Green's theorem

Def: $S \subset \mathbb{R}^n$ is a regular region if it is compact and $S = \overline{\mathring{S}}$

Remark: $S = \overline{\mathring{S}}$ means that $\forall x \in \partial S, \forall r > 0, \; B(x,r) \cap \mathring{S} \neq \emptyset$

Eg:  $\{(x,y) \in \mathbb{R}^2 : x^2+y^2 \leq 1\}$ is a regular region

 $\{(x,0) : x \in [-1,1]\}$ is not a regular region in $\mathbb{R}^2$

Def: We say that $C \subset \mathbb{R}^n$ is a simple regular piecewise-C^1 closed curve

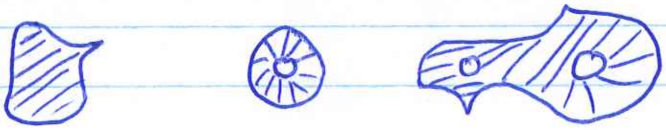
if $C = \{\sigma(t) : t \in [a,b]\}$ where $\sigma : [a,b] \longrightarrow \mathbb{R}^n$ satisfies:

1. σ is C^0
2. σ is injective on (a,b) "simple"
3. $\sigma(a) = \sigma(b)$ "closed"
4. There are finitely many $t_k \in [a,b]$, $a = t_0 < t_1 < \dots < t_K = b$ s.t.
   - σ is C^1 on (t_k, t_{k+1}) piecewise C^1
   - $\forall s \in (t_k, t_{k+1})$, $\sigma'(s) \neq \vec{0}$ regular
   - $\lim_{s \to t_k^+} \sigma'(s)$ and $\lim_{s \to t_{k+1}^-} \sigma'(s)$ exist

Remark: notice that line integrals are well-defined for piecewise C^1 curve.

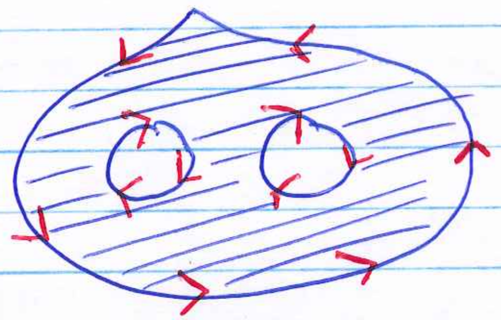
Def: We say that a regular region $S \subset \mathbb{R}^2$ has a piecewise smooth boundary if

$\partial S = C_1 \cup \dots \cup C_s$ where the C_i are disjoint simple regular piecewise-C^1 closed curves.

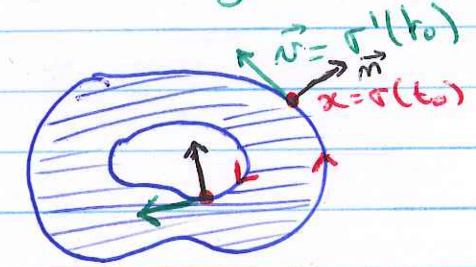
Ex: 

Def. Let $S \subset \mathbb{R}^2$ be a regular region with piecewise smooth boundary $\partial S = C_1 \cup \cdots \cup C_s$. We say that ∂S is *positively oriented* if the parametrization σ_i of C_i keeps $\mathring{S}$ on the left for $i = 1 \ldots s$.

Ex:



More formally: if $x = \sigma(t_0) = (\sigma_1(t_0), \sigma_2(t_0)) \in \partial S$
we set $\vec{v} = (v_1, v_2) = \sigma'(t_0) = (\sigma_1'(t_0), \sigma_2'(t_0))$
and we want that $\vec{n} = (v_2, -v_1)$ points outward of S
(we get $\vec{n}$ by rotating $\vec{v}$ by $\frac{\pi}{2}$ clockwise)



Definition: if $\partial S = C_1 \cup \cdots \cup C_s$ we write

$$\int_{\partial S} \vec{F} \cdot d\vec{x} := \sum_{i=1}^{s} \int_{C_i} \vec{F} \cdot d\vec{x}$$

↳ positively oriented

Theorem (Green's theorem)

Let $S \subset \mathbb{R}^2$ be a regular region with piecewise smooth boundary and $F : \mathcal{U} \to \mathbb{R}^2$ be a C^1-vector field where $\mathcal{U} \subset \mathbb{R}^2$ open satisfies $S \subset \mathcal{U}$.

