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Introduction to shifted symplectic structures

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Introduction

Generalities

The purpose of *derived geometry* is to deal with singular spaces, such as most of the moduli spaces appearing in classical physics (spaces of solutions of equations of motion - as opposed to quantum physics). Foundations of derived algebraic geometry have been laid out by Toën and Vezzosi [5] on the one hand, and by Lurie [3] on the other hand. It combines two ways of introducing more flexibility in geometry:

(A) adding a homotopical flavour, which consists in replacing rings in algebraic geometry with simplicial ones or differential graded algebras - this addresses the general problem of bad intersections or more generally fiber products;

(B) allowing gluing up to identifications, leading to the notion of stacks, very effective to deal with bad quotients, that we will not mention in these notes.

This approach can be motivated by various mathematical problems such as Bézout's theorem, or the study of solution spaces to elliptic partial differential equations.

Symplectic geometry is a natural geometric setting for the hamiltonian formulation of classical mechanics : most phase spaces indeed appear to be symplectic manifolds (or variations of these, like Poisson manifolds). Recall that a symplectic manifold is a smooth manifold X equipped with a 2-form ω that is non-degenerate (meaning that the induced bundle map $T_X \to T_X^*$ is an isomorphism) and closed (meaning that $d_{\mathrm{dR}}\omega = 0$). This definition makes sense in the algebro-geometric context only if X is a smooth algebraic variety. The cotangent bundle T_M^* of a manifold M is an example of a symplectic manifold, with $\omega_{\mathrm{can}} = d_{\mathrm{dR}}\lambda$, where λ is the tautological 1-form on T_M^*. Symplectic manifolds do not have local invariants: it follows from a theorem of Darboux that every symplectic manifold is locally symplectomorphic to $T^*\mathbb{R}^n$ equipped with ω_{can}.

Lagrangian submanifolds play a crucial role in symplectic geometry: recall that a lagrangian submanifold $L \subset X$ in a symplectic manifold (X,ω) is lagrangian if $\omega|_L = 0$ and the induced map $T_L \to T_{L/X}^*$

is an isomorphism. Generalizing Darboux's theorem, Weinstein proved that in the neighborhood of a lagrangian submanifold L every symplectic manifold is symplectomorphic to a neighborhood of the zero section of T^*_L. Thus lagrangian submanifolds can naturally be interpreted as generalized configurations of a classical mechanical system. These submanifolds pop up everywhere: graphs of closed 1-forms, graphs of symplectomorphisms (an example of which is the time $t = 1$ flow of a hamiltonian vector field), conormal bundles, zero loci of moment maps, etc. This led Weinstein to follow the symplectic creed claiming that "everything is a lagrangian submanifold", and envision a symplectic category whose objects are symplectic manifolds and morphisms are lagrangian correspondences, i.e. lagrangian submanifolds of a product $(X_1 \times X_2, p_1^*\omega_1 - p_2^*\omega_2)$. At this point, the need to deal with singular/pathological spaces in symplectic geometry should be obvious:

(i) the zero locus of a moment map μ might be singular;

(ii) the above example is actually a lagrangian correspondence between the original symplectic manifold X and its symplectic reduction X_{red}, where X_{red} is a quotient of $\mu^{-1}(0)$ and thus could be even more singular;

(iii) the composition in the symplectic category involves taking fiber products, that might not be well-behaved.

A traditional way to deal with that is by applying a small geometric perturbation, but there are issues with this approach: it cannot always be used in algebraic geometry, and geometric perturbations are not functorial.

A leitmotiv of derived geometry is to replace geometric perturbations with homological perturbations in order to compute fiber products. Homological perturbations can be made functorial (in a higher categorical sense), and make sense in the algebro-geometric context, resolving both issues.

The ancestors of *shifted symplectic structures* on derived stacks are the odd symplectic structures on super-manifolds and Q-manifolds appearing for instance in mathematical physics publications on the geometry of the Batalin–Vilkovisky (BV) formalism. These excellent works along with the beautiful treatment by Costello, using L_∞-spaces and elliptic moduli problems, have two drawbacks: none of the two defining properties of a symplectic structure (closedness and non-degeneracy) are homotopy invariant; and the geometric objects they consider only capture infinitesimal symmetries. Both issues are addressed by Pantev, Toën, Vaquié and Vezzosi [4] via the formalism of derived geometry: it indeed encompasses global symmetries (stacks have been invented for that purpose) and it is homotopy invariant by definition.

About these notes

Here we give an extended version with Exercises of a three days Masterclass given in Angers (LAREMA – Université Angers) in December 2025. They are written with the aim of being accessible for M2 students. In particular the necessary material in homological algebra is given in Section 1, as well as a few notions of Lie theory in Section 2 which are used in Examples. The third Section deals with the algebraic side which gives the material to sketch affine derived geometry, as done in the last Section. Its purpose is to give a couple of Examples of derived critical loci, which are fundamental both in traditional symplectic geometry, and contemporary geometric representation theory. To that end we insist in the third Section on lagrangian *correspondences*, a very convenient way to consider lagrangian structures, in order to use higher categories which are not presented here.

