$\mathcal{C}^m$ SOLUTIONS OF SEMIALGEBRAIC EQUATIONS

Joint work with E. Bierstone and P.D. Milman

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Motivations – Whitney's Extension Problem

Whitney's Extension Problem

Let $X \subset \mathbb{R}^n$ be closed and $f : X \to \mathbb{R}$.
Under which assumptions does f admit a C^m extension?
(i.e. $\exists F : \mathbb{R}^n \to \mathbb{R}$ which is C^m and such that $F_{|X} = f$)

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Characterization of functions $f : X \to \mathbb{R}$ admitting a C^m extension.

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Geometric version of Whitney's Extension Problem

If the data are semialgebraic, does f admit an extension preserving this condition?

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Theorem – Whitney, 1934

Let $X \subset \mathbb{R}^n$ be closed. Consider a family $(f_\alpha : X \to \mathbb{R})_{\substack{\alpha\in\mathbb{N}^n \\ |\alpha|\le m}}$ of continuous functions such that

$$\forall z \in X,\ \forall \alpha \in \mathbb{N}^n,\ |\alpha| \le m \implies f_\alpha(x) - \sum_{|\beta|\le m-|\alpha|} \frac{f_{\alpha+\beta}(y)}{\beta!}(x-y)^\beta = \underset{X\ni x,y\to z}{o}\left(\|x-y\|^{m-|\alpha|}\right) \qquad (1)$$

then there exists a C^m function $F : \mathbb{R}^n \to \mathbb{R}$ such that $D^\alpha F = f_\alpha$ on X (and F is C^ω on $\mathbb{R}^n \setminus X$).

Such a family $(f_\alpha : X \to \mathbb{R})_{\alpha\in\mathbb{N}^n,|\alpha|\le m}$ of continuous functions satisfying (1) is called a *Whitney field of order m on X*.

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Theorem – Kurdyka–Pawłucki, 1997, 2014 – Thamrongthanyalak, 2017

If the set X and the functions f_α are definable in an o-minimal structure then we may assume that F is definable too (and C^q on $\mathbb{R}^n \setminus X$ for $q \ge m$).

Motivations – The Brenner–Fefferman–Hochster–Kollár problem

1. Let $f_1, \dots, f_r \in \mathbb{R}[x_1, \dots, x_n]$.
   Which functions $\varphi : \mathbb{R}^n \to \mathbb{R}$ may be expressed as $\varphi = \sum \varphi_i f_i$ with $\varphi_i \in C^0(\mathbb{R}^n, \mathbb{R})$?

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2. Let $f_1, \dots, f_r \in \mathbb{R}[x_1, \dots, x_n]$. Let $\varphi : \mathbb{R}^n \to \mathbb{R}$ satisfying some property.
   If $\varphi = \sum_{i=1}^{r} \varphi_i f_i$ where $\varphi_i \in C^0(\mathbb{R}^n, \mathbb{R})$, does there exist $\tilde{\varphi}_i$ with the above property s.t. $\varphi = \sum_{i=1}^{r} \tilde{\varphi}_i f_i$?

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- Fefferman–Kollár (2013) – polynomial data:
  Continuous solution ⇒ **semialgebraic** continuous solution.
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- Kollár–Nowak (2015) – polynomial data:
  Continuous solution $\not\Rightarrow$ **regulous** solution.
- Adamus–Seyedinejad (2018) – polynomial data:
  Continuous solution $\not\Rightarrow$ **arc-analytic** semialgebraic solution.

Motivations – The Brenner–Fefferman–Hochster–Kollár problem

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Under which assumptions, does an equation $\varphi = \sum \varphi_i f_i$ admit a $\mathcal{C}^m$ solution?

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If the data are semialgebraic, does the equation admit a C^m solution preserving this condition?

Questions: are there solutions preserving semialgebraicity?

Question

Let $X \subset \mathbb{R}^n$ be semialgebraic and closed. Let $f : X \to \mathbb{R}$ be semialgebraic.
If f admits a C^m extension, does it admit a semialgebraic C^m extension?

