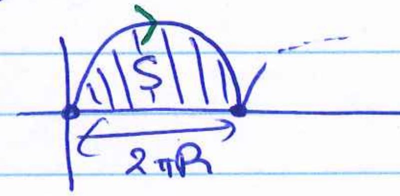
Ex 29

① Notice that $\sigma(\theta+2\pi) = (2\pi R, 0) + \sigma(\theta)$

and that $\sigma_2'(\theta) = R\sin\theta$

	0		π		2π
σ_2'	0	$+$	0	$-$	0
σ_2		$\nearrow$		$\searrow$	

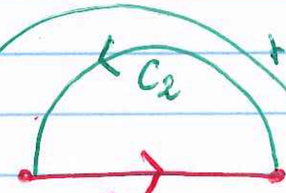
So one arch is given for $\theta \in [0, 2\pi]$:



$$\text{So } A = \iint_S 1 = \iint_S \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \quad \text{for } (P,Q) = (-y, 0)$$

$$= \int_C -y\,dx \quad \text{by Green's theorem}$$

$$= \int_{C_1} -y\,dx + \int_{C_2} -y\,dx$$



$\sigma_2(t) = (R(t-\sin t), R(1-\cos t))$, $t \in [0, 2\pi]$ with the opposite orientation

C_1: $\sigma(t) = (t, 0)$, $t \in [0, 2\pi R]$

$$= \int_0^{2\pi R} 0 = \int_0^{2\pi} -R(1-\cos t)R(1-\cos t)\,dt$$

$$= R^2 \int_0^{2\pi} (1-\cos t)^2\,dt$$

$$= R^2 \int_0^{2\pi} 1 - 2\cos t + \cos^2 t\,dt$$

$$= R^2 \int_0^{2\pi} 1 - 2\cos t + \frac{1+\cos 2t}{2}\,dt$$

$$= R^2 \times \frac{3}{2} \times 2\pi$$

$$= 3\pi R^2$$

Ex 30

Let $F(x,y) = (yx^3 + xe^y, xy^3 + ye^x - 2y)$ on $\mathbb{R}^2$

$$\frac{\partial F_2}{\partial x}(x,y) = y^3 + ye^x$$

$$\frac{\partial F_1}{\partial y}(x,y) = x^3 + xe^y$$

Hence by Green's theorem

$$\int_C \vec{F} \cdot d\vec{x} = \iint_S \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = \iint_S y^3 + ye^x - x^3 - xe^y$$

$$= \iint_S y^3 + ye^x - \iint_S x^3 + xe^y$$

$$= \iint_S y^3 + ye^x - \iint_S y^3 + ye^x$$ by symmetry:

$\Phi : S \xrightarrow{\cong} S$
$(x,y) \mapsto (y,x)$

$= 0$

Ex 31. $F(x,y,z) = (x^2, xyz, yz^2)$

$$\operatorname{div} F(x,y,z) = \sum_{i=1}^{3} \frac{\partial F_i}{\partial x_i}(x,y,z) = 2x + xz + 2yz$$

$$\operatorname{curl} F(x,y,z) = \begin{pmatrix} \partial/\partial x \\ \partial/\partial y \\ \partial/\partial z \end{pmatrix} \times \begin{pmatrix} F_1 \\ F_2 \\ F_3 \end{pmatrix}(x,y,z) = \begin{pmatrix} z^2 - xy \\ 0 \\ yz \end{pmatrix}$$

so $\operatorname{curl} F(x,y,z) = (z^2 - xy, 0, yz)$

$$\Delta F(x,y,z) = (\Delta F_1(x,y,z), \Delta F_2(x,y,z), \Delta F_3(x,y,z)) = (2,0,2y)$$

Ex 32

① $\frac{\partial}{\partial x_i}(fg) = f \cdot \frac{\partial g}{\partial x_i} + g \cdot \frac{\partial f}{\partial x_i}$

So $\nabla(fg) = \left(\frac{\partial}{\partial x_i}(fg)\right)_{i=1..m} = f\nabla g + g\nabla f$

② $\operatorname{div}(f\vec{G}) = \operatorname{div}(fG_1, fG_2, \dots, fG_m)$

$$= \sum_{i=1}^{m} \frac{\partial}{\partial x_i}(fG_i)$$

$$= \sum_{i=1}^{m}\left(\frac{\partial f}{\partial x_i} \cdot G_i + f\frac{\partial G_i}{\partial x_i}\right)$$

$$= f\left(\sum_{i=1}^{m} \frac{\partial G_i}{\partial x_i}\right) + \sum_{i=1}^{m} \frac{\partial f}{\partial x_i} \cdot G_i$$

$$= f \operatorname{div}\vec{G} + \nabla f \cdot \vec{G}$$

③ $$\operatorname{curl}(f\vec{G}) = \begin{pmatrix} \partial/\partial x \\ \partial/\partial y \\ \partial/\partial z \end{pmatrix} \times \begin{pmatrix} fG_1 \\ fG_2 \\ fG_3 \end{pmatrix}$$