Acknowledgement

These notes are widely inspired by those of Damien Calaque [2].

In this document, $\Bbbk$ denotes a field of characteristic 0.

1 Crash course in homological algebra

1.1 Complexes

▷ A *cochain complex* of $\Bbbk$-vector spaces $(V, d) = (V^\bullet, d_\bullet)$ is the data of a $\Bbbk$-vector space V^n for every $n \in \mathbb{Z}$, along with *differentials* $d_n : V^n \to V^{n+1}$ which are required to be linear and to satisfy $d_{n+1} \circ d_n = 0$

for every $n \in \mathbb{Z}$. We will sometimes drop the index on differentials. We will indiscriminately use the letter d for all complexes, that we will hence simply call V, W, etc. We denote by $|v| = n$ the *degree* of an element $v \in V^n$.

Remark 1.1.

- When differentials go "downward" $V_n \to V_{n-1}$, we speak of *chain* complexes. In this lecture we will stick to cochains.
- We might restrict ourselves in examples to the categories of modules over some algebra, representations of a Lie group, or vector bundles over a given variety.

▷ The n-th *cohomology group* $H^n = \ker(d_n)/\mathrm{im}(d_{n-1})$ is the quotient of the space $\ker(d_n)$ of *cocycles* or *closed elements* by the space $\mathrm{im}(d_{n-1})$ of *coboundaries* or *exact elements*. The category of vector spaces being abelian, these groups are actually vector spaces (false in general for vector bundles).

Example 1.2 (de Rham complex)**.** Let M be a smooth manifold. Define the space $\Omega^k(M) = \Gamma(\bigwedge^k T^*M)$ of *differential forms* of degree k as the space of sections of the k-th exterior product of the cotangent bundle of M. Note that $\Omega^0(M)$ is just the space $\mathscr{O}(M)$ of smooth functions $M \to \Bbbk$, and that $\Omega^1(M)$ is dual to the space $\mathfrak{X}(M)$ of vector fields. There exists a structure of cochain complex $(\Omega^\bullet(M), d_\bullet)$ such that d_0 is the standard differential $d : \mathscr{O}(M) \to \Omega^1(M)$, $f \mapsto df$, and satisfying the Leibniz rule

$$d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^k \alpha \wedge d\beta \in \Omega^{k+\ell+1}(M)$$

for any pair of forms $(\alpha, \beta) \in \Omega^k(M) \times \Omega^\ell(M)$. The cohomology groups of this complex are called the *de Rham cohomology groups.* ♢

▷ Complexes can be *shifted*: for a given one V we define $V[n]$ by $V[n]^k = V^{n+k}$ with differentials multiplied by $(-1)^n$. Note that if V is concentrated in degree 0, then $V[1]$ is concentrated in degree -1.

▷ There is an *internal hom* given by $\mathrm{Hom}^n(V, W) = \prod_{k \in \mathbb{Z}} \mathrm{Hom}_\Bbbk(V^k, W^{k+n})$ with differential given by $(df)(v) = d(f(v)) - (-1)^n f(dv)$. Denote by $\Bbbk$ the one dimensional complex concentrated in degree 0. Then $\mathrm{Hom}^n(V, \Bbbk) = (V^{-n})^*$ with differential $(df)(v) = (-1)^{n+1} f(dv)$. We set $V^* = \mathrm{Hom}^\bullet(V, \Bbbk)$ and we see that $H^n(V^*) = H^{-n}(V)^*$.

▷ There is also a *tensor product* given by $(V \otimes W)^n = \bigoplus_{k+\ell=n} V^k \otimes_\Bbbk W^\ell$ with differential $d(v \otimes w) = dv \otimes w + (-1)^{|v|} v \otimes dw$. The second symmetric power $\mathrm{Sym}^2(V)$ is the quotient of $V^{\otimes 2}$ by relations $v \otimes v' = (-1)^{|v|.|v'|} v' \otimes v$. The second exterior power is then defined as $\bigwedge^2 V = \mathrm{Sym}^2(V[-1])[2]$, with differential given by $d(v \wedge v') = dv \wedge v' + (-1)^{|v|+1} v \wedge dv'$.

▷ A cochain map $f_\bullet : V^\bullet \to W^\bullet$ is a collection of linear maps $f_n : V^n \to W^n$ such that $f_{n+1} d_n = d_n f_n$ for all $n \in \mathbb{Z}$. Such a map induces linear maps $H^n(f) : H^n(V) \to H^n(W)$ on cohomology groups, and we call $f_\bullet$ a *quasi-isomorphism* if all $H^n(f)$ are isomorphisms, and we write $V \simeq W$ (for usual isomorphisms of vector spaces we write $\cong$).

