$\mathcal{C}^m$ SOLUTIONS OF SEMIALGEBRAIC EQUATIONS

Joint work with E. Bierstone and P.D. Milman

Jean-Baptiste Campesato

University of Toronto

August 19, 2021

Semialgebraic geometry – Definitions

Semialgebraic geometry – Definitions

Definition: semialgebraic sets

Semialgebraic subsets of $\mathbb{R}^n$ are elements of the boolean algebra spanned by sets of the form

$$\{x \in \mathbb{R}^n \ : \ f(x) \geq 0\}$$

where $f \in \mathbb{R}[x_1, \dots, x_n]$.

Semialgebraic geometry – Definitions

Definition: semialgebraic sets

Semialgebraic subsets of $\mathbb{R}^n$ are elements of the boolean algebra spanned by sets of the form

$$\{x \in \mathbb{R}^n \ : \ f(x) \geq 0\}$$

where $f \in \mathbb{R}[x_1, \dots, x_n]$.

Remark

Given $f \in \mathbb{R}[x_1, \dots, x_n]$, the following sets are semialgebraic

$$\{x \in \mathbb{R}^n \ : \ f(x) > 0\}, \{x \in \mathbb{R}^n \ : \ f(x) \leq 0\}, \{x \in \mathbb{R}^n \ : \ f(x) < 0\}, \{x \in \mathbb{R}^n \ : \ f(x) = 0\}, \{x \in \mathbb{R}^n \ : \ f(x) \neq 0\}$$

Semialgebraic geometry – Definitions

Definition: semialgebraic sets

Semialgebraic subsets of $\mathbb{R}^n$ are elements of the boolean algebra spanned by sets of the form

$$\{x \in \mathbb{R}^n \ : \ f(x) \geq 0\}$$

where $f \in \mathbb{R}[x_1, \dots, x_n]$.

Remark

Given $f \in \mathbb{R}[x_1, \dots, x_n]$, the following sets are semialgebraic

$$\{x \in \mathbb{R}^n \ : \ f(x) > 0\}, \{x \in \mathbb{R}^n \ : \ f(x) \leq 0\}, \{x \in \mathbb{R}^n \ : \ f(x) < 0\}, \{x \in \mathbb{R}^n \ : \ f(x) = 0\}, \{x \in \mathbb{R}^n \ : \ f(x) \neq 0\}$$

Definition: semialgebraic functions

Let $X \subset \mathbb{R}^n$. A function $f : X \to \mathbb{R}^p$ is semialgebraic if its graph $\Gamma_f \subset \mathbb{R}^{n+p}$ is semialgebraic.

Semialgebraic geometry – Tarski–Seidenberg theorem

Theorem (Tarski–Seidenberg): semialgebraic sets are closed under projections

If $S \subset \mathbb{R}^{n+1}$ is semialgebraic then so is $\pi(S)$, where $\pi : \mathbb{R}^{n+1} \to \mathbb{R}^n$, $\pi(x_1, \dots, x_{n+1}) = (x_1, \dots, x_n)$.

Semialgebraic geometry – Tarski–Seidenberg theorem

Theorem (Tarski–Seidenberg): semialgebraic sets are closed under projections

If $S \subset \mathbb{R}^{n+1}$ is semialgebraic then so is $\pi(S)$, where $\pi : \mathbb{R}^{n+1} \to \mathbb{R}^n$, $\pi(x_1, \dots, x_{n+1}) = (x_1, \dots, x_n)$.

Remark

If $f : X \to \mathbb{R}^p$ is semialgebraic then so is X.

Semialgebraic geometry – Tarski–Seidenberg theorem

Theorem (Tarski–Seidenberg): semialgebraic sets are closed under projections

If $S \subset \mathbb{R}^{n+1}$ is semialgebraic then so is $\pi(S)$, where $\pi : \mathbb{R}^{n+1} \to \mathbb{R}^n$, $\pi(x_1, \dots, x_{n+1}) = (x_1, \dots, x_n)$.

Remark

If $f : X \to \mathbb{R}^p$ is semialgebraic then so is X.

Corollary: elimination of quantifiers

Let $S \subset \mathbb{R}^{n+1}$ be semialgebraic, then the following sets are too

$$\left\{(x_1, \dots, x_n) \in \mathbb{R}^n \;:\; \exists y,\, (x_1, \dots, x_n, y) \in S\right\} = \pi(S)$$

$$\left\{(x_1, \dots, x_n) \in \mathbb{R}^n \;:\; \forall y,\, (x_1, \dots, x_n, y) \in S\right\} = \mathbb{R}^n \setminus \pi(\mathbb{R}^{n+1} \setminus S)$$

Semialgebraic geometry – Tarski–Seidenberg theorem

Theorem (Tarski–Seidenberg): semialgebraic sets are closed under projections

If $S \subset \mathbb{R}^{n+1}$ is semialgebraic then so is $\pi(S)$, where $\pi : \mathbb{R}^{n+1} \to \mathbb{R}^n$, $\pi(x_1, \dots, x_{n+1}) = (x_1, \dots, x_n)$.

Remark

If $f : X \to \mathbb{R}^p$ is semialgebraic then so is X.

Corollary: elimination of quantifiers

Let $S \subset \mathbb{R}^{n+1}$ be semialgebraic, then the following sets are too

$$\left\{(x_1, \dots, x_n) \in \mathbb{R}^n \ : \ \exists y,\ (x_1, \dots, x_n, y) \in S\right\} = \pi(S)$$
$$\left\{(x_1, \dots, x_n) \in \mathbb{R}^n \ : \ \forall y,\ (x_1, \dots, x_n, y) \in S\right\} = \mathbb{R}^n \setminus \pi(\mathbb{R}^{n+1} \setminus S)$$

Example

If $A \subset \mathbb{R}^n$ is semialgebraic, then so is $\overline{A} := \left\{ x \in \mathbb{R}^n \ : \ \forall \varepsilon \in (0, +\infty),\ \exists y \in A,\ \sum_{i=1}^{n} (x_i - y_i)^2 < \varepsilon^2 \right\}$.

