Seminario de Singularidades Online

Arc spaces, motivic measure and Lipschitz geometry of real algebraic sets

joint work with T. Fukui, K. Kurdyka and A. Parusiński

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Introduction: goal

- $X = \left\{x \in \mathbb{R}^N \mid P_1(x) = \cdots = P_r(x) = 0\right\}$ a real algebraic set.
- $\mathcal{L}(X) := \left\{ \gamma : (\mathbb{R}, 0) \xrightarrow{C^\omega} \mathbb{R}^N \,\middle|\, \mathrm{Im}(\gamma) \subset X \right\}$ the space of arcs on X

Goal

Construct a *measure* $\mathcal{L}(X) \supset A \rightsquigarrow \mu(A) \in R \neq \mathbb{R}$ having some good properties, for instance:

- Measurable sets form a boolean algebra
- $\mu(A \sqcup B) = \mu(A) + \mu(B)$
- $\mathcal{L}(X)$ is measurable
- A change of variables formula

Introduction: history

- Premotivic world:
  - The Nash problem (1968, published in 1995): study the singularities of a complex algebraic varieties through its space of holomorphic arcs.
  - In the 80s and early 90s: Hironaka, Lejeune-Jalabert, Kurdyka, Hickel, …
  - 2012: Fernández de Bobadilla and Pe Pereira.
  - 2016: de Fernex and Docampo.

- Motivic avent:
  - Kontsevich 95: a first motivic measure in the non-singular case to prove Batyrev's conjecture.
  - Foundations of the motivic measure developed: Denef and Loeser, Batyrev, Looijenga, …
  - Applications to birational geometry: Ein, Mustaţă, …

- The real counterpart:
  - Koike and Parusiński 2004, Fichou 2005: motivic measure in the non-singular case to study singularities of real analytic functions.
  - C. 2016: in the singular case to study blow-Nash maps.
  - C.–Fukui–Kurdyka–Parusiński 2019: in the singular case to study inner-Lipschitz maps.

Problems $/\mathbb{R}$**:** no Nullstellensatz and no Chevalley's theorem!

Arcs & Jets: the arc space

Fix $X = \left\{x \in \mathbb{R}^N \,\middle|\, P_1(x) = \cdots = P_r(x) = 0\right\}$ a real algebraic set.

Definition – Arc space

$$\mathcal{L}(X) := \left\{ \gamma : (\mathbb{R}, 0) \xrightarrow{C^\omega} \mathbb{R}^N \,\middle|\, \mathrm{Im}(\gamma) \subset X \right\}$$

It behaves like an infinite dimensional algebraic set:

Example

Set $X = \left\{y^2 - x^3 = 0\right\} \subset \mathbb{R}^2$.

$$\begin{aligned}\mathcal{L}(X) &= \left\{\gamma(t) = (a_0 + a_1 t + a_2 t^2 + \cdots, b_0 + b_1 t + b_2 t^2 + \cdots) \in \mathbb{R}\{t\}^2 \,\middle|\, \gamma_2(t)^2 - \gamma_1(t)^3 = 0\right\} \\ &= \left\{\gamma(t) \in \mathbb{R}\{t\}^2 \,\middle|\, \left(b_0^2 - a_0^3\right) + \left(2b_0b_1 - 3a_1a_0^2\right)t + \left(b_1^2 + 2b_0b_2 - 3a_0a_1^2 - 3a_0^2a_2\right)t^2 + \cdots = 0\right\} \\ &= \left\{(a_0, a_1, \dots, b_0, b_1, \dots) \,\middle|\, b_0^2 = a_0^3,\ 3a_1a_0^2 = 2b_0b_1,\ b_1^2 + 2b_0b_2 = 3a_0a_1^2 + 3a_0^2a_2, \dots\right\}\end{aligned}$$

Arcs & Jets: the jet spaces

Fix $X = \left\{x \in \mathbb{R}^N \,\middle|\, P_1(x) = \cdots = P_r(x) = 0\right\}$ a real algebraic set.