Question

Let $A : \mathbb{R}^n \to \mathcal{M}_{p,q}(\mathbb{R})$ and $f : \mathbb{R}^n \to \mathbb{R}^p$ be semialgebraic.
If there exists a C^m function $g : \mathbb{R}^n \to \mathbb{R}^q$ such that $f(x) = A(x)g(x)$,
then does there exist a semialgebraic C^m function $\tilde{g} : \mathbb{R}^n \to \mathbb{R}^q$ such that $f(x) = A(x)\tilde{g}(x)$?

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- Fefferman–Luli (in preparation): $\forall m$ but $n = 2$.

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- Bierstone–C.–Milman (2020): $\forall n$, $\forall m$, but with a loss of differentiability.

Presentation of the results

Theorem – Bierstone–C.–Milman, 2020

Given $A : \mathbb{R}^n \to \mathcal{M}_{p,q}(\mathbb{R})$ semialgebraic, there exists $r : \mathbb{N} \to \mathbb{N}$ such that:
If $F : \mathbb{R}^n \to \mathbb{R}^p$ semialgebraic may be written $F(x) = A(x)G(x)$ where G is $C^{r(m)}$, then $F(x) = A(x)\tilde{G}(x)$ where $\tilde{G}(x)$ is semialgebraic and C^m.

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Besides, if A is C^∞ then $r(m) = \alpha m + \beta$.

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Theorem – Bierstone–C.–Milman, 2020

Given $X \subset \mathbb{R}^n$ closed and semialgebraic, there exists $r : \mathbb{N} \to \mathbb{N}$ satisfying the following property: if $f : X \to \mathbb{R}$ is semialgebraic and admits a $C^{r(m)}$ extension, then it admits a C^m extension which is semialgebraic.

Presentation of the results - A common generalization

The equation problem

Consider an equation

$$\forall x \in \mathbb{R}^n,\ A(x)G(x) = F(x)$$

By resolution of singularities, there exists $\varphi : M \to \mathbb{R}^n$ Nash and proper defined on a Nash manifold such that after composition, we get an equation

$$\forall y \in M,\ \tilde{A}(y)G(\varphi(y)) = \tilde{F}(y)$$

where $\tilde{A}$ is now Nash.

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The extension problem

Let $X \subset \mathbb{R}^n$ be semialgebraic and closed.

By resolution of singularities, there exists $\varphi : M \to \mathbb{R}^n$ Nash and proper defined on a Nash manifold such that $X = \varphi(M)$.

Given $g : \mathbb{R}^n \to \mathbb{R}$ and $f : X \to \mathbb{R}$, we have $g_{|X} = f$ if and only if

$$\forall y \in M,\ g(\varphi(y)) = \tilde{f}(y)$$

where $\tilde{f} = f \circ \varphi$.

Presentation of the results - The main result

Theorem – Bierstone–C.–Milman, 2020

Let $A:\mathbb{R}^n\to\mathcal{M}_{p,q}(\mathbb{R})$ be Nash and let $\varphi:M\to\mathbb{R}^n$ be Nash and proper defined on $M\subset\mathbb{R}^N$ a Nash submanifold.
There exists $r:\mathbb{N}\to\mathbb{N}$ satisfying the following property.
If $f:M\to\mathbb{R}^p$ semialgebraic may be written

$$f(x)=A(x)g(\varphi(x))$$

for a $C^{r(m)}$ function $g:\mathbb{R}^n\to\mathbb{R}^q$ then

$$f(x)=A(x)\tilde{g}(\varphi(x))$$

for a semialgebraic C^m function $\tilde{g}$.

Heart of the proof: induction on dimension

Proposition: the induction step

Let $B \subset \varphi(M)$ be semialgebraic and closed.
There exist $B' \subset B$ semialgebraic satisfying $\dim B' < \dim B$ and $t : \mathbb{N} \to \mathbb{N}$ such that if

1. $f : M \to \mathbb{R}^p$ is $C^{t(k)}$, semialgebraic and $t(k)$-flat on $\varphi^{-1}(B')$, and
2. $f = A \cdot (g \circ \varphi)$ admits a $C^{t(k)}$ solution g,

then there exists a semialgebraic C^k function $\tilde{g} : \mathbb{R}^n \to \mathbb{R}^q$ s.t. $f - A \cdot (\tilde{g} \circ \varphi)$ is k-flat on $\varphi^{-1}(B)$.