A semialgebraic version of Whitney's extension theorem

**Theorem – Kurdyka–Pawłucki, 1997, 2014
Thamrongthanyalak, 2017
Kocel-Cynk–Pawłucki–Valette, 2019**

Given a semialgebraic C^m Whitney field on a closed subset $X \subset \mathbb{R}^n$,
i.e. a family $(f_\alpha : X \to \mathbb{R})_{\substack{\alpha\in\mathbb{N}^n \\ |\alpha|\leq m}}$ of continuous semialgebraic functions such that

$$\forall z \in X, \forall \alpha \in \mathbb{N}^n, |\alpha| \leq m \implies f_\alpha(x) - \sum_{|\beta|\leq m-|\alpha|} \frac{f_{\alpha+\beta}(y)}{\beta!}(x-y)^\beta \underset{X\ni x,y\to z}{=} o\left(\|x-y\|^{m-|\alpha|}\right),$$

there exists a C^m semialgebraic function $F : \mathbb{R}^n \to \mathbb{R}$ such that $D^\alpha F_{|X} = f_\alpha$ and F is Nash on $\mathbb{R}^n \setminus X$.

Nash := semialgebraic and analytic.

Are there solutions preserving semialgebraicity?

For Whitney's Extension Problem

For the Brenner–Fefferman–Hochster–Kollár Problem

Are there solutions preserving semialgebraicity?

For Whitney's Extension Problem

Let $f : X \to \mathbb{R}$ be a semialgebraic function where $X \subset \mathbb{R}^n$ is closed.
If f admits a C^m extension $F : \mathbb{R}^n \to \mathbb{R}$, does it admit a semialgebraic C^m extension $\tilde{F} : \mathbb{R}^n \to \mathbb{R}$?

For the Brenner–Fefferman–Hochster–Kollár Problem

Are there solutions preserving semialgebraicity?

For Whitney's Extension Problem

Let $f : X \to \mathbb{R}$ be a semialgebraic function where $X \subset \mathbb{R}^n$ is closed.
If f admits a C^m extension $F : \mathbb{R}^n \to \mathbb{R}$, does it admit a semialgebraic C^m extension $\tilde{F} : \mathbb{R}^n \to \mathbb{R}$?

For the Brenner–Fefferman–Hochster–Kollár Problem

Let $f_1, \dots, f_r, \varphi : \mathbb{R}^n \to \mathbb{R}$ be semialgebraic functions.
If the equation $\varphi = \sum \varphi_i f_i$ admit a C^m solution $(\varphi_i)_i$, does it admit a semialgebraic C^m solution?

Are there solutions preserving semialgebraicity?

For Whitney's Extension Problem

Let $f : X \to \mathbb{R}$ be a semialgebraic function where $X \subset \mathbb{R}^n$ is closed.
If f admits a C^m extension $F : \mathbb{R}^n \to \mathbb{R}$, does it admit a semialgebraic C^m extension $\tilde{F} : \mathbb{R}^n \to \mathbb{R}$?

For the Brenner–Fefferman–Hochster–Kollár Problem

Let $f_1, \dots, f_r, \varphi : \mathbb{R}^n \to \mathbb{R}$ be semialgebraic functions.
If the equation $\varphi = \sum \varphi_i f_i$ admit a C^m solution $(\varphi_i)_i$, does it admit a semialgebraic C^m solution?

- Aschenbrenner–Thamrongthanyalak (2019): $\forall n$, for $m = 1$ and $m = 0$, respectively.
- Fefferman–Luli (2021): $\forall m$, for $n = 2$.
- Bierstone–C.–Milman (2021): $\forall n$, $\forall m$, with a loss of differentiability.

Strategy for the semialgebraic C^1 extension problem (Aschenbrenner–Thamrongthanyalak, 2019)

Let $f : X \to \mathbb{R}$ be a semialgebraic function admitting a C^1 extension $F : \mathbb{R}^n \to \mathbb{R}$.

Strategy for the semialgebraic C^1 extension problem (Aschenbrenner–Thamrongthanyalak, 2019)

Let $f : X \to \mathbb{R}$ be a semialgebraic function admitting a C^1 extension $F : \mathbb{R}^n \to \mathbb{R}$.

- Set $S := \left\{ (x, y, v) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \;:\; \begin{array}{l} x \in X,\ y = f(x), \\ \forall \varepsilon > 0,\ \forall \delta > 0,\ \forall a, b \in B_\delta(x),\ |f(b) - f(a) - v \cdot (b - a)| \leq \varepsilon \|b - a\| \end{array} \right\}.$

Strategy for the semialgebraic C^1 extension problem (Aschenbrenner–Thamrongthanyalak, 2019)

Let $f : X \to \mathbb{R}$ be a semialgebraic function admitting a C^1 extension $F : \mathbb{R}^n \to \mathbb{R}$.

- Set $S \coloneqq \left\{ (x, y, v) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \ : \ \begin{array}{l} x \in X,\ y = f(x), \\ \forall \varepsilon > 0,\ \forall \delta > 0,\ \forall a, b \in B_\delta(x),\ |f(b) - f(a) - v \cdot (b-a)| \leq \varepsilon \|b-a\| \end{array} \right\}$.
- Then S is semialgebraic,

Strategy for the semialgebraic C^1 extension problem (Aschenbrenner–Thamrongthanyalak, 2019)

Let $f : X \to \mathbb{R}$ be a semialgebraic function admitting a C^1 extension $F : \mathbb{R}^n \to \mathbb{R}$.

- Set $S := \left\{ (x, y, v) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \;:\; \begin{array}{l} x \in X,\ y = f(x), \\ \forall \varepsilon > 0,\ \forall \delta > 0,\ \forall a, b \in B_\delta(x),\ |f(b) - f(a) - v \cdot (b-a)| \leq \varepsilon \|b - a\| \end{array} \right\}$.
- Then S is semialgebraic, and, $\forall x \in X$, $(x, F(x), \nabla F(x)) \in S$.

Strategy for the semialgebraic C^1 extension problem (Aschenbrenner–Thamrongthanyalak, 2019)

Let $f : X \to \mathbb{R}$ be a semialgebraic function admitting a C^1 extension $F : \mathbb{R}^n \to \mathbb{R}$.

- Set $S := \left\{ (x, y, v) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \ : \begin{array}{l} x \in X,\ y = f(x), \\ \forall \varepsilon > 0,\ \forall \delta > 0,\ \forall a, b \in B_\delta(x),\ |f(b) - f(a) - v \cdot (b - a)| \leq \varepsilon \|b - a\| \end{array} \right\}$.
- Then S is semialgebraic, and, $\forall x \in X$, $(x, F(x), \nabla F(x)) \in S$.
- Semialgebraic Michael's Selection Lemma:
  there exists $\sigma : X \to S$ semialgebraic and continuous such that $\pi_x \circ \sigma = \mathrm{id}$ where $\pi_x(x, y, v) = x$.