Definition – Jet space

$$\mathcal{L}_n(X) := \left\{\gamma \in \mathcal{L}\left(\mathbb{R}^N\right) / \mathrm{mod}\, t^{n+1} \,\middle|\, \forall f \in I(X),\ f(\gamma(t)) \equiv 0 \mod t^{n+1}\right\}$$

It is an algebraic subset of $\mathbb{R}^{N(n+1)}$:

Example

Set $X = \left\{y^2 - x^3 = 0\right\} \subset \mathbb{R}^2$ then $I(X) = \left\langle y^2 - x^3 \right\rangle$.

$$\begin{aligned}
\mathcal{L}_1(X) &= \left\{\gamma \in \mathbb{R}\{t\}^2 / \mathrm{mod}\, t^2 \,\middle|\, \gamma_2(t)^2 \equiv \gamma_1(t)^3 \mod t^2\right\} \\
&= \left\{(a_0 + a_1 t, b_0 + b_1 t) \,\middle|\, (b_0 + b_1 t)^2 \equiv (a_0 + a_1 t)^3 \mod t^2\right\} \\
&= \left\{(a_0, a_1, b_0, b_1) \,\middle|\, b_0^2 = a_0^3,\ 3a_1 a_0^2 = 2b_0 b_1\right\}
\end{aligned}$$

Arcs & Jets: truncations

Definition – Truncation morphisms

They are the natural projection maps

- $\pi_n : \mathcal{L}(X) \to \mathcal{L}_n(X)$
- $\pi_n^m : \mathcal{L}_m(X) \to \mathcal{L}_n(X)$ for $m \geq n$

Proposition

TFAE

1. X is non-singular
2. $\forall n,\ \pi_n$ is surjective
3. $\forall n,\ \pi_n^{n+1}$ is surjective

Example: 3 $\Longrightarrow$ 1

Set $X = \left\{y^2 - x^3 = 0\right\} \subset \mathbb{R}^2$ then $\alpha = (0,1) \in T_{\mathbf{0}}X \setminus C_{\mathbf{0}}X$.
Assume that $\alpha t + t^2\eta(t) \in \mathcal{L}_2(X)$. Then $(t + t^2\eta_2(t))^2 - (t\eta_1(t))^3 = t^2 + \cdots$.
Hence $\alpha t \in \mathcal{L}_1(X) \setminus \pi_1^2(\mathcal{L}_2(X))$.

Beware

When X is singular: $\pi_n(\mathcal{L}(X)) \neq \mathcal{L}_n(X)$.

The Grothendieck ring of $\mathcal{AS}$-sets: $\mathcal{AS}$-sets

Definition – Arc-symmetric sets – Kurdyka 1988

We say that $S \subset M$ is arc-symmetric if for all real analytic arcs $\gamma : (-1,1) \to M$, we have

$$\operatorname{Int}(\gamma^{-1}(S)) \neq \varnothing \implies \gamma^{-1}(S) = (-1,1).$$

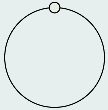
Definition – $\mathcal{AS}$-sets – Parusiński 2004

$A \subset \mathbb{P}^n_{\mathbb{R}}$ is $\mathcal{AS}$ if it is semialgebraic and for all real analytic arcs $\gamma : (-1,1) \to \mathbb{P}^n_{\mathbb{R}}$, we have

$$\gamma((-1,0)) \subset A \implies \exists \varepsilon > 0,\ \gamma((0,\varepsilon)) \subset A.$$

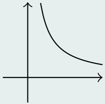
Example

$\{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\} \setminus \{(0,1)\} \in \mathcal{AS}$



Non-example

$\{(x,y) \in \mathbb{R}^2 \mid xy = 1,\ x \geq 0\} \notin \mathcal{AS}$



$\mathcal{AS}$ is the boolean algebra spanned by semialgebraic sets which are arc-symmetric at ∞.