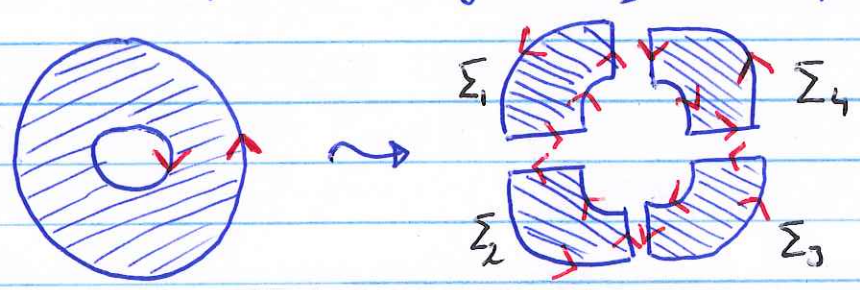
Then $$\int_{\partial S} \vec{F} \cdot d\vec{x} = \iint_S \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) \quad \text{where } \vec{F} = (F_1, F_2)$$
$\hookrightarrow$ positively oriented

or using the convenient (but dangerous) notation:
$$\int_{\partial S} P\,dx + Q\,dy = \iint_S \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \quad \text{where } \vec{F} = (P, Q)$$
$\hookrightarrow$ positively oriented

Remark: It's another special case of a general result:
Stokes theorem: $\int_\Omega d\omega = \int_{\partial \Omega} \omega$

Δ We prove Green's theorem in a special case: for S that can be broken into finitely many elementary regions.

We say that $\Sigma \subset \mathbb{R}^2$ is elementary if
$$\Sigma = \{(x,y) : a \leq x \leq b,\ \varphi_1(x) \leq y \leq \varphi_2(x)\} \quad \varphi_i\ C^0 \text{ and piecewise } C^1$$
and $$= \{(x,y) : c \leq y \leq d,\ \psi_1(y) \leq x \leq \psi_2(y)\} \quad \psi_i\ C^0 \text{ and piecewise } C^1$$

eg: S


Notice that: $\iint_S \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = \sum_{i=1}^{N} \iint_{\Sigma_i} \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$ (additivity of the domain)

and that $\int_{\partial S} \vec{F} \cdot d\vec{x} = \sum_{i=1}^{N} \int_{\partial \Sigma_i} \vec{F} \cdot d\vec{x}$ since the additional edges are added twice with opposite orientation and cancel each other

Hence, it is enough to prove the result for $\Sigma \subset \mathbb{R}^2$ elementary region:

$$\iint_\Sigma \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) = \iint_\Sigma \frac{\partial F_2}{\partial x} - \iint_\Sigma \frac{\partial F_1}{\partial y}$$

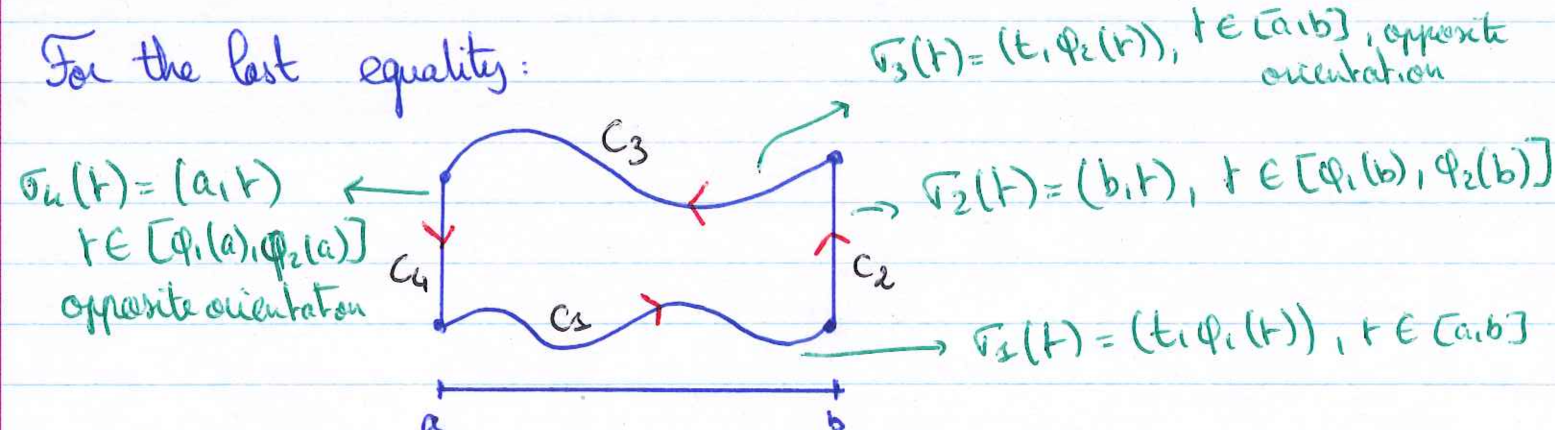
$$= \int_c^d \int_{\psi_1(y)}^{\psi_2(y)} \frac{\partial F_2}{\partial x}(x,y)\, dx\, dy - \int_a^b \int_{\varphi_1(x)}^{\varphi_2(x)} \frac{\partial F_1}{\partial y}(x,y)\, dy\, dx$$