$$= \begin{pmatrix} \dfrac{\partial(fG_3)}{\partial y} - \dfrac{\partial(fG_2)}{\partial z} \\[2ex] \dfrac{\partial(fG_1)}{\partial z} - \dfrac{\partial(fG_3)}{\partial x} \\[2ex] \dfrac{\partial(fG_2)}{\partial x} - \dfrac{\partial(fG_1)}{\partial y} \end{pmatrix}$$

$$= \begin{pmatrix} \dfrac{\partial f}{\partial y} \cdot G_3 + f\dfrac{\partial G_3}{\partial y} - \dfrac{\partial f}{\partial z} G_2 - f\dfrac{\partial G_2}{\partial z} \\[2ex] \dfrac{\partial f}{\partial z} G_1 + f\dfrac{\partial G_1}{\partial z} - \dfrac{\partial f}{\partial x} G_3 - f\dfrac{\partial G_3}{\partial x} \\[2ex] \dfrac{\partial f}{\partial x} G_2 + f\dfrac{\partial G_2}{\partial x} - \dfrac{\partial f}{\partial y} G_1 - f\dfrac{\partial G_1}{\partial y} \end{pmatrix}$$

$$= f \operatorname{curl}\vec{G} + \overrightarrow{\nabla f \cdot G}$$

④ $$\operatorname{div}(\vec{F}\times\vec{G}) = \operatorname{div}\begin{pmatrix} F_2G_3 - F_3G_2 \\ F_3G_1 - F_1G_3 \\ F_1G_2 - F_2G_1 \end{pmatrix}$$

$$= \frac{\partial}{\partial x}(F_2G_3 - F_3G_2) + \frac{\partial}{\partial y}(F_3G_1 - F_1G_3) + \frac{\partial}{\partial z}(F_1G_2 - F_2G_1)$$

$$= \frac{\partial F_2}{\partial x} \cdot G_3 + F_2\frac{\partial G_3}{\partial x} - \frac{\partial F_3}{\partial x}G_2 - F_3\frac{\partial}{\partial x}G_2 + \frac{\partial}{\partial y}F_3 G_1 + F_3\frac{\partial G_1}{\partial y} - \frac{\partial F_1}{\partial y}G_3 - F_1\frac{\partial G_3}{\partial y}$$
$$+ \frac{\partial F_1}{\partial z}G_2 + F_1\frac{\partial G_2}{\partial z} - \frac{\partial F_2}{\partial z}G_1 - F_2\frac{\partial G_1}{\partial z}$$

$$= G_1\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) + G_2\left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right) + G_3\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)$$
$$- F_1\left(\frac{\partial G_3}{\partial y} - \frac{\partial G_2}{\partial z}\right) - F_2\left(\frac{\partial G_1}{\partial z} - \frac{\partial G_3}{\partial x}\right) - F_3\left(\frac{\partial G_2}{\partial x} - \frac{\partial G_1}{\partial y}\right)$$

$$= \vec{G}\cdot\operatorname{curl}\vec{F} - \vec{F}\cdot\operatorname{curl}\vec{G}$$

the formula can't be symmetric in $\vec{F}$ and $\vec{G}$ since $\operatorname{div}(\vec{G}\times\vec{F}) = \operatorname{div}(-\vec{F}\times\vec{G}) = -\operatorname{div}(\vec{F}\times\vec{G})$

that's why there is a − here

Ex 33: This question seems very difficult/long: we have to find several antiderivatives

But let's think first: Assume $\exists\, \vec{F} : \mathcal{U} \to \mathbb{R}^3$ C^2 ($\mathcal{U} \subset \mathbb{R}^3$ open)

s.t. $\quad \operatorname{curl} \vec{F} = (x, y, z)$

then $\quad \operatorname{div}(\operatorname{curl} \vec{F}) = \dfrac{\partial x}{\partial x} + \dfrac{\partial y}{\partial y} + \dfrac{\partial z}{\partial z} = 3$

$\qquad \qquad \qquad \qquad \qquad \overset{\shortparallel}{0}$ by a result of the lecture

Contradiction

So there is no such $\vec{F}$.

Ex 34: First, let me give you a mnemonic device:

you know from the first term that $a \times (b \times c) = (a \cdot c) b - (a \cdot b) c$

so if you let $a = b = \nabla$, $c = F$ you get:

$$\nabla \times (\nabla \times F) = \nabla(\nabla \cdot F) - \nabla^2 F$$

However, it is not a proof since ∇ is not really a vector

Go back to the exo:

$$\nabla \times (\nabla \times F) = \nabla \times \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}, \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}, \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right)$$

$$= \left(\frac{\partial^2 F_2}{\partial y \partial x} - \frac{\partial^2 F_1}{\partial y^2} - \frac{\partial^2 F_1}{\partial z^2} + \frac{\partial^2 F_3}{\partial z \partial x}, \right.$$

$$\frac{\partial^2 F_3}{\partial z \partial y} - \frac{\partial^2 F_2}{\partial z^2} - \frac{\partial^2 F_2}{\partial x^2} + \frac{\partial^2 F_1}{\partial x \partial y},$$

$$\left. \frac{\partial^2 F_1}{\partial x \partial z} - \frac{\partial^2 F_3}{\partial x^2} - \frac{\partial^2 F_3}{\partial y^2} + \frac{\partial^2 F_2}{\partial y \partial z} \right)$$

for these 2 I used the symmetry to permute the variables from ...