Strategy for the semialgebraic C^1 extension problem (Aschenbrenner–Thamrongthanyalak, 2019)

Let $f : X \to \mathbb{R}$ be a semialgebraic function admitting a C^1 extension $F : \mathbb{R}^n \to \mathbb{R}$.

- Set $S := \left\{ (x,y,v) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \ : \ \begin{array}{l} x \in X,\ y = f(x), \\ \forall \varepsilon > 0,\ \forall \delta > 0,\ \forall a,b \in B_\delta(x),\ |f(b) - f(a) - v \cdot (b-a)| \le \varepsilon \|b-a\| \end{array} \right\}$.
- Then S is semialgebraic, and, $\forall x \in X$, $(x, F(x), \nabla F(x)) \in S$.
- Semialgebraic Michael's Selection Lemma:
  there exists $\sigma : X \to S$ semialgebraic and continuous such that $\pi_x \circ \sigma = \mathrm{id}$ where $\pi_x(x,y,v) = x$.
- Set $G := \pi_v \circ \sigma : \mathbb{R}^n \to \mathbb{R}^n$ where $\pi_v(x,y,v) = v$, then G is semialgebraic, continuous and satisfies
$$\forall c \in X,\ f(b) = f(a) + G(a) \cdot (b-a) + \underset{X \ni a,b \to c}{o}(\|b-a\|).$$

Strategy for the semialgebraic C^1 extension problem (Aschenbrenner–Thamrongthanyalak, 2019)

Let $f : X \to \mathbb{R}$ be a semialgebraic function admitting a C^1 extension $F : \mathbb{R}^n \to \mathbb{R}$.

- Set $S := \left\{ (x,y,v) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \ : \ \begin{array}{l} x \in X,\ y = f(x), \\ \forall \varepsilon > 0,\ \forall \delta > 0,\ \forall a,b \in B_\delta(x),\ |f(b) - f(a) - v \cdot (b-a)| \le \varepsilon \|b-a\| \end{array} \right\}$.
- Then S is semialgebraic, and, $\forall x \in X,\ (x, F(x), \nabla F(x)) \in S$.
- Semialgebraic Michael's Selection Lemma:
  there exists $\sigma : X \to S$ semialgebraic and continuous such that $\pi_x \circ \sigma = \mathrm{id}$ where $\pi_x(x,y,v) = x$.
- Set $G := \pi_v \circ \sigma : \mathbb{R}^n \to \mathbb{R}^n$ where $\pi_v(x,y,v) = v$, then G is semialgebraic, continuous and satisfies
  $$\forall c \in X,\ f(b) = f(a) + G(a) \cdot (b-a) + \underset{X \ni a,b \to c}{o}(\|b-a\|). \qquad \blacksquare$$

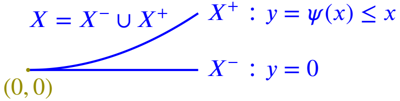
This strategy does not generalize to $m > 1$ since the unknown $(f_\alpha)_{\substack{\alpha \in \mathbb{N}^n \setminus \{\mathbf{0}\} \\ |\alpha| \le m}}$ can't be described as a section.

For instance, if $m = 2$, $f_{\mathbf{e}_i}$ needs to satisfy

$$f_{\mathbf{e}_i}(b) = f_{\mathbf{e}_i}(a) + \sum_{j=1}^{n} f_{\mathbf{e}_i + \mathbf{e}_j}(a)(b_j - a_j) + \underset{X \ni a,b \to c}{o}(\|b-a\|).$$

Strategy for the planar semialgebraic extension problem (Fefferman–Luli, 2021)

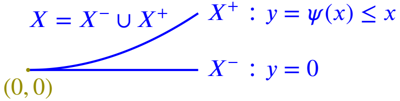
Let $f : X \to \mathbb{R}$ be semialgebraic.



Strategy for the planar semialgebraic extension problem (Fefferman–Luli, 2021)

Let $f : X \to \mathbb{R}$ be semialgebraic.

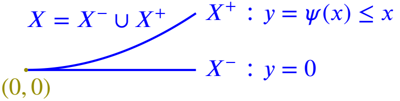
1. Let $F : \mathbb{R}^2 \to \mathbb{R}$ be a $\mathcal{C}^m$ function such that $F_{|X} = f$ and $J_{(0,0)}F = 0$.



Strategy for the planar semialgebraic extension problem (Fefferman–Luli, 2021)

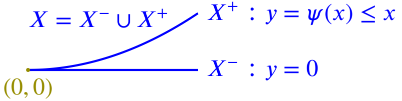
Let $f : X \to \mathbb{R}$ be semialgebraic.

1. Let $F : \mathbb{R}^2 \to \mathbb{R}$ be a C^m function such that $F_{|X} = f$ and $J_{(0,0)}F = 0$.
   Set $f_l^-(x) := \partial_y^l F(x,0)$ and $f_l^+(x) := \partial_y^l F(x,\psi(x))$.



Strategy for the planar semialgebraic extension problem (Fefferman–Luli, 2021)

Let $f : X \to \mathbb{R}$ be semialgebraic.

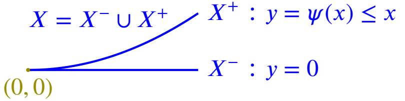


1. Let $F : \mathbb{R}^2 \to \mathbb{R}$ be a C^m function such that $F_{|X} = f$ and $J_{(0,0)}F = 0$.
   Set $f_l^-(x) := \partial_y^l F(x,0)$ and $f_l^+(x) := \partial_y^l F(x,\psi(x))$. Then

$$(*)\begin{cases} (i) & f_0^-(x) = f(x,0) \\ (ii) & f_0^+(x) = f(x,\psi(x)) \\ (iii) & f_l^+(x) = \displaystyle\sum_{k=0}^{m-l} \frac{\psi(x)^k}{k!} f_{l+k}^-(x) + \underset{x\to 0+}{o}\left(\psi(x)^{m-l}\right) \\ (iv) & f_l^-(x) = \underset{x\to 0+}{o}\left(x^{m-l}\right) \\ (v) & f_l^+(x) = \underset{x\to 0+}{o}\left(x^{m-l}\right) \end{cases}$$

Strategy for the planar semialgebraic extension problem (Fefferman–Luli, 2021)

Let $f : X \to \mathbb{R}$ be semialgebraic.