$$= \int_c^d F_2(\psi_2(y),y) - F_2(\psi_1(y),y)\, dy$$

$$- \int_a^b F_1(x,\varphi_2(x)) - F_1(x,\varphi_1(x))\, dx$$

$$= \int_{\partial\Sigma} F_2\, dy + \int_{\partial\Sigma_1} F_1\, dx = \int_{\partial\Sigma_1} F_1 dx + F_2 dy$$

For the last equality:



so $\int_{\partial\Sigma} F_1 dx = \int_{C_1} F_1 dx + \int_{C_2} F_1 dx + \int_{C_3} F_1 dx + \int_{C_4} F_1 dx$

$$= \int_a^b F_1(t,\varphi_1(t)) \cdot 1\, dt + \int_{\varphi_1(b)}^{\varphi_2(b)} F_1(b,t) \cdot 0\, dt$$

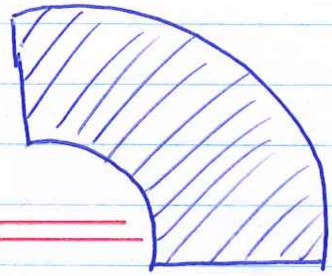
$$- \int_a^b F_1(t,\varphi_2(t)) \cdot 1\, dt - \int_{\varphi_1(a)}^{\varphi_2(a)} F_1(a,t) \cdot 0\, dt$$

$$= \int_a^b F_1(t,\varphi_1(t)) - F_1(t,\varphi_2(t))\, dt$$

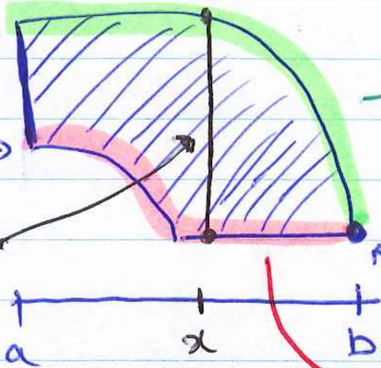
And similarly for $\int_{\partial\Sigma} F_2\, dy = \int_c^d F_2(\psi_2(t),t) - F_2(\psi_1(t),t)\, dt$

using $\Sigma_1 = \{(x,y) : c \leq y \leq d,\ \psi_1(y) \leq x \leq \psi_2(y)\}$ $\square$

ADDENDUM:

Why is $\Sigma =$  an elementary region?

$\varphi_2(a) \geq \varphi_1(a)$



(x,y) where $\varphi_1(x) \leq y \leq \varphi_2(x)$

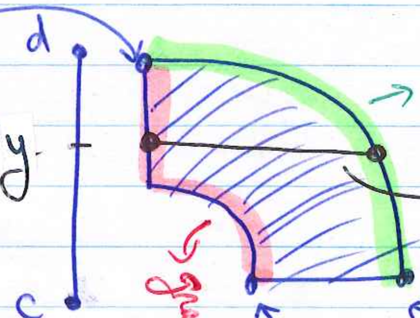
→ that's the graph of some function $y = \varphi_1(x)$ C^0 and piecewise C^1

that's the graph of some function $y = \varphi_2(x)$ C^0 and piecewise C^1

$\varphi_1(b) = \varphi_2(b)$

So $\Sigma = \{(x,y) : a \leq x \leq b,\ \varphi_1(x) \leq y \leq \varphi_2(x)\}$