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By substracting $A \cdot (\tilde{g} \circ \varphi)$ on both side, we get an equation

$$f = A \cdot (g \circ \varphi)$$

where f is now k-flat on $\varphi^{-1}(B)$.

Heart of the proof: induction on dimension

Strategy: construction of a semialgebraic Whitney field

$$G(b,\mathbf{y}) = \sum_{|\alpha|\leq l} \frac{g_\alpha(b)}{\alpha!}\mathbf{y}^\alpha \in C^0(B)[\mathbf{y}]$$

vanishing on B' such that

$$\forall b \in B \setminus B',\, \forall a \in \varphi^{-1}(b),\, T_a^l f(\mathbf{x}) \equiv T_a^l A(\mathbf{x})\, G(b, T_a^l \varphi(\mathbf{x})) \mod (\mathbf{x})^{l+1}\mathbb{R}[\![\mathbf{x}]\!]^p$$

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A - Whitney regularity

Given B, there exists $\rho \in \mathbb{N}$ such that for b in a neighborhood of a, there exists a path on B from a to b whose length is $\leq C\|b-a\|^{1/\rho}$.

If G is a Whitney field of order $l \geq k\rho$ on $B \setminus B'$ then it is a Whitney field of order k on B.

The module of relations at $b \in \varphi(M)$

We consider the equation at the level of Taylor polynomials:

$$T_a^r f(\mathbf{x}) \equiv T_a^r A(\mathbf{x})\, G(b, T_a^r \varphi(\mathbf{x})) \mod (\mathbf{x})^{r+1}\mathbb{R}[\![\mathbf{x}]\!]^p$$

The *module of relations of order* r *at* $b \in \varphi(M)$ is

$$\mathcal{R}_r(b) := \left\{ W \in \mathbb{R}[\![\mathbf{y}]\!]^q \ :\ \forall a \in \varphi^{-1}(b),\, T_a^r A(\mathbf{x}) W\left(\tilde{T}_a^r \varphi(\mathbf{x})\right) \equiv 0 \mod \mathfrak{m}_{\mathbf{x}}^{r+1}\mathbb{R}[\![\mathbf{x}]\!]^p \right\}$$

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B - Chevalley's function

Given $l \in \mathbb{N}$, there exists $r \geq l$ such that the derivatives of g of order $\leq l$ can be expressed in terms of the derivatives of f of order $\leq r$.

The module of relations at $b \in \varphi(M)$

Writing

$$W = \sum_{\substack{\beta \in \mathbb{N}^n \\ j=1,\dots,q}} W_{(\beta,j)} \mathbf{y}^{(\beta,j)} \quad \text{où} \quad \mathbf{y}^{(\alpha,j)} = \left(0,\dots,0, y_1^{\alpha_1} \cdots y_n^{\alpha_n}, 0, \dots, 0\right)$$

We have that $W \in \mathcal{R}_r(b)$, i.e. $\forall a \in \varphi^{-1}(b)$, $T_a^r A(\mathbf{x}) W\left(\tilde{T}_a^r \varphi(\mathbf{x})\right) \equiv 0 \mod \mathfrak{m}_{\mathbf{x}}^{r+1} \mathbb{R}[\![\mathbf{x}]\!]^p$, if and only if

$$\sum_{\substack{(\beta,j) \\ |\beta| \le r}} L_{(\alpha,i)}^{(\beta,j)}(a) W_{(\beta,j)} = 0, \text{ où } \quad \begin{cases} |\alpha| \le r, \\ i = 1, \dots, p, \\ a \in \varphi^{-1}(b). \end{cases}$$

Set $\quad s := \#\{(\beta, j) \ : \ |\beta| \le r\} = \binom{n+r}{r} q, \qquad \underline{a} \in \left\{(a_1, \dots, a_s) \in M^s \ : \ \varphi(a_1) = \cdots = \varphi(a_s)\right\},$

$$\rho_r^0(\underline{a}) := \operatorname{rank}\left\{\sum_{\substack{(\beta,j) \\ |\beta| \le r}} L_{(\alpha,i)}^{(\beta,j)}(a_\nu) W_{(\beta,j)} = 0 \ : \ \begin{array}{l} |\alpha| \le r, \\ i = 1, \dots, p, \\ \nu = 1, \dots, s. \end{array}\right\} \text{ and } \rho_{r,l}^1(\underline{a}) := \operatorname{rank}\left\{\sum_{\substack{(\beta,j) \\ l \le |\beta| \le r}} L_{(\alpha,i)}^{(\beta,j)}(a_\nu) W_{(\beta,j)} = 0 \ : \ \begin{array}{l} |\alpha| \le r, \\ i = 1, \dots, p, \\ \nu = 1, \dots, s. \end{array}\right\}$$