The Grothendieck ring of $\mathcal{AS}$-sets: $\mathcal{AS}$-sets

Definition – $\mathcal{AS}$-sets – Parusiński 2004

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$$\gamma((-1,0)) \subset A \implies \exists \varepsilon > 0,\ \gamma((0,\varepsilon)) \subset A.$$

Example

Zariski-constructible sets.

The Grothendieck ring of $\mathcal{AS}$-sets: $\mathcal{AS}$-sets

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Non-example

Set $X = \left\{(x,y) \in \mathbb{R}^2 \mid f(x,y) := x^2 - zy^2 = 0\right\}$, then $\pi_2(\mathcal{L}(X)) \notin \mathcal{AS}$.

Indeed, define $\Gamma : (-1,1) \to \mathcal{L}_2(X)$ by $\Gamma(a) = \gamma_a(t) = (0, t^2, at^2)$ then

- If $a \geq 0$ then $\gamma_a(t) = \pi_2\left(\sqrt{a}t^3, t^2, at^2\right) \in \pi_2(\mathcal{L}(X))$.
- If $a < 0$ then $\gamma_a \notin \pi_2(\mathcal{L}(X))$, since $f(bt^3 + t^4\eta_1(t), t^2 + t^3\eta_2(t), at^2 + t^3\eta_3(t)) = (b^2 - a)t^6 + \cdots$.

Notice that over an algebraically closed field of characteristic zero, the spaces of truncated arcs are Zariski-constructible thanks to a theorem of Greenberg together with a theorem of Chevalley. This is one of the reasons why the original construction doesn't hold as it is over $\mathbb{R}$.

The Grothendieck ring of $\mathcal{AS}$-sets: $K_0(\mathcal{AS})$

Definition – The Grothendieck ring of $\mathcal{AS}$-sets

We define $K_0(\mathcal{AS})$ as the free abelian group spanned by $[A]$ for $A \in \mathcal{AS}$ modulo

1. If there is $A \xrightarrow{\sim} B$ a bijection with $\mathcal{AS}$ graph then $[A] = [B]$.
2. $A \subset B \implies [B \setminus A] = [B] - [A]$.

We obtain a ring using the cartesian product:

3. $[A \times B] = [A][B]$

Notation

- $0 = [\varnothing]$
- $1 = [\{*\}]$
- $\mathbb{L} := [\mathbb{R}]$
- $\mathcal{M} := K_0(\mathcal{AS})\left[\mathbb{L}^{-1}\right]$

$\mathcal{M}$ is the localization of $K_0(\mathcal{AS})$ with respect to $\left\{1, \mathbb{L}, \mathbb{L}^2, \dots\right\}$.

The Grothendieck ring of $\mathcal{AS}$-sets: the virtual Poincaré polynomial

Theorem (the virtual Poincaré polynomial) – McCrory–Parusiński 2004, 2011

There exists a unique ring morphism $\beta : K_0(\mathcal{AS}) \to \mathbb{Z}[u]$ such that for A compact non-singular

$$\beta([A]) = \sum \dim H_i(A, \mathbb{Z}_2) u^i$$

Example

$\beta(\mathbb{L}) = \beta\left(\left[\mathbb{P}^1_{\mathbb{R}} \setminus \{*\}\right]\right) = \beta\left(\left[\mathbb{P}^1_{\mathbb{R}}\right]\right) - \beta(1) = u + 1 - 1 = u$

Propositions

- For $A \neq \varnothing$, we have $\deg \beta([A]) = \dim A$ and the leading coefficient is positive.
- $\beta([A])_{|u=-1} = \chi_c(A)$.
- β is actually an isomorphism (Fichou, 2018), so $K_0(\mathcal{AS}) \simeq \mathbb{Z}[u]$ and $\mathcal{M} \simeq \mathbb{Z}\left[u, u^{-1}\right]$.

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The virtual Poincaré polynomial is one of the main reasons why $\mathcal{AS}$-sets are convenient for our purpose: because of the cell decomposition, all the additive invariants of semialgebraic sets up to semialgebraic homeomorphisms factorize through χ_c but $\chi_c(S^1) = 0$.

