Motivic, logarithmic, and topological Milnor fibrations

Joint work with Goulwen Fichou and Adam Parusiński

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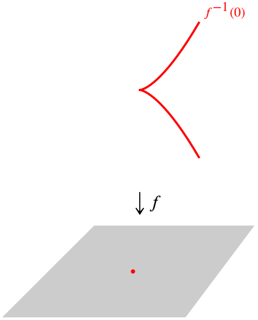
Introduction: the topological Milnor fibration

Let $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ be an analytic function germ.



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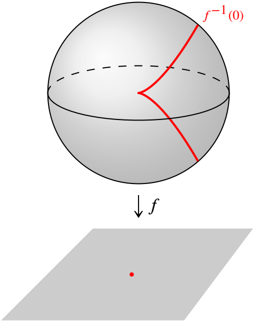

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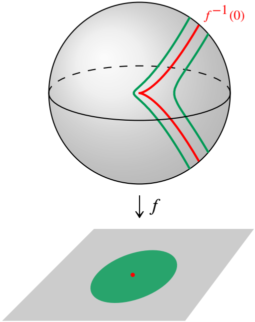

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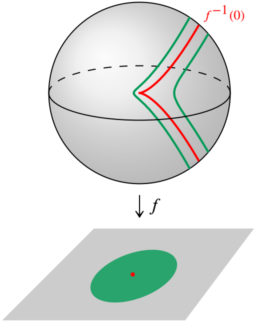

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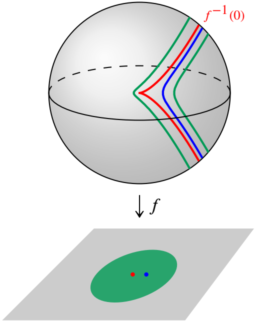

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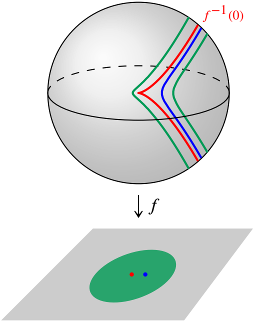

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$\mathcal{M}_f$ is not an algebraic variety.

Introduction: the motivic Milnor fibre (or fibration?)

Definition: the Grothendieck ring of algebraic varieties

We denote by $K_0\left(\mathrm{Var}_{\mathbb{C}}^{\mathbb{C}^*}\right)$ the free abelian group spanned by isomorphism classes $[f : X \to \mathbb{C}^*]$ of complex algebraic varieties over $\mathbb{C}^*$ modulo the relation

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- $0 := [\varnothing \to \mathbb{C}^*]$,
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Example

$$[\mathrm{pr} : \mathbb{P}^n \times \mathbb{C}^* \to \mathbb{C}^*] = \mathbb{L}^n + \mathbb{L}^{n-1} + \cdots + 1$$

Introduction: the motivic Milnor fibre (or fibration?)

Definition: the motivic zeta function

Given $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ a regular function, we set

$$Z_f(\mathbb{L}^{-s}) := \int_{\mathcal{L}(\mathbb{C}^{n+1},0)} (\mathrm{ac}_f, \mathbb{L}^{-\mathrm{ord}_t f \cdot s}) \in \mathcal{M}[\![\mathbb{L}^{-s}]\!] \qquad \text{where } \mathcal{M} = K_0\left(\mathrm{Var}_{\mathbb{C}}^{\mathbb{C}^*}\right)[\mathbb{L}^{-1}].$$

Definition: the motivic Milnor fibre

$$\mathcal{S}_f := -\lim_{s \to -\infty} Z_f(\mathbb{L}^{-s}) \in \mathcal{M}$$

Introduction

Question

Given a regular function $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$,
how are the topological and motivic Milnor fibres related to each other?