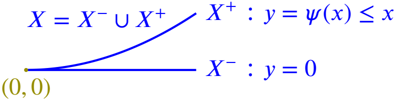
1. Let $F : \mathbb{R}^2 \to \mathbb{R}$ be a C^m function such that $F_{|X} = f$ and $J_{(0,0)}F = 0$. Set $f_l^-(x) := \partial_y^l F(x,0)$ and $f_l^+(x) := \partial_y^l F(x,\psi(x))$. Then

$$(*)\begin{cases} (i) & f_0^-(x) = f(x,0) \\ (ii) & f_0^+(x) = f(x,\psi(x)) \\ (iii) & f_l^+(x) = \displaystyle\sum_{k=0}^{m-l} \frac{\psi(x)^k}{k!} f_{l+k}^-(x) + \underset{x\to0+}{o}\left(\psi(x)^{m-l}\right) \\ (iv) & f_l^-(x) = \underset{x\to0+}{o}\left(x^{m-l}\right) \\ (v) & f_l^+(x) = \underset{x\to0+}{o}\left(x^{m-l}\right) \end{cases}$$

2. According to the definable choice: there exist $\tilde{f}_l^{\pm}$ semialgebraic satisfying $(*)$.

Strategy for the planar semialgebraic extension problem (Fefferman–Luli, 2021)

Let $f : X \to \mathbb{R}$ be semialgebraic.



1. Let $F : \mathbb{R}^2 \to \mathbb{R}$ be a C^m function such that $F_{|X} = f$ and $J_{(0,0)}F = 0$. Set $f_l^-(x) \coloneqq \partial_y^l F(x,0)$ and $f_l^+(x) \coloneqq \partial_y^l F(x,\psi(x))$. Then

$$(*)\begin{cases} (i) & f_0^-(x) = f(x,0) \\ (ii) & f_0^+(x) = f(x,\psi(x)) \\ (iii) & f_l^+(x) = \displaystyle\sum_{k=0}^{m-l} \frac{\psi(x)^k}{k!} f_{l+k}^-(x) + \underset{x\to 0+}{o}\left(\psi(x)^{m-l}\right) \\ (iv) & f_l^-(x) = \underset{x\to 0+}{o}\left(x^{m-l}\right) \\ (v) & f_l^+(x) = \underset{x\to 0+}{o}\left(x^{m-l}\right) \end{cases}$$

2. According to the definable choice: there exist $\tilde{f}_l^{\pm}$ semialgebraic satisfying $(*)$.

3. Then $\tilde{F}(x,y) = \theta^-(x,y)\left(\displaystyle\sum_{l=0}^{m} \frac{\tilde{f}_l^-(x)}{l!} y^l\right) + \theta^+(x,y)\left(\displaystyle\sum_{l=0}^{m} \frac{\tilde{f}_l^+(x)}{l!}(y-\psi(x))^l\right)$ is a semialgebraic C^m extension of f in a neighborhood of the origin such that $J_{(0,0)}\tilde{F} = 0$.

The main results: statements

Theorem – Bierstone–C.–Milman, 2021

Given $X \subset \mathbb{R}^n$ closed and semialgebraic, there exists $r : \mathbb{N} \to \mathbb{N}$ satisfying the following property: if $f : X \to \mathbb{R}$ semialgebraic admits a $\mathcal{C}^{r(m)}$ extension, then it admits a semialgebraic $\mathcal{C}^m$ extension.

The main results: statements

Theorem – Bierstone–C.–Milman, 2021

Given $X \subset \mathbb{R}^n$ closed and semialgebraic, there exists $r : \mathbb{N} \to \mathbb{N}$ satisfying the following property: if $f : X \to \mathbb{R}$ semialgebraic admits a $\mathcal{C}^{r(m)}$ extension, then it admits a semialgebraic $\mathcal{C}^m$ extension.

Theorem – Bierstone–C.–Milman, 2021

Given $A : \mathbb{R}^n \to \mathcal{M}_{p,q}(\mathbb{R})$ semialgebraic, there exists $r : \mathbb{N} \to \mathbb{N}$ such that:
if $F : \mathbb{R}^n \to \mathbb{R}^p$ semialgebraic may be written $F(x) = A(x)G(x)$ where G is $\mathcal{C}^{r(m)}$,
then $F(x) = A(x)\tilde{G}(x)$ where $\tilde{G}(x)$ is semialgebraic and $\mathcal{C}^m$.

Main results: towards a common generalization

The extension problem

Let $X \subset \mathbb{R}^n$ be semialgebraic and closed.

By resolution of singularities, there exists $\varphi : M \to \mathbb{R}^n$ Nash and proper defined on a Nash manifold such that $X = \varphi(M)$.

Given $g : \mathbb{R}^n \to \mathbb{R}$ and $f : X \to \mathbb{R}$, we have $g_{|X} = f$ if and only if

$$\forall y \in M,\ g(\varphi(y)) = \tilde{f}(y)$$

where $\tilde{f} := f \circ \varphi$.

Main results: towards a common generalization

The extension problem

Let $X \subset \mathbb{R}^n$ be semialgebraic and closed.

By resolution of singularities, there exists $\varphi : M \to \mathbb{R}^n$ Nash and proper defined on a Nash manifold such that $X = \varphi(M)$.

Given $g : \mathbb{R}^n \to \mathbb{R}$ and $f : X \to \mathbb{R}$, we have $g_{|X} = f$ if and only if

$$\forall y \in M,\ g(\varphi(y)) = \tilde{f}(y)$$

where $\tilde{f} := f \circ \varphi$.

The equation problem

Consider an equation

$$A(x)G(x) = F(x),\ x \in \mathbb{R}^n.$$

By resolution of singularities, there exists $\varphi : M \to \mathbb{R}^n$ Nash and proper defined on a Nash manifold such that after composition, we get

$$\tilde{A}(y)G(\varphi(y)) = \tilde{F}(y),\ y \in M$$

where $\tilde{A} := A \circ \varphi$ is now Nash and $\tilde{F} := F \circ \varphi$.

The main result

Theorem – Bierstone–C.–Milman, 2021

Let $A : \mathbb{R}^n \to \mathcal{M}_{p,q}(\mathbb{R})$ be Nash and let $\varphi : M \to \mathbb{R}^n$ be Nash and proper defined on $M \subset \mathbb{R}^N$ a Nash submanifold.
Then there exists $r : \mathbb{N} \to \mathbb{N}$ satisfying the following property.
If $f : M \to \mathbb{R}^p$ semialgebraic may be written

$$f(x) = A(x)g(\varphi(x))$$

for a $C^{r(m)}$ function $g : \mathbb{R}^n \to \mathbb{R}^q$ then

$$f(x) = A(x)\tilde{g}(\varphi(x))$$

for a semialgebraic C^m function $\tilde{g}$.