Set $\omega^{r,l} := \binom{n+l}{l} q + \max_{\underline{a}} \rho^0(\underline{a}) - \max_{\underline{a}} \rho^1(\underline{a})$. Then $l \le r \le r' \implies \omega^{r',l} \le \omega^{r,l}$.

Besides, for $b = \varphi(a_i)$ where the max is achieved, we have $\dim \pi_l(\mathcal{R}_r(b)) = \omega^{r,l}$.

The module of relations at $b \in \varphi(M)$

Lemma – Chevalley's function

Let $l \in \mathbb{N}$.
There exists $(\Lambda_\tau)_\tau$ a stratification of B such that given a stratum Λ_τ, there exists $r \geq l$ satisfying

$$\forall b \in \Lambda_\tau,\ \pi_l(\mathcal{R}_r(b)) = \pi_l(\mathcal{R}_{r-1}(b))$$

where π_l if the truncation up to degree l.

A stratification of B is a finite partition $\mathcal{S}$ of B into connected Nash submanifolds such that if $S \in \mathcal{S}$ then $\left(\overline{S} \backslash S\right) \cap B$ is the union of strata $T \in \mathcal{S}$ satisfying $\dim T < \dim S$.

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Can we obtain a result uniform with respect to b?

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From a pointwise situation to a uniform one

Can we obtain a result uniform with respect to b?

$\rightsquigarrow$ division with respect to the relations.

Hironaka's formal division

- $F = \sum F_{(\alpha,j)} \mathbf{y}^{(\alpha,j)} \in \mathbb{R}[\![y_1, \dots, y_n]\!]^p$ where $\mathbf{y}^{(\alpha,j)} = \left(0, \dots, 0, y_1^{\alpha_1} \cdots y_n^{\alpha_n}, 0, \dots, 0\right)$.
- The set $\mathbb{N}^n \times \{1, \dots, p\} \ni (\alpha, j)$ is totally ordered by $\operatorname{lex}(|\alpha|, j, \alpha_1, \dots, \alpha_n)$.

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- $\exp F \coloneqq \min(\operatorname{supp} F)$

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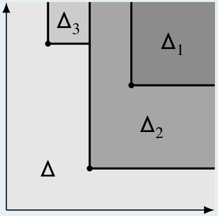
- $\exp F \coloneqq \min(\operatorname{supp} F)$

Theorem – Hironaka 1964, Bierstone–Milman 1987

Let $\Phi_1,\dots,\Phi_q \in \mathbb{R}[\![\mathbf{y}]\!]^p$. Set $(\alpha_i,j_i) \coloneqq \exp\Phi_i$.

Set $\Delta_1 \coloneqq (\alpha_1,j_1)+\mathbb{N}^n$, $\Delta_i \coloneqq \left((\alpha_i,j_i)+\mathbb{N}^n\right) \setminus \bigcup_{k=1}^{i-1} \Delta_k$, and $\Delta \coloneqq (\mathbb{N}^n \times \{1,\dots,p\}) \setminus \bigcup_{k=1}^{q} \Delta_k$.



Then $\forall F \in \mathbb{R}[\![\mathbf{y}]\!]^p$, $\exists! Q_i \in \mathbb{R}[\![\mathbf{y}]\!]$, $R \in \mathbb{R}[\![\mathbf{y}]\!]^p$ such that

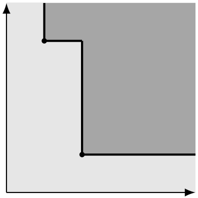
- $F = \sum_{i=1}^{q} Q_i\Phi_i + R$
- $(\alpha_i,j_i) + \operatorname{supp} Q_i \subset \Delta_i$
- $\operatorname{supp} R \subset \Delta$

Besides $(\alpha_i,j_i) + \exp Q_i \geq \exp F$ and $\exp R \geq \exp F$.