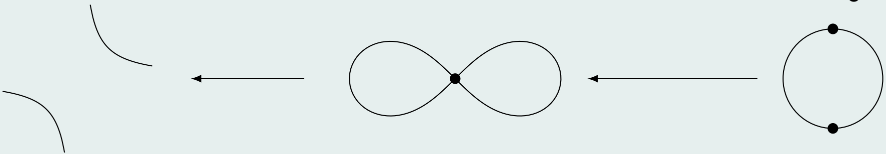
The Grothendieck ring of $\mathcal{AS}$-sets: the virtual Poincaré polynomial

Recipe to compute the virtual Poincaré polynomial:

1. Compactify
2. Resolution of singularities

Example

$X = \{xy = 1\} \subset \mathbb{R}^2$, $\dot{X} = X \sqcup \{\infty\}$, $\tilde{X} = \mathrm{Bl}_{\mathbf{0}}(\dot{X})$.

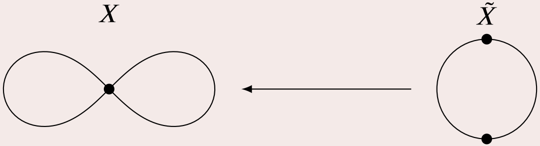


Then

- $\beta([X]) = \beta\left(\left[\dot{X}\right]\right) - 1$
- $\beta\left(\left[\dot{X}\right]\right) - 1 = \beta\left(\left[S^1\right]\right) - 2 = u - 1$
- $\beta([X]) = u - 1$

The Grothendieck ring of $\mathcal{AS}$-sets: the virtual Poincaré polynomial

Remark – The virtual Poincaré polynomial is not a topological invariant:



So $\beta([X]) - 1 = \beta\left(\left[S^1\right]\right) - 2 = u - 1$ and hence $\beta([X]) = u$.

So $\beta([Y]) - 1 = 2\beta\left(\left[S^1\right]\right) - 2 = 2u$ and hence $\beta([Y]) = 2u + 1$.

The motivic measure: stable sets

Definition – Piecewise trivial fibration

We say that $\pi : A \to B$ is a p.t.f. with fibre $F \in \mathcal{AS}$ if its graph is an $\mathcal{AS}$-set and $B = \bigsqcup_{i=1}^{r} C_i$ with $C_i \in \mathcal{AS}$ and $\pi^{-1}(C_i) \simeq C_i \times F$.

Then $[A] = [B][F]$.

Definition – Stable sets

We say that $A \subset \mathcal{L}(X)$ is stable if

- $m \gg 0 \implies \pi_m(A) \in \mathcal{AS}$
- $m \gg 0 \implies \pi_m^{-1}(\pi_m(A)) = A$
- $m \gg 0 \implies \pi_{m+1}(A) \to \pi_m(A)$ is a p.t.f. with fibre $\mathbb{R}^{\dim X}$

Definition – Measure of a stable sets

For $A \subset \mathcal{L}(X)$ a stable set, we set $\mu(A) := \left[\pi_m(A)\right] \mathbb{L}^{-(m+1)\dim X}$ for $m \gg 0$.

The motivic measure: cylinders

Definition – Cylinders

A cylinder is a subset of the form $\pi_n^{-1}(C)$ where C is an $\mathcal{AS}$-subset of $\mathcal{L}_n(X)$.

Examples

- $\mathcal{L}(X) = \pi_0^{-1}(X)$
- $\mathcal{L}(X, x) = \pi_0^{-1}(\{x\})$

Remark

When X is non-singular, the truncation maps are locally trivial for the $\mathcal{AS}$-topology (and hence are p.t.f. since it is a noetherian topology). Therefore **the measure of a cylinder is well defined**.

However, **it is no longer the case when X is singular**: $\mathcal{L}(X)$ may not even be measurable. Hence we need to enlarge the set of measurable sets.