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Theorem (Denef–Loeser, 1998)

They share the same following numerical invariants:

- Compactly supported Euler characteristic: $\chi_c(\mathcal{M}_f) = \chi_c(\mathcal{S}_f) \in \mathbb{Z}$,
- Hodge–Deligne polynomial: $E(\mathcal{M}_f) = E(\mathcal{S}_f) \in \mathbb{Z}[u, v]$.

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Results

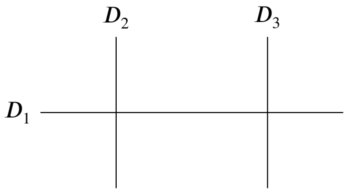
We introduce a common generalisation of the topological and motivic Milnor fibres (fibrations) for which we give two constructions:

- Using logarithmic geometry,
- Using a real oriented version of the deformation to normal cone.

Towards a common generalization: set-up

Set-up

Let $f : (M, D) \to (\mathbb{C}, 0)$ be a regular function with $D := f^{-1}(0) = \bigcup_{i \in I} D_i$ a divisor with simple normal crossings.



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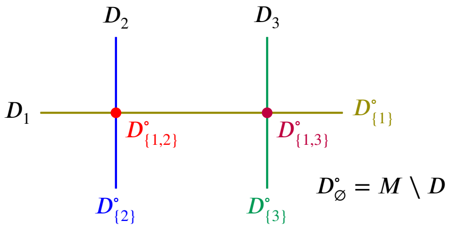
Definition: canonical stratification induced by D

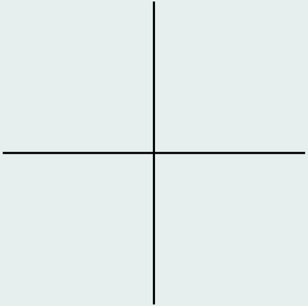
For $J \subset I$, we set $D_J^\circ := \bigcap_{j\in J} D_j \setminus \bigcup_{i\in I\setminus J} D_i$.



The logarithmic construction

Divisorial sheaf of monoids: $\mathcal{M}(U) := \left\{ f \in \mathcal{O}_M(U) \ : \ f_{|U\cap(M\setminus D)} \text{ is invertible} \right\} \supset \mathcal{O}_M^*(U)$.

Example: $D = \{z_1 z_2 = 0\} \subset \mathbb{C}^2$



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$$(M, D)^{\log} := \left\{ (x, \varphi) \;:\; x \in M,\, \varphi \in \mathrm{Hom}_{\mathrm{mon}}(\mathcal{M}_x, S^1),\, \forall g \in \mathcal{O}_x^*,\, \varphi(g) = \frac{g(x)}{|g(x)|} \right\}$$

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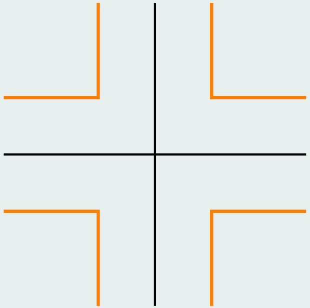

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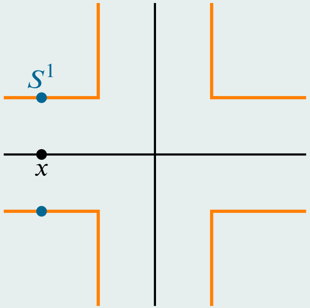

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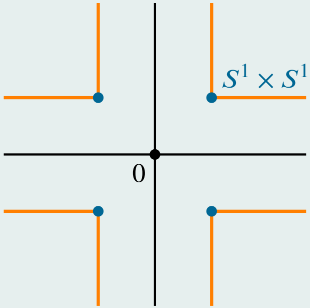

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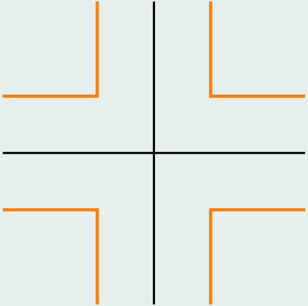