Heart of the proof: induction on dimension

Proposition: the induction step

Let $B \subset \varphi(M)$ be semialgebraic and closed.
There exist $B' \subset B$ semialgebraic satisfying $\dim B' < \dim B$ and $t : \mathbb{N} \to \mathbb{N}$ such that if

1. $f : M \to \mathbb{R}^p$ is $C^{t(k)}$, semialgebraic and $t(k)$-flat on $\varphi^{-1}(B')$, and
2. $f = A \cdot (g \circ \varphi)$ admits a $C^{t(k)}$ solution g,

then there exists a semialgebraic C^k function $\tilde{g} : \mathbb{R}^n \to \mathbb{R}^q$ s.t. $f - A \cdot (\tilde{g} \circ \varphi)$ is k-flat on $\varphi^{-1}(B)$.

Heart of the proof: induction on dimension

Proposition: the induction step

Let $B \subset \varphi(M)$ be semialgebraic and closed.
There exist $B' \subset B$ semialgebraic satisfying $\dim B' < \dim B$ and $t : \mathbb{N} \to \mathbb{N}$ such that if

1. $f : M \to \mathbb{R}^p$ is $C^{t(k)}$, semialgebraic and $t(k)$-flat on $\varphi^{-1}(B')$, and
2. $f = A \cdot (g \circ \varphi)$ admits a $C^{t(k)}$ solution g,

then there exists a semialgebraic C^k function $\tilde{g} : \mathbb{R}^n \to \mathbb{R}^q$ s.t. $f - A \cdot (\tilde{g} \circ \varphi)$ is k-flat on $\varphi^{-1}(B)$.

Then, up to subtracting by $A \cdot (\tilde{g} \circ \varphi)$ on both side, we get an equation

$$f = A \cdot (g \circ \varphi)$$

where f is now semialgebraic and k-flat on $\varphi^{-1}(B)$.

Heart of the proof: induction on dimension

Strategy: construction of a semialgebraic Whitney field

$$G(b,\mathbf{y}) = \sum_{|\alpha|\leq l} \frac{g_\alpha(b)}{\alpha!}\mathbf{y}^\alpha \in C^0(B)[\mathbf{y}]$$

vanishing on B' such that

$$\forall b \in B \setminus B',\, \forall a \in \varphi^{-1}(b),\, T_a^l f(\mathbf{x}) \equiv T_a^l A(\mathbf{x})\, G(b, T_a^l\varphi(\mathbf{x})) \mod (\mathbf{x})^{l+1}\mathbb{R}[\![\mathbf{x}]\!]^p$$

Heart of the proof: induction on dimension

Strategy: construction of a semialgebraic Whitney field

$$G(b,\mathbf{y}) = \sum_{|\alpha|\leq l} \frac{g_\alpha(b)}{\alpha!}\mathbf{y}^\alpha \in C^0(B)[\mathbf{y}]$$

vanishing on B' such that

$$\forall b \in B \setminus B',\ \forall a \in \varphi^{-1}(b),\ T_a^l f(\mathbf{x}) \equiv T_a^l A(\mathbf{x})\, G(b, T_a^l \varphi(\mathbf{x})) \mod (\mathbf{x})^{l+1}\mathbb{R}[\![\mathbf{x}]\!]^p$$

A - Whitney regularity

Given B, there exists $\rho \in \mathbb{N}$ such that if G is a Whitney field of order $l \geq k\rho$ on $B \setminus B'$ then it is a Whitney field of order k on B.

The module of relations at $b \in \varphi(M)$

We consider the equation at the level of Taylor polynomials:

$$T_a^r f(\mathbf{x}) \equiv T_a^r A(\mathbf{x})\, G(b, T_a^r \varphi(\mathbf{x})) \mod (\mathbf{x})^{r+1}\mathbb{R}[\![\mathbf{x}]\!]^p \tag{1}$$

The module of relations at $b \in \varphi(M)$

We consider the equation at the level of Taylor polynomials:

$$T_a^r f(\mathbf{x}) \equiv T_a^r A(\mathbf{x})\, G(b, T_a^r \varphi(\mathbf{x})) \mod (\mathbf{x})^{r+1}\mathbb{R}[\![\mathbf{x}]\!]^p \tag{1}$$

B - Chevalley's function

Given $l \in \mathbb{N}$, there exists $r \geq l$ such that the derivatives of g of order $\leq l$ can be expressed in terms of the derivatives of f of order $\leq r$.

The module of relations at $b \in \varphi(M)$

We consider the equation at the level of Taylor polynomials:

$$T_a^r f(\mathbf{x}) \equiv T_a^r A(\mathbf{x})\, G(b, T_a^r \varphi(\mathbf{x})) \mod (\mathbf{x})^{r+1}\mathbb{R}[\![\mathbf{x}]\!]^p \tag{1}$$

B - Chevalley's function

Given $l \in \mathbb{N}$, there exists $r \geq l$ such that the derivatives of g of order $\leq l$ can be expressed in terms of the derivatives of f of order $\leq r$.

Formally, we stratify $B = \bigsqcup_{\tau=1}^{\tau_{\max}} \Lambda_\tau$ such that for each τ, there exists $r \geq l$ satisfying

$$\forall b \in \Lambda_\tau,\ \pi_l(\mathcal{R}_r(b)) = \pi_l(\mathcal{R}_{r-1}(b))$$

where

- $\mathcal{R}_r(b)$ is the *module of relations at* $b \in \varphi(M)$ formed by the $G \in \mathbb{R}[\![\mathbf{y}]\!]^q$ satisfying the homogeneous equation associated to (1) for all $a \in \varphi^{-1}(b)$, and,
- π_l is the truncation up to degree l.

Hironaka's formal division

- $F = \sum F_{(\alpha,j)} \mathbf{y}^{(\alpha,j)} \in \mathbb{R}[\![y_1, \dots, y_n]\!]^p$ where $\mathbf{y}^{(\alpha,j)} = \left(0, \dots, 0, y_1^{\alpha_1} \cdots y_n^{\alpha_n}, 0, \dots, 0\right)$.