$\psi_1(d) = \psi_2(d)$



→ graph of some $x = \psi_2(y)$ C^0 and piecewise C^1

→ $(x,y),\ \psi_1(y) \leq x \leq \psi_2(y)$

→ graph of some $x = \psi_1(y)$

$\psi_1(c) < \psi_2(c)$

So we also have that $\Sigma = \{(x,y) : c \leq y \leq d,\ \psi_1(y) \leq x \leq \psi_2(y)\}$

Ccl: $\exists\, \varphi_1, \varphi_2, \psi_1, \psi_2$ C^0 and piecewise C^1 such that

$\Sigma = \{(x,y) : a \leq x \leq b,\ \varphi_1(x) \leq y \leq \varphi_2(x)\} = \{(x,y) : c \leq y \leq d,\ \psi_1(y) \leq x \leq \psi_2(y)\}$

Comment: Unfortunately, some regular regions with piecewise smooth boundaries can't be broken into finitely many elementary regions: We only proved a special case of Green's theorem:

Ex: $\{(x,y) : 0 \leq x \leq 1,\ 0 \leq y \leq 1 + x^3 \sin(1/x)\}$

because of the oscillations around $x = 0^+$

Here is an alternative statement:

Theorem: "Green's theorem for flux $\int_{\partial S} \vec{F} \cdot \vec{n} = \iint \operatorname{div}(\vec{F})$"

$S \subset \mathbb{R}^2$ regular region with piecewise smooth boundaries

$F : U \to \mathbb{R}^2$ C^1, U open, $S \subset U \subset \mathbb{R}^2$

$$\iint_S \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} = \int_{\partial S} \vec{F} \cdot \vec{n}$$

line integral for the real-valued function $x \mapsto F(x) \cdot n(x)$

positively oriented

where $\vec{n}(x)$ is the normal outward pointing unit vector at $\vec{x} \in \partial S$

Δ $\partial S = \bigcup C_i$, σ a parametrization of $C := C_i$ positively oriented

then $\vec{n}(\sigma(t)) = \frac{1}{\|\sigma'(t)\|} \left(\sigma_2'(t), -\sigma_1'(t) \right)$

Hence $\int_{\partial S} \vec{F} \cdot \vec{n} = \int_a^b \left(\frac{F_1(\sigma(t))\, \sigma_2'(t) - F_2(\sigma(t))\, \sigma_1'(t)}{\cancel{\|\sigma'(t)\|}} \right) \cancel{\|\sigma'(t)\|}\, dt$

$\partial S \to \mathbb{R}$
$x \mapsto \vec{F}(x) \cdot \vec{n}(x)$
line integral for real valued

$$= \int_{\partial S} -F_2\, dx + F_1\, dy = \iint_S \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} \qquad \square$$

line integral for vector field $\vec{F} = (-F_2, F_1)$

Green's thm

Ex: Use Green's theorem to compute a difficult $\int_C$ thanks to an easy $\iint_S$: $S = \overline{B}(0,1)$, $C = \partial S$ positively oriented

$$\int_C y e^{-x} dx + \left(\tfrac{1}{2} x^2 - e^{-x} \right) dy = \iint_S (x + e^{-x}) - e^{-x} = \iint_S x = \int_{-\pi}^{\pi} \int_0^1 r^2 \cos\theta \, dr\, d\theta = 0$$

Ex 2. Green's theorem allow to compute an area thanks to a line integral!

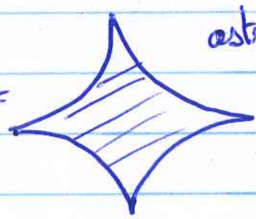
⤳ Type "planimeter" on google or your favorite search engine.

Take P, Q s.t. $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1$, then

$$\text{Area}(S) = \iint_S 1 = \iint_S \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = \int_{\partial S} P\,dx + Q\,dy$$