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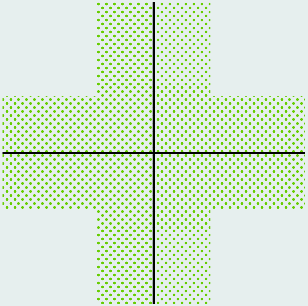

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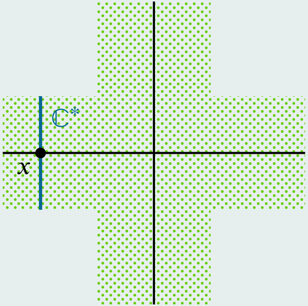

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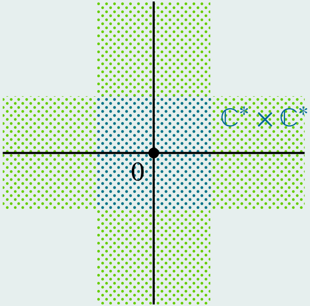

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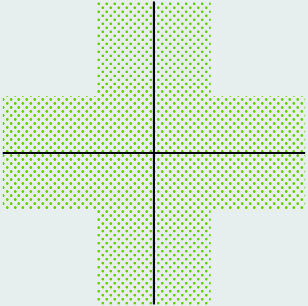

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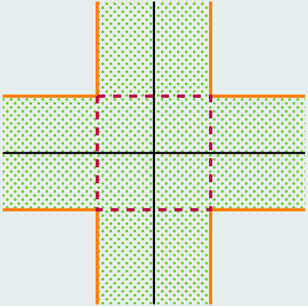

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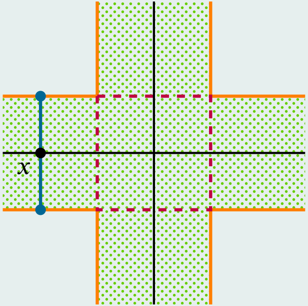

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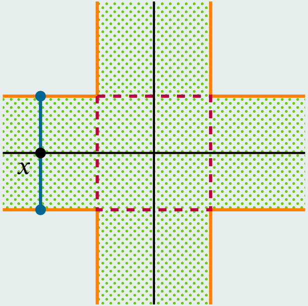

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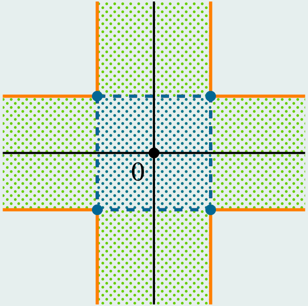

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The logarithmic construction

Divisorial sheaf of monoids: $\mathcal{M}(U) := \left\{ f \in \mathcal{O}_M(U) \ : \ f_{|U\cap(M\setminus D)} \text{ is invertible} \right\} \supset \mathcal{O}_M^*(U)$.

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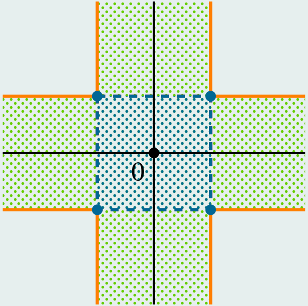
Example: $D = \{z_1 z_2 = 0\} \subset \mathbb{C}^2$



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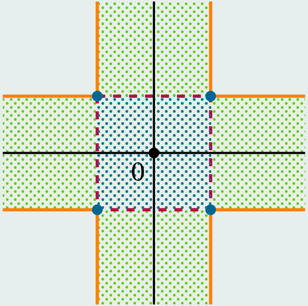

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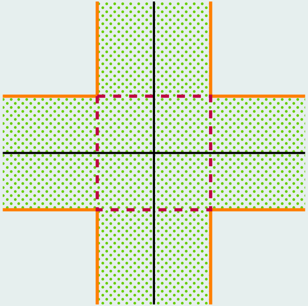