Hironaka's formal division

- $F = \sum F_{(\alpha,j)} \mathbf{y}^{(\alpha,j)} \in \mathbb{R}[\![y_1, \dots, y_n]\!]^p$ where $\mathbf{y}^{(\alpha,j)} = \left(0, \dots, 0, y_1^{\alpha_1} \cdots y_n^{\alpha_n}, 0, \dots, 0\right)$.
- The set $\mathbb{N}^n \times \{1, \dots, p\} \ni (\alpha, j)$ is totally ordered by $\operatorname{lex}(|\alpha|, j, \alpha_1, \dots, \alpha_n)$.
- $\operatorname{supp} F \coloneqq \left\{(\alpha, j) \;:\; F_{(\alpha,j)} \neq 0\right\}$
- $\exp F \coloneqq \min(\operatorname{supp} F)$

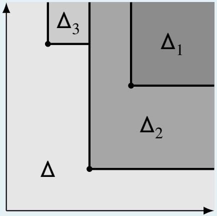
Hironaka's formal division

- $F = \sum F_{(\alpha,j)}\mathbf{y}^{(\alpha,j)} \in \mathbb{R}[\![y_1,\dots,y_n]\!]^p$ where $\mathbf{y}^{(\alpha,j)} = \left(0,\dots,0,y_1^{\alpha_1}\cdots y_n^{\alpha_n},0,\dots,0\right)$.
- The set $\mathbb{N}^n\times\{1,\dots,p\}\ni(\alpha,j)$ is totally ordered by $\operatorname{lex}(|\alpha|,j,\alpha_1,\dots,\alpha_n)$.
- $\operatorname{supp} F \coloneqq \left\{(\alpha,j)\ :\ F_{(\alpha,j)}\neq 0\right\}$
- $\exp F \coloneqq \min(\operatorname{supp} F)$

Theorem – Hironaka 1964, Bierstone–Milman 1987

Let $\Phi_1,\dots,\Phi_q\in\mathbb{R}[\![\mathbf{y}]\!]^p$.

Set $\Delta_1 \coloneqq \exp\Phi_1+\mathbb{N}^n$, $\Delta_i \coloneqq \left(\exp\Phi_i+\mathbb{N}^n\right)\setminus\bigcup_{k=1}^{i-1}\Delta_k$, and $\Delta \coloneqq (\mathbb{N}^n\times\{1,\dots,p\})\setminus\bigcup_{k=1}^{q}\Delta_k$.



Then $\forall F\in\mathbb{R}[\![\mathbf{y}]\!]^p$, $\exists! Q_i\in\mathbb{R}[\![\mathbf{y}]\!]$, $R\in\mathbb{R}[\![\mathbf{y}]\!]^p$ such that

- $F=\sum_{i=1}^{q}Q_i\Phi_i+R$
- $\exp\Phi_i+\operatorname{supp}Q_i\subset\Delta_i$
- $\operatorname{supp}R\subset\Delta$

Diagram of initial exponents

Let $M \subset \mathbb{R}[\![\mathbf{y}]\!]^p$ be a $\mathbb{R}[\![\mathbf{y}]\!]$-submodule.
The *diagram of initial exponents* of M is

$$\mathcal{N}(M) := \{\exp F \ : \ F \in M \setminus \{0\}\} \subset \mathbb{N}^n \times \{1, \dots, p\}$$

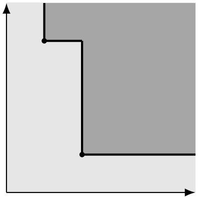


Diagram of initial exponents

Let $M \subset \mathbb{R}[\![\mathbf{y}]\!]^p$ be a $\mathbb{R}[\![\mathbf{y}]\!]$-submodule.
The *diagram of initial exponents* of M is

$$\mathcal{N}(M) := \{\exp F \ : \ F \in M \setminus \{0\}\} \subset \mathbb{N}^n \times \{1, \dots, p\}$$

Note that $\mathcal{N}(M)$ has finitely many vertices (α_i, j_i), $i = 1, \dots, q$.
For each one, we pick a representative $\Phi_i \in M$, i.e. $\exp \Phi_i = (\alpha_i, j_i)$.

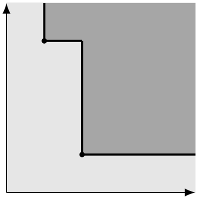
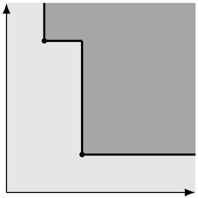


Diagram of initial exponents

Let $M \subset \mathbb{R}[\![\mathbf{y}]\!]^p$ be a $\mathbb{R}[\![\mathbf{y}]\!]$-submodule.
The *diagram of initial exponents* of M is

$$\mathcal{N}(M) := \{\exp F \ : \ F \in M \setminus \{0\}\} \subset \mathbb{N}^n \times \{1, \dots, p\}$$



Note that $\mathcal{N}(M)$ has finitely many vertices (α_i, j_i), $i = 1, \dots, q$.
For each one, we pick a representative $\Phi_i \in M$, i.e. $\exp \Phi_i = (\alpha_i, j_i)$.

Corollary

Let $F \in \mathbb{R}[\![\mathbf{y}]\!]^p$. Then $F \in M$ if and only if its remainder by the formal division w.r.t. the Φ_i is 0.

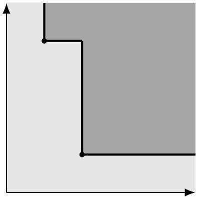
Particularly $\Phi_1, \dots, \Phi_q$ generate M.

Diagram of initial exponents

Let $M \subset \mathbb{R}[\![\mathbf{y}]\!]^p$ be a $\mathbb{R}[\![\mathbf{y}]\!]$-submodule.
The *diagram of initial exponents* of M is

$$\mathcal{N}(M) := \{\exp F \ : \ F \in M \setminus \{0\}\} \subset \mathbb{N}^n \times \{1, \dots, p\}$$

Note that $\mathcal{N}(M)$ has finitely many vertices (α_i, j_i), $i = 1, \dots, q$.
For each one, we pick a representative $\Phi_i \in M$, i.e. $\exp \Phi_i = (\alpha_i, j_i)$.



Corollary

Let $F \in \mathbb{R}[\![\mathbf{y}]\!]^p$. Then $F \in M$ if and only if its remainder by the formal division w.r.t. the Φ_i is 0.

Particularly $\Phi_1, \dots, \Phi_q$ generate M.

Proof. Write $F = \sum_{i=1}^{q} Q_i \Phi_i + R$ with $\operatorname{supp} R \subset \mathcal{N}(M)^c$. ∎

Diagram of initial exponents and module of relations

Lemma – Chevalley's function

Let $l \in \mathbb{N}$.
There exists $(\Lambda_\tau)_\tau$ a stratification of B such that given a stratum Λ_τ, there exists $r \geq l$ satisfying

- $\forall b \in \Lambda_\tau,\ \pi_l(\mathcal{R}_r(b)) = \pi_l(\mathcal{R}_{r-1}(b))$,
- $\mathcal{N}(\mathcal{R}_r(b))$ is constant on Λ_τ.