Ex: $\int_{\partial S} x\,dy$, $\int_{\partial S} -y\,dx$, $\int_{\partial S} \frac{-y}{2}dx + \frac{x}{2}dy$

ie $\vec{F} = (0, x)$, $\vec{F} = (-y, 0)$, $\vec{F} = (-y/2, x/2)$

For instance: $S =$  astroid, $\partial S : (\cos^3 t, \sin^3 t)$

$$\text{Area}(S) = \frac{1}{2}\int_{\partial S} -y\,dx + x\,dy = \frac{1}{2}\int_0^{2\pi} \frac{3}{2}\sin^4 t \cos^2 t + \frac{3}{2}\cos^4 t \sin^2 t \, dt$$

$$= \frac{3}{2}\int_0^{2\pi} \cos^2 t \sin^2 t \, dt$$

$$= \frac{3}{8}\int_0^{2\pi} \sin^2(2t)\, dt$$

$$= \frac{3}{16}\int_0^{2\pi} 1 - \cos(4t)\, dt$$

$$= \frac{3\pi}{8}$$

Ex3: Gradient fields / Conservative fields

Green's theorem allows to prove the following result to check if a vector field is conservative or not

Proposition: $\cdot$ $F : \mathbb{R}^2 \to \mathbb{R}^2$ a C^1 vector field

$\cdot$ $F = \nabla f$ for some $f : \mathbb{R}^2 \to \mathbb{R}$ C^2

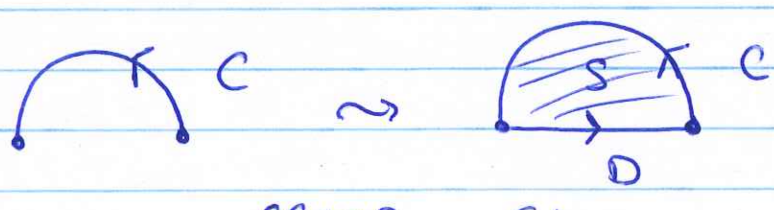
if and only if $\dfrac{\partial F_2}{\partial x} = \dfrac{\partial F_1}{\partial y}$

Δ $\underline{\Rightarrow}$: $$\frac{\partial F_2}{\partial x} = \frac{\partial^2 f}{\partial x \partial y} \underset{\text{Clairaut}}{=} \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial F_1}{\partial y}$$

$\underline{\Leftarrow}$: $$\oint_C \vec{F} \cdot d\vec{x} \underset{\text{Green}}{=} \iint_S \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) = 0$$ so $\oint_C \vec{F} \cdot d\vec{x}$ for any closed curve

$\Rightarrow \oint_C \vec{F} \cdot d\vec{x}$ independent on C for any curve C $\square$

Ex4:



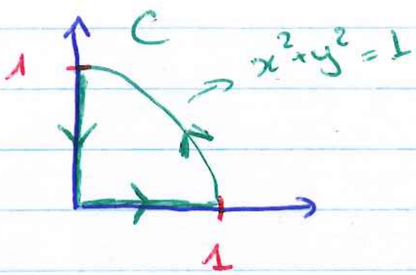
$$\int_{\partial S} \vec{F} \cdot d\vec{x} = \iint_S \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right)$$

$$\shortparallel \quad \int_C \vec{F} \cdot d\vec{x} + \int_D \vec{F} \cdot d\vec{x}$$

$$\Rightarrow \int_C \vec{F} \cdot d\vec{x} = \iint_S \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) - \int_D \vec{F} \cdot d\vec{x}$$

that could be easier to compute rather than

Ex 5 :



Compute $\int_C xy^2\,dx + 2xy\,dy$

Method 1 : directly, we have to parametrize the 3 edges $(t,0)$, $(\cos t, \sin t)$, $(0,t)$ and compute 3 integrals.

Method 2 : with Green :

$$\int_C xy^2\,dx + 2xy\,dy = \iint_S 2y - 2xy$$

since C is positively oriented for $C = \partial S$ where $S = \{x \geq 0, y \geq 0, x^2+y^2 \leq 1\}$

$$= \int_0^1\int_0^{\pi/2} 2r^2\sin\theta - r^3\sin(2\theta)\,d\theta\,dr$$

$$= \int_0^1 2r^2 - r^3\,dr = 5/12$$

Ex 6 : $S = \left\{ \frac{x^2}{a^2} + \frac{y^2}{b^2} \leq 1 \right\}$, $\partial S = \{(a\cos\theta, b\sin\theta), \theta \in [-\pi, \pi]\}$

$$A(S) = \int_{\partial S} x\,dy$$

$$= \int_{-\pi}^{\pi} ab\cos^2\theta\,d\theta = \int_{-\pi}^{\pi} ab\,\frac{\cos(2\theta)+1}{2}\,d\theta$$

$$= \frac{ab}{2}\int_{-\pi}^{\pi}(\cos(2\theta)+1)\,d\theta = \pi ab$$