Diagram of initial exponents

Let $M \subset \mathbb{R}[\![\mathbf{y}]\!]^p$ be a $\mathbb{R}[\![\mathbf{y}]\!]$-submodule.
The *diagram of initial exponents* of M is

$$\mathcal{N}(M) \coloneqq \{\exp F \;:\; F \in M \setminus \{0\}\} \subset \mathbb{N}^n \times \{1, \dots, p\}$$



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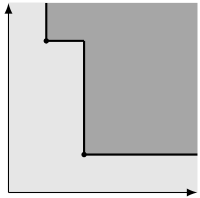

Note that $\mathcal{N}(M)$ has finitely many vertices (α_i, j_i), $i = 1, \dots, q$.
For each one, we pick a representative $\Phi_i \in M$, i.e. $\exp \Phi_i = (\alpha_i, j_i)$.

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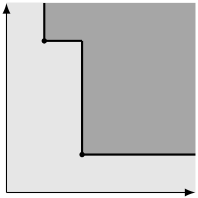

Corollary

Let $F \in \mathbb{R}[\![\mathbf{y}]\!]^p$. Then $F \in M$ if and only if its remainder by the formal division w.r.t. the Φ_i is 0.

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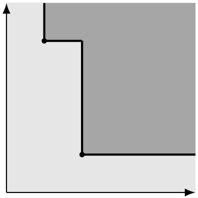

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Proof. Write $F = \sum_{i=1}^{q} Q_i \Phi_i + R$ with $\operatorname{supp} R \subset \Delta = \mathcal{N}(M)^c$. ∎

Diagram of initial exponents and module of relations

Lemma – Chevalley's function

Let $l \in \mathbb{N}$.
There exists $(\Lambda_\tau)_\tau$ a stratification of B such that given a stratum Λ_τ, there exists $r \geq l$ satisfying

- $\forall b \in \Lambda_\tau,\ \pi_l(\mathcal{R}_r(b)) = \pi_l(\mathcal{R}_{r-1}(b))$,
- $\mathcal{N}(\mathcal{R}_r(b))$ is constant on Λ_τ.

We define B' as the union of the strata of $< \dim B$.

Construction of G on Λ_τ

For $b \in B \setminus B'$ and $t \geq r$, there exists a formal solution

$$T_a^t f(\mathbf{x}) \equiv T_a^t A(\mathbf{x})\, W_b(T_a^t \varphi(\mathbf{x})) \mod (\mathbf{x})^{t+1}\mathbb{R}[\![\mathbf{x}]\!]^p, \quad \forall a \in \varphi^{-1}(b),$$

where $W_b \in \mathbb{R}[\mathbf{y}]^q$.

Let's fix a stratum Λ_τ and $b \in \Lambda_\tau$. Then, by formal division, there exists a unique polynomial of degree $\leq r$

$$V_\tau(b,\mathbf{y}) = \sum_{(\beta,j)\in\Delta_\tau} V_\tau^{\beta,j}(b)\mathbf{y}^{\beta,j} \in \mathbb{R}[\mathbf{y}]^q$$

such that

$$W_b(\mathbf{y}) - V_\tau(b,\mathbf{y}) \in \mathcal{R}_r(b) \quad \text{et} \quad \operatorname{supp} V_\tau(b,\mathbf{y}) \subset \Delta_\tau.$$

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Lemma

$$G_\tau(b,\mathbf{y}) := \pi_l\left(V_\tau(b,\mathbf{y})\right) = \sum_{\substack{(\beta,j)\in\Delta_\tau \\ |\beta|\leq l}} V_\tau^{\beta,j}(b)\,\mathbf{y}^{\beta,j}$$

is a semialgebraic Whitney field of order l on Λ_τ.

G is a Whitney field of order l on Λ_τ

Thanks to the following lemma, it is enough to check that $D_{b,v}G_\tau^{l-1}(b,\mathbf{y}) = D_{\mathbf{y},v}G_\tau(b,\mathbf{y})$.

Borel's lemma with parameter

Let Λ be a C^m- submanifold and $F = \sum_{|\alpha|\leq m} \frac{f_\alpha(a)}{\alpha!}\mathbf{x}^\alpha \in C^0(\Lambda)[\mathbf{x}]$.