Diagram of initial exponents and module of relations

Lemma – Chevalley's function

Let $l \in \mathbb{N}$.
There exists $(\Lambda_\tau)_\tau$ a stratification of B such that given a stratum Λ_τ, there exists $r \geq l$ satisfying

- $\forall b \in \Lambda_\tau,\ \pi_l(\mathcal{R}_r(b)) = \pi_l(\mathcal{R}_{r-1}(b))$,
- $\mathcal{N}(\mathcal{R}_r(b))$ is constant on Λ_τ.

We set

$$B' := \bigcup_{\dim \Lambda_\tau < \dim B} \Lambda_\tau$$

so that $\forall \tau,\ \overline{\Lambda_\tau} \setminus \Lambda_\tau \subset B'$.

Construction of G on Λ_τ

For $b \in B$ and $t \geq r$, by assumption there exists $W_b \in \mathbb{R}[\mathbf{y}]^q$ such that

$$T_a^t f(\mathbf{x}) \equiv T_a^t A(\mathbf{x})\, W_b(T_a^t \varphi(\mathbf{x})) \mod (\mathbf{x})^{t+1}\mathbb{R}[\![\mathbf{x}]\!]^p, \quad \forall a \in \varphi^{-1}(b).$$

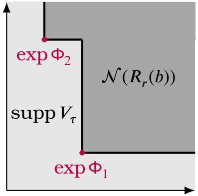
Construction of G on Λ_τ

For $b \in B$ and $t \geq r$, by assumption there exists $W_b \in \mathbb{R}[\mathbf{y}]^q$ such that

$$T_a^t f(\mathbf{x}) \equiv T_a^t A(\mathbf{x})\, W_b(T_a^t \varphi(\mathbf{x})) \mod (\mathbf{x})^{t+1}\mathbb{R}[\![\mathbf{x}]\!]^p, \quad \forall a \in \varphi^{-1}(b).$$

Let's fix a stratum Λ_τ and $b \in \Lambda_\tau$.
By formal division, we may write $W_b(\mathbf{y}) = \sum Q_i(\mathbf{y})\Phi_i(\mathbf{y}) + V_\tau(b,\mathbf{y})$ where the Φ_i are as above for $\mathcal{R}_r(b)$.
Note that the remainder $V_\tau(b,\mathbf{y}) \in \mathbb{R}[\mathbf{y}]^q$ satisfies

$$W_b(\mathbf{y}) - V_\tau(b,\mathbf{y}) \in \mathcal{R}_r(b) \quad \text{and} \quad \operatorname{supp} V_\tau(b,\mathbf{y}) \subset \mathcal{N}(\mathcal{R}_r(b))^c.$$



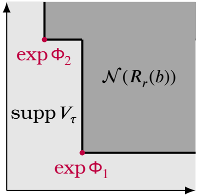
Construction of G on Λ_τ

For $b \in B$ and $t \geq r$, by assumption there exists $W_b \in \mathbb{R}[\mathbf{y}]^q$ such that

$$T_a^t f(\mathbf{x}) \equiv T_a^t A(\mathbf{x})\, W_b(T_a^t \varphi(\mathbf{x})) \mod (\mathbf{x})^{t+1}\mathbb{R}[\![\mathbf{x}]\!]^p, \quad \forall a \in \varphi^{-1}(b).$$

Let's fix a stratum Λ_τ and $b \in \Lambda_\tau$.
By formal division, we may write $W_b(\mathbf{y}) = \sum Q_i(\mathbf{y})\Phi_i(\mathbf{y}) + V_\tau(b,\mathbf{y})$ where the Φ_i are as above for $\mathcal{R}_r(b)$.
Note that the remainder $V_\tau(b,\mathbf{y}) \in \mathbb{R}[\mathbf{y}]^q$ satisfies

$$W_b(\mathbf{y}) - V_\tau(b,\mathbf{y}) \in \mathcal{R}_r(b) \quad \text{and} \quad \operatorname{supp} V_\tau(b,\mathbf{y}) \subset \mathcal{N}(\mathcal{R}_r(b))^c.$$



Lemma

$G_\tau(b,\mathbf{y}) := \pi_l\left(V_\tau(b,\mathbf{y})\right)$ is a semialgebraic Whitney field of order l on Λ_τ satisfying (1).

G is a Whitney field of order l on Λ_τ

To simplify the situation, we omit φ.
Thanks to Borel's lemma with parameter, it is enough to check that $D_{b,v}G_\tau^{l-1}(b,\mathbf{y}) = D_{\mathbf{y},v}G_\tau(b,\mathbf{y})$.

G is a Whitney field of order l on Λ_τ

To simplify the situation, we omit φ.
Thanks to Borel's lemma with parameter, it is enough to check that $D_{b,v}G_\tau^{l-1}(b,\mathbf{y}) = D_{\mathbf{y},v}G_\tau(b,\mathbf{y})$.

Applying $D_{b,v} - D_{\mathbf{y},v}$ to

$$T_a^r f(\mathbf{y}) \equiv T_a^r A(\mathbf{y})\, V_\tau\,(b,\mathbf{y}) \mod (\mathbf{y})^{r+1}\mathbb{R}[\![\mathbf{y}]\!]^p$$

we get

$$0 \equiv T_a^r A(\mathbf{y}) \left(D_{b,v}V_\tau^{r-1}\,(b,\mathbf{y}) - D_{\mathbf{y},v}V_\tau\,(b,\mathbf{y})\right) \mod (\mathbf{y})^{r+1}\mathbb{R}[\![\mathbf{y}]\!]^p$$

G is a Whitney field of order l on Λ_τ

To simplify the situation, we omit φ.
Thanks to Borel's lemma with parameter, it is enough to check that $D_{b,v}G_\tau^{l-1}(b,\mathbf{y}) = D_{\mathbf{y},v}G_\tau(b,\mathbf{y})$.

Applying $D_{b,v} - D_{\mathbf{y},v}$ to

$$T_a^r f(\mathbf{y}) \equiv T_a^r A(\mathbf{y})\, V_\tau(b,\mathbf{y}) \mod (\mathbf{y})^{r+1}\mathbb{R}[\![\mathbf{y}]\!]^p$$

we get

$$0 \equiv T_a^r A(\mathbf{y}) \left(D_{b,v}V_\tau^{r-1}(b,\mathbf{y}) - D_{\mathbf{y},v}V_\tau(b,\mathbf{y})\right) \mod (\mathbf{y})^{r+1}\mathbb{R}[\![\mathbf{y}]\!]^p$$

therefore

$$D_{b,v}V_\tau^{r-1}(b,\mathbf{y}) - D_{\mathbf{y},v}V_\tau(b,\mathbf{y}) \in \mathcal{R}_{r-1}(b)$$

G is a Whitney field of order l on Λ_τ

To simplify the situation, we omit φ.
Thanks to Borel's lemma with parameter, it is enough to check that $D_{b,v}G_\tau^{l-1}(b,\mathbf{y}) = D_{\mathbf{y},v}G_\tau(b,\mathbf{y})$.