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log: $S^1 \simeq \{+\infty\} \times S^1$

alog: $\mathbb{C}^* \simeq (0,+\infty) \times S^1$

clog: $(0,+\infty] \times S^1$

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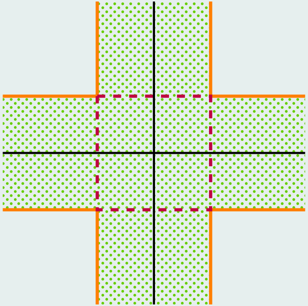

$(M,D)^{\mathrm{alog}} \subset (M,D)^{\mathrm{clog}}$ given by $\mathbb{C}^* \simeq (0,+\infty) \times S^1$

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$$\begin{tikzcd}
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Recovering the Milnor fibrations

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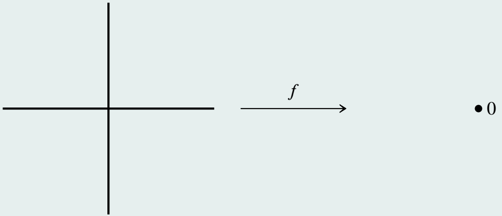

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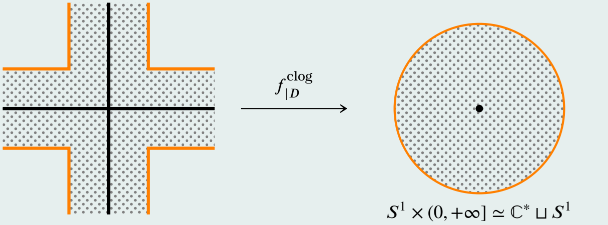

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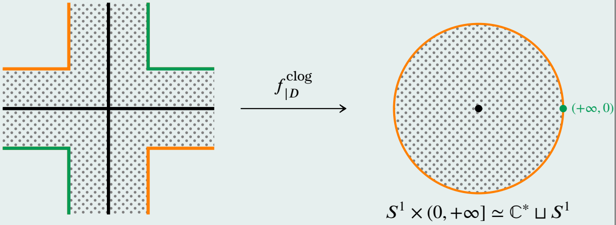

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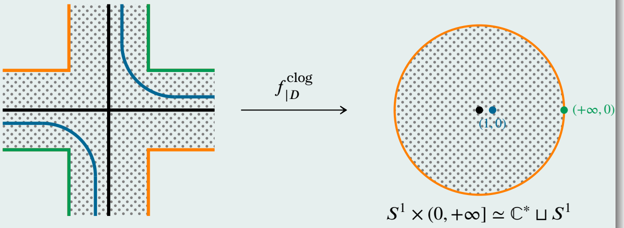

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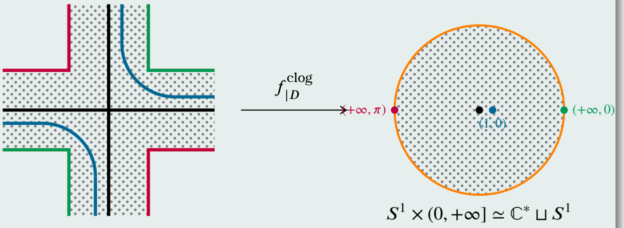

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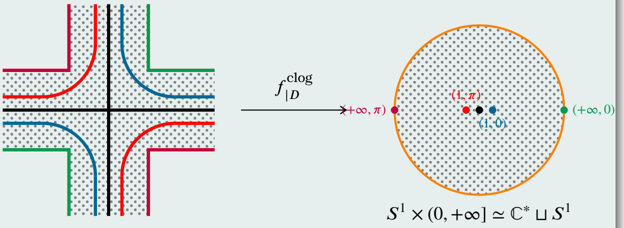

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Theorem

$f^{\mathrm{log}}_{|D} : (M, D)^{\mathrm{log}}_{|D} \to S^1$ coincides with A'Campo's model for the topological Milnor fibration.