Then F is a Whitney field of order m on Λ if and only if

$$\begin{cases} F^{m-1} \in C^1(\Lambda)[\mathbf{x}] \\ \forall a \in \Lambda,\ \forall u \in T_a\Lambda,\ D_{a,u}F^{m-1}(a,\mathbf{x}) = D_{\mathbf{x},u}F(a,\mathbf{x}) \end{cases}$$

G is a Whitney field of order l on Λ_τ

To simplify the situation, we omit φ. Applying $D_{b,v} - D_{\mathbf{y},v}$ to

$$T_a^r f(\mathbf{y}) \equiv T_a^r A(\mathbf{y}) V_\tau(b,\mathbf{y}) \mod (\mathbf{y})^{r+1}\mathbb{R}[\![\mathbf{y}]\!]^p$$

we get

$$0 \equiv T_a^r A(\mathbf{y}) \left(D_{b,v} V_\tau^{r-1}(b,\mathbf{y}) - D_{\mathbf{y},v} V_\tau(b,\mathbf{y}) \right) \mod (\mathbf{y})^{r+1}\mathbb{R}[\![\mathbf{y}]\!]^p$$

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therefore

$$D_{b,v} V_\tau^{r-1}(b, \mathbf{y}) - D_{\mathbf{y},v} V_\tau(b, \mathbf{y}) \in \mathcal{R}_{r-1}(b)$$

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hence, by Chevalley's function,

$$D_{b,v} G_\tau^{l-1}(b,\mathbf{y}) - D_{\mathbf{y},v} G_\tau(b,\mathbf{y}) \in \pi_{l-1}(\mathcal{R}_{r-1}(b)) = \pi_{l-1}(\mathcal{R}_r(b))$$

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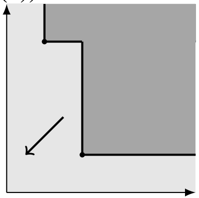
but

$$\operatorname{supp}\left(D_{b,v} G_\tau^{l-1}(b,\mathbf{y}) - D_{\mathbf{y},v} G_\tau(b,\mathbf{y})\right) \subset \Delta_\tau$$

consequently, $D_{b,v} G_\tau^{l-1}(b,\mathbf{y}) = D_{\mathbf{y},v} G_\tau(b,\mathbf{y})$.



Gluing between strata

C - gluing between strata: the Łojasiewicz inequality

Fix a stratum Λ_τ. There exists $\sigma \in \mathbb{N}$ such that if $t \geq r + \sigma$ then $\lim_{b \to \overline{\Lambda_\tau} \setminus \Lambda_\tau} V_\tau^{\beta,j}(b) = 0$.

The constant term of the equation is vanishing on B' hence on $\overline{\Lambda_\tau} \setminus \Lambda_\tau$.

Summary

We constructed $G(b,\mathbf{y}) = \sum_{|\alpha|\leq k} \frac{g_\alpha(b)}{\alpha!}\mathbf{y}^\alpha$ a semialgebraic Whitney field of order k on B such that

$$\forall b \in B,\ \forall a \in \varphi^{-1}(b),\ T_a^k f(\mathbf{x}) \equiv T_a^k A(\mathbf{x})\, G(b, T_a^k\varphi(\mathbf{x})) \mod (\mathbf{x})^{k+1}\mathbb{R}[\![\mathbf{x}]\!]^p$$

Using Kurdyka–Pawłucki–Thamrongthanyalak theorem, we obtain a C^k semialgebraic solution $g : \mathbb{R}^n \to \mathbb{R}^q$ such that $f - A\cdot(g\circ\varphi)$ is k-flat on $\varphi^{-1}(B)$. ■

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Loss of differentiability

For $k \in \mathbb{N}$, we set $l \ge k\rho$, then $r \ge r(l)$ and finally $t(k) := t \ge r + \sigma$ where

A. ρ is an upper bound of Whitney's loss of differentiability (induction step).
B. $r : \mathbb{N} \to \mathbb{N}$ is an upper bound of the Chevalley functions on the various strata.
C. σ is an upper bound of Łojasiewicz's loss of differentiability on each stratum.