Applying $D_{b,v} - D_{\mathbf{y},v}$ to

$$T_a^r f(\mathbf{y}) \equiv T_a^r A(\mathbf{y})\, V_\tau\,(b,\mathbf{y}) \quad \text{mod} \quad (\mathbf{y})^{r+1}\mathbb{R}[\![\mathbf{y}]\!]^p$$

we get

$$0 \equiv T_a^r A(\mathbf{y}) \left(D_{b,v}V_\tau^{r-1}\,(b,\mathbf{y}) - D_{\mathbf{y},v}V_\tau\,(b,\mathbf{y})\right) \quad \text{mod} \quad (\mathbf{y})^{r+1}\mathbb{R}[\![\mathbf{y}]\!]^p$$

therefore

$$D_{b,v}V_\tau^{r-1}\,(b,\mathbf{y}) - D_{\mathbf{y},v}V_\tau\,(b,\mathbf{y}) \in \mathcal{R}_{r-1}(b)$$

hence, by Chevalley's function,

$$D_{b,v}G_\tau^{l-1}(b,\mathbf{y}) - D_{\mathbf{y},v}G_\tau(b,\mathbf{y}) \in \pi_{l-1}(\mathcal{R}_{r-1}(b)) = \pi_{l-1}(\mathcal{R}_r(b))$$

G is a Whitney field of order l on Λ_τ

To simplify the situation, we omit φ.
Thanks to Borel's lemma with parameter, it is enough to check that $D_{b,v}G_\tau^{l-1}(b,\mathbf{y}) = D_{\mathbf{y},v}G_\tau(b,\mathbf{y})$.

Applying $D_{b,v} - D_{\mathbf{y},v}$ to

$$T_a^r f(\mathbf{y}) \equiv T_a^r A(\mathbf{y})\, V_\tau\,(b,\mathbf{y}) \quad \text{mod} \quad (\mathbf{y})^{r+1}\mathbb{R}[\![\mathbf{y}]\!]^p$$

we get

$$0 \equiv T_a^r A(\mathbf{y})\left(D_{b,v}V_\tau^{r-1}\,(b,\mathbf{y}) - D_{\mathbf{y},v}V_\tau\,(b,\mathbf{y})\right) \quad \text{mod} \quad (\mathbf{y})^{r+1}\mathbb{R}[\![\mathbf{y}]\!]^p$$

therefore

$$D_{b,v}V_\tau^{r-1}\,(b,\mathbf{y}) - D_{\mathbf{y},v}V_\tau\,(b,\mathbf{y}) \in \mathcal{R}_{r-1}(b)$$

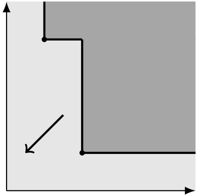
hence, by Chevalley's function,

$$D_{b,v}G_\tau^{l-1}(b,\mathbf{y}) - D_{\mathbf{y},v}G_\tau(b,\mathbf{y}) \in \pi_{l-1}(\mathcal{R}_{r-1}(b)) = \pi_{l-1}(\mathcal{R}_r(b))$$

but

$$\operatorname{supp}\left(D_{b,v}G_\tau^{l-1}(b,\mathbf{y}) - D_{\mathbf{y},v}G_\tau(b,\mathbf{y})\right) \subset \mathcal{N}(\mathcal{R}_r(b))^c$$

consequently, $D_{b,v}G_\tau^{l-1}(b,\mathbf{y}) - D_{\mathbf{y},v}G_\tau(b,\mathbf{y}) = 0$.



Gluing between strata

C - gluing between strata: the Łojasiewicz inequality

Fix a stratum Λ_τ. There exists $\sigma \in \mathbb{N}$ such that if $t \geq r + \sigma$ then $\lim\limits_{b \to \overline{\Lambda_\tau} \setminus \Lambda_\tau} G_\tau(b, \mathbf{y}) = 0$.

The constant term of the equation is flat on B' hence on $\overline{\Lambda_\tau} \setminus \Lambda_\tau \subset B'$.

Summary

We constructed $G(b,\mathbf{y}) = \sum_{|\alpha|\leq k} \frac{g_\alpha(b)}{\alpha!}\mathbf{y}^\alpha$ a semialgebraic Whitney field of order k on B such that

$$\forall b \in B,\ \forall a \in \varphi^{-1}(b),\ T_a^k f(\mathbf{x}) \equiv T_a^k A(\mathbf{x})\, G(b, T_a^k \varphi(\mathbf{x})) \mod (\mathbf{x})^{k+1}\mathbb{R}[\![\mathbf{x}]\!]^p$$

Hence we obtain $g : \mathbb{R}^n \to \mathbb{R}^q$ semialgebraic and C^k such that $f - A \cdot (g \circ \varphi)$ is k-flat on $\varphi^{-1}(B)$.

Summary

We constructed $G(b,\mathbf{y}) = \sum_{|\alpha|\leq k} \frac{g_\alpha(b)}{\alpha!}\mathbf{y}^\alpha$ a semialgebraic Whitney field of order k on B such that

$$\forall b \in B,\ \forall a \in \varphi^{-1}(b),\ T_a^k f(\mathbf{x}) \equiv T_a^k A(\mathbf{x})\, G(b, T_a^k \varphi(\mathbf{x})) \mod (\mathbf{x})^{k+1}\mathbb{R}[\![\mathbf{x}]\!]^p$$

Hence we obtain $g : \mathbb{R}^n \to \mathbb{R}^q$ semialgebraic and C^k such that $f - A \cdot (g \circ \varphi)$ is k-flat on $\varphi^{-1}(B)$.

Loss of differentiability

For $k \in \mathbb{N}$, we set $l \geq k\rho$, then $r \geq r(l)$ and finally $t(k) := t \geq r + \sigma$ where

A. ρ is an upper bound of Whitney's loss of differentiability (induction step).

B. $r : \mathbb{N} \to \mathbb{N}$ is an upper bound of the Chevalley functions on the various strata.

C. σ is an upper bound of Łojasiewicz's loss of differentiability on each stratum.