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$$S_f = - \sum_{\varnothing \neq J \subset I} (-1)^{|J|} \left[f^{\mathrm{alog}}_{|D_J^\circ} : (M, D)^{\mathrm{alog}}_{D_J^\circ} \to \mathbb{C}^* \right]$$

Applications

Theorem: the motivic Milnor fibration determines the topological Milnor fibration

We obtain $f^{\log}_{|D} : (M, D)^{\log}_{|D} \to S^1$ by dividing $f^{\mathrm{alog}}_{|D} : (M, D)^{\mathrm{alog}}_{|D} \to \mathbb{C}^*$ over each stratum D°_J by $(\mathbb{R}_{>0})^{|J|}$ in the domain and by $\mathbb{R}_{>0}$ in the codomain.

$$\begin{tikzcd}
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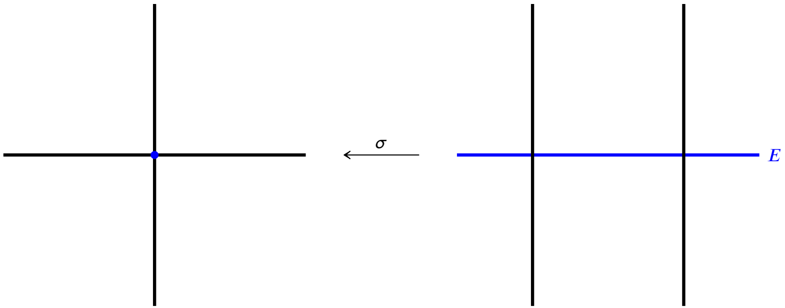
The motivic Milnor fibre S_f is determined by the stratified topological Milnor fibration $f^{\log}_{|D} : (M,D)^{\log}_{|D} \to S^1$.

Use that $f^{\log}_{|D_J^\circ}(x,\theta_i) = \arg(u(x)) \prod_{i \in J} \theta_i^{N_i}$ to recover the $N_i' s$ and $u(x)$ up to a positive constant.

Then $\left[f^{\mathrm{alog}}_{|D_J^\circ} : (M,D)^{\mathrm{alog}}_{D_J^\circ} \to \mathbb{C}^* \right] \in K_0\left(\mathrm{Var}_{\mathbb{C}}^{\mathbb{C}^*}\right)$ is entirely determined.

Effect of a blowing-up on $(M, D)^{\mathrm{clog}}$

Let $D = \{z_1 z_2 = 0\} \subset \mathbb{C}^2$ and $\sigma : (\tilde{M}, \tilde{D}) \to (\mathbb{C}^2, 0)$ be the blowing-up at 0.



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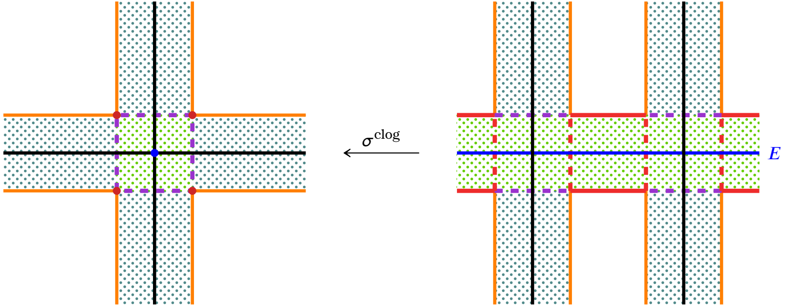

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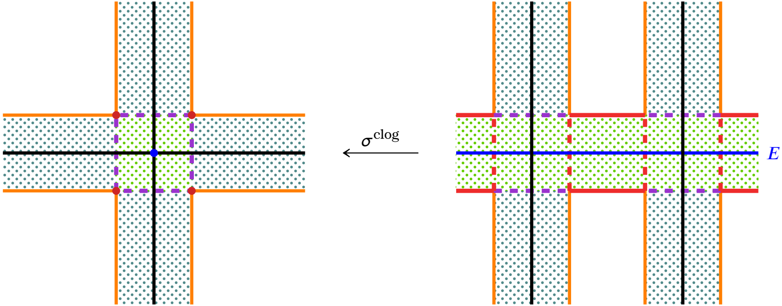

