MAT137Y1 – LEC0501

Calculus!

One-to-one functions & inverse trig functions

November 14th, 2018

For next week

For Monday (Nov 19), watch the videos:

- Local extrema: 5.1, 5.2, 5.3, 5.4

For Wednesday (Nov 21), watch the videos:

- Rolle's Theorem: 5.5, 5.6
- The MVT: 5.7, 5.8, 5.9

One-to-one functions

1. Write the formal definition of "$f : D \to \mathbb{R}$ is one-to-one".
2. Let f be the function defined on $\mathbb{R}$ by $f(x) = 2x^3 + 7$.
   Prove that f is one-to-one.
3. Let g be the function defined on $\mathbb{R}$ by $g(x) = 2x^2 + 7$.
   Prove that g is not one-to-one.

Strictly increasing functions are one-to-one

Let $f : D \to \mathbb{R}$ a function whose domain D is a subset of $\mathbb{R}$.

1. Recall the definition of "f is strictly increasing".
2. Recall the definition of "f is one-to-one" (from the previous slide).
3. Prove the following result.

Theorem

If f is strictly increasing then f is one-to-one.

4. Now let $D = \mathbb{R} \setminus \left\{ \frac{\pi}{2} + n\pi,\ n \in \mathbb{R} \right\}$.
   Is $\tan : D \to \mathbb{R}$ strictly increasing? Is it one-to-one?
5. What about $\tan_{\left|\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)\right.}$, the restriction of $\tan$ to $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$?

Composition of one-to-one functions

Assume for simplicity that all functions in this problem have domain $\mathbb{R}$. Prove the following theorem.

Theorem

Let f and g be functions.
IF f and g are one-to-one,
THEN $f \circ g$ is one-to-one.

Suggestion:

1. Write the definition of what you want to prove.
2. Figure out the formal structure of the proof.
3. Complete the proof (use the hypotheses!)

Composition of one-to-one functions – 2

Assume for simplicity that all functions in this problem have domain $\mathbb{R}$.

Is the following claim TRUE or FALSE?
Prove it or give a counterexample.

Claim

Let f and g be functions.
IF $f \circ g$ is one-to-one,
THEN f is one-to-one.

Composition of one-to-one functions – 3

Assume for simplicity that all functions in this problem have domain $\mathbb{R}$.

Is the following claim TRUE or FALSE?
Prove it or give a counterexample.

Claim

Let f and g be functions.
IF $f \circ g$ is one-to-one,
THEN g is one-to-one.

Composition and inverses

Assume for simplicity that all functions in this problem have domain $\mathbb{R}$.
Let f and g be functions. Assume they each have an inverse.
Is $(f \circ g)^{-1} = f^{-1} \circ g^{-1}$?

- If YES, prove it.
- If NO, fix the statement.

If you do not know how to start, experiment with the functions

$$f(x) = x + 1, \qquad g(x) = 2x.$$

Definition of $\arctan$

1. Sketch the graph of $\tan$.
2. Prove that $\tan$ is not one-to-one.
3. Select the largest interval containing 0 such that the restriction of $\tan$ to it is one-to-one.
   Briefly explain why $\tan$ is one-to-one on this interval.
4. What's the range of $\tan$ restricted to the above interval?
5. We define $\arctan$ as the inverse of $\tan$ restricted to the above interval.
   What are the domain and the range of $\arctan$?
6. Sketch the graph of $\arctan$.

Definition of $\arctan$ – 2

Remember from the previous slide that $\arctan : \mathbb{R} \to \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ is defined as the inverse of $\tan : \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \to \mathbb{R}$.

1. Fill: $\forall x \in \ldots, \forall y \in \ldots, \big(y = \tan(x) \Leftrightarrow x = \arctan(y)\big)$.
2. What can you say about $\tan(\arctan(x))$?
   And about $\arctan(\tan(x))$?
   (For which x are these functions defined? What are they equal to? Sketch their graphs.)
3. Compute:
   1. $\tan(\arctan(0))$
   2. $\tan\left(\arctan\left(\sqrt{2}+\pi\right)\right)$
   3. $\arctan(\tan(1))$
   4. $\arctan(\tan(3))$
   5. $\arctan\left(\tan\left(\frac{\pi}{2}\right)\right)$
   6. $\arctan(\tan(-6))$

Derivative of $\arctan$

Obtain (and prove) a formula for the derivative of $\arctan$.

Hint: Differentiate the identity

$$\forall t \in \ldots \qquad \tan(\arctan(t)) = t$$

Computation

Compute the derivative of

$$f(x) = 2x^2 \arctan(x^2) - \ln(x^4 + 1)$$

and simplify it as much as possible.