University of Toronto – MAT137Y1 – LEC0501

Calculus!

Characterization of the sup/inf (slide 6)

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Recall the following definitions from the videos.

Definition 1. Let $A \subseteq \mathbb{R}$ and $U \in \mathbb{R}$. We say that U is an **upper bound** of A if

$$\forall x \in A,\ x \leq U$$

Definition 2. Let $A \subseteq \mathbb{R}$ and $L \in \mathbb{R}$. We say that L is a **lower bound** of A if

$$\forall x \in A,\ L \leq x$$

Definition 3. We say that a subset $A \subseteq \mathbb{R}$ is **bounded from above** if it admits an upper bound.

Definition 4. We say that a subset $A \subseteq \mathbb{R}$ is **bounded from below** if it admits a lower bound.

Definition 5. Let $A \subseteq \mathbb{R}$ and $S \in \mathbb{R}$.
We say that S is the **supremum** (or **least upper bound**) of A if
1. S is an upper bound of A, and,
2. for all upper bounds T of A, $S \leq T$.

Then we use the notation $S = \sup(A)$.

Definition 6. Let $A \subseteq \mathbb{R}$ and $I \in \mathbb{R}$.
We say that I is the **infimum** (or **greatest lower bound**) of A if
1. I is a lower bound of A, and,
2. for all lower bounds J of A, $J \leq I$.

Then we use the notation $I = \inf(A)$.

Remark 7. Notice that we talk about **the** supremum of a set but about **an** upper bound of a set. It is because, as seen during the lecture (slide 4), if a set admits a supremum then it is unique. Beware, it is possible for a set to not have a supremum.

The real line $\mathbb{R}$ satisfies two very fundamental properties.

Theorem 8 (The least upper bound property)**.**
If a non-empty subset of $\mathbb{R}$ is bounded from above then it admits a least upper bound (supremum).

Theorem 9 (The greatest lower bound property)**.**
If a non-empty subset of $\mathbb{R}$ is bounded from below then it admits a greatest lower bound (infimum).

Remark 10. As seen during the lecture (slide 5), the "non-empty" assumption is essential here!

We have seen the following characterizations of the supremum and of the infimum (slide 6). These characterizations may be useful when writing proofs: do not hesitate to use them!

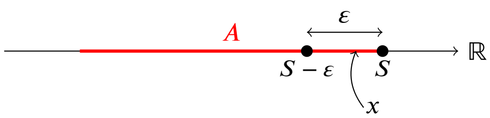
Proposition 11. *Let* $A \subseteq \mathbb{R}$ *and* $S \in \mathbb{R}$. *Then*

$$S = \sup(A) \Leftrightarrow \begin{cases} \forall x \in A,\ x \leq S \\ \forall \varepsilon > 0,\ \exists x \in A,\ S - \varepsilon < x \end{cases}$$

Proposition 12. *Let* $A \subseteq \mathbb{R}$ *and* $I \in \mathbb{R}$. *Then*

$$I = \inf(A) \Leftrightarrow \begin{cases} \forall x \in A,\ I \leq x \\ \forall \varepsilon > 0,\ \exists x \in A,\ x < I + \varepsilon \end{cases}$$

We will only focus on the characterization of the supremum (that's similar for the infimum).
Notice that the first line simply means that S is an upper bound.
Then the second line of the characterization means that S is the smallest one!
Indeed, for any $\varepsilon > 0$, even a very very very small one, $S - \varepsilon < S$. So the fact that S is the least upper bound means exactly that $S - \varepsilon$ isn't an upper bound, or, equivalently, that there is at least one $x \in A$ such that $S - \varepsilon < x$.



Beware, for simplicity I represented A as an interval in the above figure, but A may not be an interval!

Proof of proposition 11. Let $A \subseteq \mathbb{R}$ and $S \in \mathbb{R}$.

1. Proof of $\Rightarrow$.
   Assume that $S = \sup(A)$.

   Then S is a upper bound of A so $\forall x \in A,\ x \leq S$.

   We know that if T is an upper bound of A then $S \leq T$.
   So, by taking the contrapositive, if $T < S$ then T isn't an upper bound of A.

   Let $\varepsilon > 0$. Since $S - \varepsilon < S$, we know that $S - \varepsilon$ is not an upper bound of A, meaning that there exists $x \in A$ such that $S - \varepsilon < x$.

2. Proof of $\Leftarrow$.
   We assume that
   $$\begin{cases} \forall x \in A,\ x \leq S \\ \forall \varepsilon > 0,\ \exists x \in A,\ S - \varepsilon < x \end{cases}$$
   The first part of the characterization ensures that S is an upper bound of A.

   We still have to prove that if T is an upper bound of A then $S \leq T$.
   We will show the contrapositive: if $T < S$ then T isn't an upper bound.
   Let $T \in \mathbb{R}$. Assume that $T < S$. Let $\varepsilon = S - T > 0$. Then there exists $x \in A$ such that $S - \varepsilon < x$, i.e. $T < x$.
   Hence T isn't an upper bound.

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Remark 13. If you prefer, you can write proofs by contradiction instead of using the contrapositive.