The motivic measure: the completed Grothendieck ring $\hat{\mathcal{M}}$

Heuristic: a subset $A \subset \mathcal{L}(X)$ is measurable if it can be approximated by stable sets.

Definition (algebraic version) – $\hat{\mathcal{M}}$: completion w.r.t. the dimension

We set $\hat{\mathcal{M}} := \varprojlim \mathcal{M}/\mathcal{F}^m\mathcal{M}$ where $\mathcal{F}^m\mathcal{M} := \left\langle [S]\mathbb{L}^{-i},\ i - \dim S \geq m \right\rangle$.

It defines a ring filtration so that $\hat{\mathcal{M}}$ has a ring structure:

- $\mathcal{F}^{m+1}\mathcal{M} \subset \mathcal{F}^m\mathcal{M}$
- $\mathcal{F}^m\mathcal{M} \cdot \mathcal{F}^n\mathcal{M} \subset \mathcal{F}^{m+n}\mathcal{M}$

Definition – Virtual dimension

The virtual dimension $\dim \alpha$ of $\alpha \in \mathcal{M}$ is the unique integer m such that $\alpha \in \mathcal{F}^{-m} \setminus \mathcal{F}^{-m+1}$.

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Definition – Virtual dimension

The virtual dimension $\dim \alpha$ of $\alpha \in \mathcal{M}$ is the unique integer m such that $\alpha \in \mathcal{F}^{-m} \setminus \mathcal{F}^{-m+1}$.

Proposition

- $\dim \alpha = \deg \beta(\alpha)$
- $\hat{\mathcal{M}} = \mathbb{Z}\left[u\right]\left[\!\left[u^{-1}\right]\!\right]$

Definition (topological version) – $\hat{\mathcal{M}}$

We define $\hat{\mathcal{M}}$ as the completion of $\mathcal{M}$ for the non-archimedean norm $\|\alpha\| := e^{\deg \beta(\alpha)}$.

The motivic measure: definition

Heuristic: a subset $A \subset \mathcal{L}(X)$ is measurable if it can be approximated by stable sets.

Definition – Measurable set

$A \subset \mathcal{L}(X)$ is measurable if $\forall m \in \mathbb{Z}_{<0}$ there exist A_m, C_m stable sets such that

- $A \Delta A_m \subset C_m$
- $\dim \mu(C_m) < m$

Then the measure of A is $\mu(A) := \lim_{m \to -\infty} \mu(A_m) \in \hat{\mathcal{M}}$.

Propositions

- Measurable sets form a boolean algebra.
- $\mu(A \sqcup B) = \mu(A) + \mu(B)$
- If $\dim \mu(A_n) \to -\infty$ then $A = \cup A_n$ is measurable too and $\mu(A) = \lim_{n \to \infty} \mu\left(\cup_{k \leq n} A_k\right)$.
- Cylinders are measurable sets.

The motivic measure: in practice

Heuristic idea: everything works well for arcs away from singularities, so we take arcs outside a neighborhood of the singular locus and then we collapse this neighborhood to the singular locus.

Set $\mathcal{L}^{(e)}(X) = \left\{\gamma \in \mathcal{L}(X) \,\middle|\, \exists h \in I(X_{\mathrm{sing}}),\ h(\gamma(t)) \not\equiv 0 \mod t^{e+1}\right\}$.
Notice that $\mathcal{L}(X) = \mathcal{L}(X_{\mathrm{sing}}) \bigsqcup \cup \mathcal{L}^{(e)}(X)$ where $\mu(\mathcal{L}(X_{\mathrm{sing}})) = 0$.

Theorem

Let $A \subset \mathcal{L}(X)$ be a cylinder then

- $A \cap \mathcal{L}^{(e)}(X)$ is stable for $e \gg 0$.
- $\lim_{e \to +\infty} \dim \mu\left(\mathcal{L}^{(e)}(X) \setminus \mathcal{L}^{(e-1)}(X)\right) = -\infty$.