At the alog-level: $(\mathbb{C}^2, 0)_0^{\mathrm{alog}} = \mathbb{C}^* \times \mathbb{C}^*$

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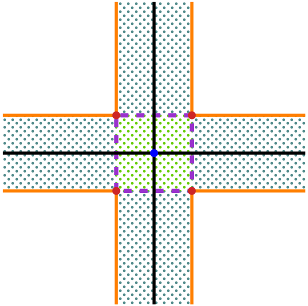

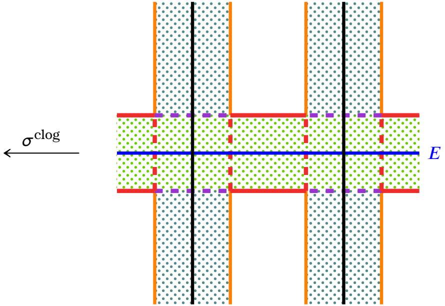

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Therefore $\mathcal{S}_f$ stays unchanged.

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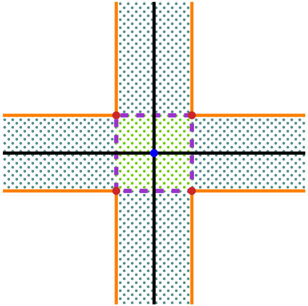

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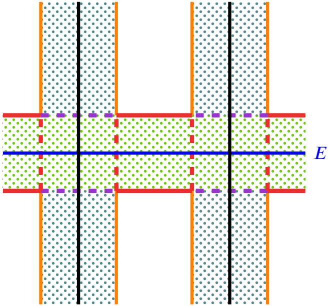

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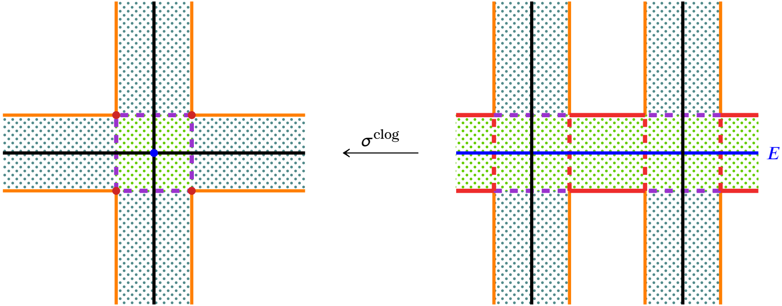

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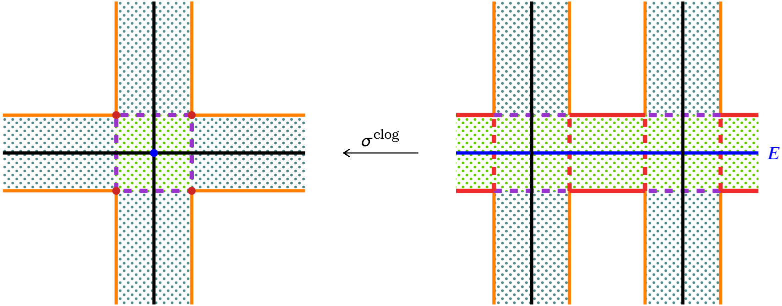

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Above 0, $\sigma^{\log}$ gives:

$$S^1 \times S^1 \leftarrow (S^1 \times S^1) \sqcup (S^1 \times S^1) \sqcup (\mathbb{C}^* \times S^1) = (S^1 \times S^1) \times [0, +\infty]$$

Hence the fiber is contractible and $\sigma^{\log} : (\tilde{M}, \tilde{D})_{|\tilde{D}}^{\log} \to (\mathbb{C}^2, 0)_{|0}^{\log}$ is a homotopy equivalence.