I start to have some regrets about this question ___

It's time for a 5 min coffee break! BRB

I am back!

$$= \left(\left(\frac{\partial^2 F_1}{\partial x^2} + \frac{\partial^2 F_2}{\partial x \partial y} + \frac{\partial^2 F_3}{\partial x \partial z} \right) - \left(\frac{\partial^2 F_1}{\partial x^2} + \frac{\partial^2 F_2}{\partial y^2} + \frac{\partial^2 F_3}{\partial z^2} \right), \right.$$

$$\left(\frac{\partial^2 F_1}{\partial y \partial x} + \frac{\partial^2 F_2}{\partial y^2} + \frac{\partial^2 F_3}{\partial y \partial z} \right) - \left(\frac{\partial^2 F_2}{\partial x^2} + \frac{\partial^2 F_2}{\partial y^2} + \frac{\partial^2 F_3}{\partial z^2} \right),$$

$$\left. \left(\frac{\partial^2 F_1}{\partial z \partial x} + \frac{\partial^2 F_2}{\partial z \partial y} + \frac{\partial^3 F_3}{\partial z^2} \right) - \left(\frac{\partial^2 F_3}{\partial x^2} + \frac{\partial^2 F_3}{\partial y^2} + \frac{\partial^2 F_3}{\partial z^2} \right) \right)$$

$$= \nabla \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} \right) - \Delta F$$

$$= \nabla(\nabla \cdot F) - \Delta F$$

That was not that bad!

Ex35

① $\operatorname{div} \vec{r} = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} = 3$

$\nabla(\|\vec{r}\|^2) = \nabla(x^2+y^2+z^2) = (2x, 2y, 2z) = 2\vec{r}$

$\operatorname{curl} \vec{r} = \begin{pmatrix} \partial/\partial x \\ \partial/y \\ \partial/z \end{pmatrix} \times \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \vec{0}$

② • $\operatorname{curl}(\vec{a}\times\vec{r}) = \operatorname{curl}\begin{pmatrix} bz - cy \\ cx - az \\ ay - bx \end{pmatrix} = \begin{pmatrix} 2a \\ 2b \\ 2c \end{pmatrix} = 2\vec{a}$

for some reasons, I used $\vec{a} = (a, b, c)$...

• $\operatorname{div}((\vec{a}\cdot\vec{r})\vec{a}) = \operatorname{div}\begin{pmatrix} a_1^2 x \\ a_2^2 y \\ a_3^2 z \end{pmatrix} = a_1^2 + a_2^2 + a_3^2 = \|a\|^2$

• $\operatorname{div}((\vec{a}\times\vec{r})\times\vec{a}) = \vec{a}\cdot\operatorname{curl}(\vec{a}\times\vec{r}) - (\vec{a}\times\vec{r})\cdot\operatorname{curl}\vec{a}$

(4) of Exo 32

$= \vec{a}\cdot(2\vec{a}) - 0$

$= 2\,\vec{a}\cdot\vec{a} = 2\|\vec{a}\|^2$

(2)

• $\operatorname{div}(\|\vec{r}\|^m(\vec{a}\times\vec{r})) = \|r\|^m \operatorname{div}(\vec{a}\times\vec{r}) + \nabla(\|\vec{r}\|^m)\cdot(\vec{a}\times\vec{r})$

$= \|r\|^m\cdot 0 + m\|\vec{r}\|^{m-2}\,\vec{r}\cdot(a\times\vec{r})$

orthogonal to $\vec{r}$

$= 0$

Ex 36

① $\frac{\partial F_2}{\partial x} = 3x^2 + 2xy \qquad \frac{\partial F_1}{\partial y} = 2xy + 3x^2$

The domain $\mathbb{R}^2$ is starshaped so we can apply Poincaré Lemma to conclude tha $\vec{F}$ is conservative

ie $\exists f : \mathbb{R}^2 \to \mathbb{R}$ C^1 s.t. $\vec{F} = \nabla f$

② Using the formula seen in class (seen the domain is a rectangle)

$$f(x,y) = \int_0^x F_1(t,0)\,dt + \int_0^y F_2(x,t)\,dt$$

$$= \int_0^x 0\,dt + \int_0^y (x^3 + tx^2)\,dt$$

$f(x,y) = x^3 y + \frac{x^2 y^2}{2}$ is a suitable potential.

(We could have used a "guess and check" method here)

③ Use the Gradient theorem

Ex 37

$\vec{F}$ is defined on $\mathbb{R}^2 \setminus \{0\}$ which is not starshaped hence the assumptions of Poincaré lemma are not satisfied and there is no contradiction — OUF!

↳ "phew" in French!

(Actually Poincaré lemma holds when the domain is "contractible", but this notion is not part of MAT237 and $\mathbb{R}^2 \setminus \{0\}$ is not contractible) (Contractible is more general since starshaped $\Rightarrow$ contractible)