Hence $\mu(A) = \lim_{e \to +\infty} \mu\left(A \cap \mathcal{L}^{(e)}(X)\right)$

The motivic measure: the change of variables formula

Theorem – Change of variables formula, C. 2016

Let $\sigma : M \to X$ be a generically 1-to-1 proper Nash map where M is non-singular.
If $A \subset \mathcal{L}(X)$ is measurable then $\sigma_*^{-1}(A)$ is too and

$$\mu\left(A \cap \mathrm{Im}(\sigma_*)\right) = \sum_{e \in \mathbb{N}} \mu\left(\gamma \in \sigma_*^{-1}(A), \mathrm{ord}_t\left(\mathrm{jac}_\sigma \gamma\right) = e\right) \mathbb{L}^{-e}$$

where $\sigma_* : \mathcal{L}(M) \to \mathcal{L}(X)$ is induced by σ.

Remark: the assumptions are weaker than the original theorem over $\mathbb{C}$ since σ is not assumed to be birational.

Application 1: a blow-Nash inverse mapping theorem

Theorem

Let $f : (X, x) \to (Y, y)$ a semialgebraic local homeomorphism between two real algebraic set germs of pure dimension.
Assume that $\mu(\mathcal{L}(X, x)) = \mu(\mathcal{L}(Y, y))$, then
f is blow-Nash and $\exists c > 0, |\det(\operatorname{Jac} f)| > c \Leftrightarrow f^{-1}$ is blow-Nash and $\exists c' > 0, |\det(\operatorname{Jac} f^{-1})| > c'$.

Proof.

$$\begin{tikzcd}
 & (M, E) \arrow[dl, "\sigma"'] \arrow[dr, "\tilde{\sigma}"] & \\
(X, x) \arrow[rr, "f"'] & & (Y, y)
\end{tikzcd}$$

$$\begin{array}{ccc}
\mu(\mathcal{L}(X, x)) = \mu\left(\operatorname{Im}\sigma_*\right) = \sum \mu\left(\gamma \in \mathcal{L}(M, E)\right. & : & \left.\operatorname{ord}_t \operatorname{jac}_\sigma(\gamma) \le e\right) \mathbb{L}^{-e} \\
\parallel & & \text{\rotatebox{90}{$\ge$}} \\
\mu(\mathcal{L}(Y, y)) \ge \mu\left(\operatorname{Im}\tilde{\sigma}_*\right) = \sum \mu\left(\gamma \in \mathcal{L}(M, E)\right. & : & \left.\operatorname{ord}_t \operatorname{jac}_{\tilde{\sigma}}(\gamma) \le e\right) \mathbb{L}^{-e}
\end{array}$$

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Proof.

$$\begin{tikzcd}
& (M,E) \arrow[dl, "\sigma"'] \arrow[dr, "\tilde{\sigma}"] & & \\
(X,x) \arrow[rr, "f"'] & & (Y,y) & (\mathbb{R},0) \arrow[l, "\gamma"] \arrow[ull, bend right, "\tilde{\gamma}"']
\end{tikzcd}$$

- So $\mu\left(\mathcal{L}(Y,y)\right) = \mu\left(\operatorname{Im}\tilde{\sigma}_*\right)$.
- Fact (C. 2016): blow-Nash $\Leftrightarrow$ generically arc-analytic.

Application 2: a lipschitz inverse mapping theorem

Theorem

Let $f : (X, x) \to (Y, y)$ a semialgebraic local homeomorphism between two real algebraic set germs of pure dimension.
Assume that $\mu(\mathcal{L}(X, x)) = \mu(\mathcal{L}(Y, y))$, then
f is inner-Lipschitz and f^{-1} is blow-Nash $\Leftrightarrow$ f^{-1} is inner-Lipschitz and f is blow-Nash.

Proof.

- In the non-singular case: easy using the comatrix formula.
- General case: L-regular decomposition theorem (Parusiński, Kurdyka, Kurdyka–Orro)

∎