The local case

Let $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ be regular.
Let $\rho : M \to \mathbb{C}^{n+1}$ be a finite sequence of blowings-up with smooth algebraic centres such that $D := \tilde{f}^{-1}(0) = \bigcup_{i \in I} D_i$ is a divisor with simple normal crossings where $\tilde{f} = f \circ \rho$.

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The motive

$$S_f := - \sum_{\emptyset \neq J \subset I} (-1)^{|J|} \left[\tilde{f}^{\mathrm{alog}}_{|D_J^\circ} : (M, D)^{\mathrm{alog}}_{D_J^\circ} \to \mathbb{C}^* \right] \in K_0 \left(\mathrm{Var}_{\mathbb{C}}^{\mathbb{C}^*} \right)$$

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Theorem

The fibration $\tilde{f}^{\log} : (M, D)^{\log}_{|D} \to S^1$ is homotopic to the Milnor fibration of f at 0.

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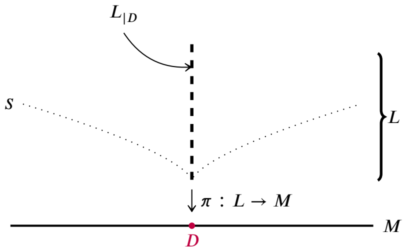
1. Case of a single smooth hypersurface $D \subset M$.
   Fix $\pi : L \to M$ a line bundle together with a section $s : M \to L$ such that $D = s^{-1}(0)$.



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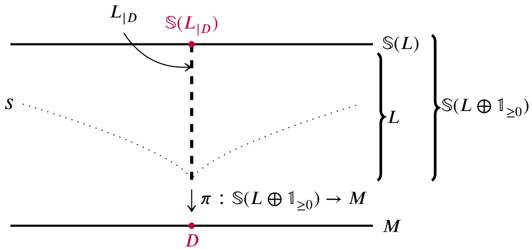

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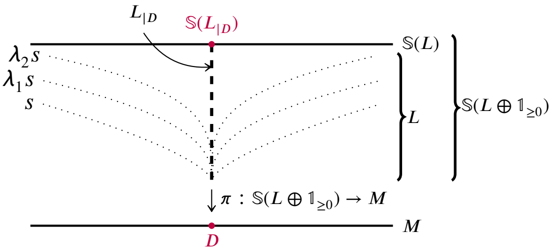

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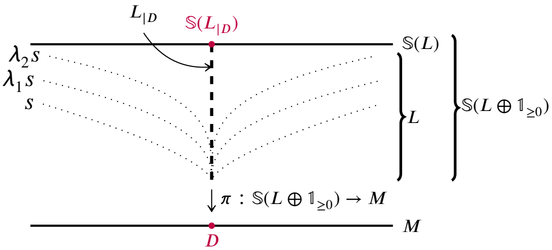

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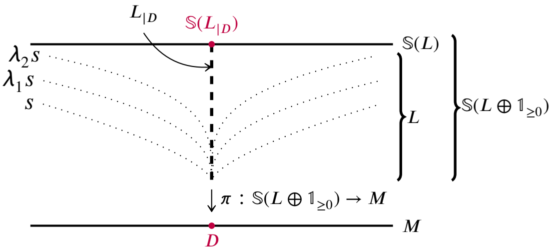

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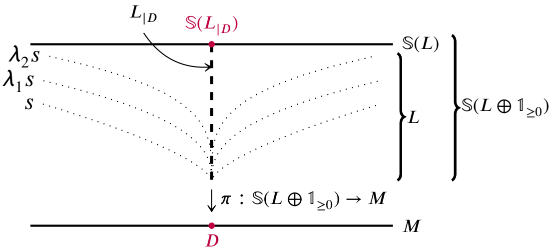

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2. General case when $D = \bigcup_{i \in I} D_i$:

$$(M, D)^{\mathrm{clog}} = \prod_{i \in I}{}_M (M, D_i)^{\mathrm{clog}}$$