Exercise 1. Consider $n \in \mathbb{Z}$. Then any cochain map $\omega : \bigwedge^2 V \to \Bbbk[n]$ induces a cochain map

$$\omega^\flat : V \to V^*[n] \quad ; \quad v \mapsto \omega(v, -).$$

1.2 Homotopies

▷ The *cocone* or *homotopy fiber* of a cochain map $\varphi : V \to W$ is the cochain complex $\mathrm{hofib}(\varphi) = V \oplus W[-1]$ with differential

$$\delta = \begin{pmatrix} d_V & 0 \\ \varphi & -d_W \end{pmatrix}.$$

It fits in a short exact sequence $0 \to W[-1] \to C := \mathrm{hofib}(\varphi) \to V \to 0$ that leads to a long one in cohomology

$$\cdots \to H^{n-1}(W) \to H^n(C) \to H^n(V) \to H^n(W) \to H^{n+1}(C) \to \cdots$$

▷ A *homotopy* h between cochain maps $f, g : V \to W$ is a collection of linear maps $h_n : V^n \to W^{n-1}$ such that for all $n \in \mathbb{Z}$ we have $f_n - g_n = h_{n+1} d_n + d_{n-1} h_n$. We write $h : f \sim g$, and it implies $H^\bullet(f) = H^\bullet(g)$. If h_1 and h_2 are homotopies $f_1 \sim f_2$ and $f_2 \sim f_3$, then $h_1 + h_2 : f_1 \sim f_3$.

Exercise 2. Consider a *null-homotopy* $h : 0 \sim f : V \to W$.

(i) Prove that for any cochain map $p : U \to V$, we have $hp : fp \sim 0$.

(ii) Prove that
$$\tilde{h} : \textstyle\bigwedge^2 V \to \left(\bigwedge^2 W\right)[-1] \quad ; \quad (x, y) \mapsto \frac{1}{2}\left(hx \wedge fy + (-1)^{|x|+1} fx \wedge hy\right)$$
is a null-homotopy $f^{\wedge 2} := \bigwedge^2 f \sim 0$.

(iii) Consider a cochain map $\omega : \bigwedge^2 W \to \Bbbk[d]$ for a given d. Prove that $\omega\tilde{h} : \omega f^{\wedge 2} \sim 0$.

Exercise 3 (universality)**.** Consider cochain maps $U \xrightarrow{\psi} V \xrightarrow{\varphi} W$ and a degree preserving linear map $\eta : U \to W[-1]$. Prove that $(\psi, \eta) : U \to \mathrm{hofib}(\varphi)$ defines a cochain map if and only if η is a null-homotopy $\varphi\psi \sim 0$.

We say that a square

$$\begin{array}{ccc} A & \xrightarrow{p} & B \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle b} \\ C & \xrightarrow{c} & D \end{array}$$

- is a *homotopy pullback* if $A \simeq \mathrm{hofib}\left(B \oplus C \xrightarrow{b-c} D\right) =: F$;
- *homotopy commutes* if $bp \sim cq$, in which case there is a unique $a : A \to F$ such that $p_B a = p$ and $p_C a = q$ where $p_B : F \to B$ and $p_C : F \to C$ are the obvious projections.

The following is very standard.

Lemma 1.3 (pullback lemma)**.** *If both squares in*

$$\begin{array}{ccccc} A & \longrightarrow & B & \longrightarrow & C \\ \downarrow & & \downarrow & & \downarrow \\ X & \longrightarrow & Y & \longrightarrow & Z \end{array}$$

are homotopy pullbacks, so is the outer rectangle. If the rectangle and the right square are homotopy pullbacks, so is the left square.

Proposition 1.4 (stability)**.** *Consider the above data, and assume that* $U \simeq \mathrm{hofib}(\varphi)$*. Then* $W[-1] \simeq \mathrm{hofib}(\psi)$.

Proof. The cochain map

$$(\psi, \eta, \mathrm{id}) : X := \mathrm{hofib}(\psi) \to \mathrm{hofib}\left(\mathrm{hofib}(\varphi) \to V\right) =: Y.$$

induces a morphism of long exact sequences

$$\begin{array}{ccccccccccccc} \cdots & \longrightarrow & H^{n-1}(U) & \longrightarrow & H^{n-1}(V) & \longrightarrow & H^n(X) & \longrightarrow & H^n(U) & \longrightarrow & H^n(V) & \longrightarrow & \cdots \\ & & \| & & \| & & \downarrow & & \| & & \| & & \\ \cdots & \longrightarrow & H^{n-1}(U) & \longrightarrow & H^{n-1}(V) & \longrightarrow & H^n(Y) & \longrightarrow & H^n(U) & \longrightarrow & H^{n+1}(V) & \longrightarrow & \cdots \end{array}$$

hence $X \simeq Y$. Now define a cochain map $W[-1] \to Y$ by $w \mapsto (0, w, 0)$. We have a short exact sequence

$$0 \to W[-1] \to Y \to \mathrm{hofib}(\mathrm{id}_V) \to 0$$

where $H^n(\mathrm{hofib}(\mathrm{id}_V)) = 0$ hence $W[-1] \simeq Y$. □

2 A dash of Lie theory

Fix G a Lie group.

⊳ For $g \in G$, we set $i_g : G \to G$, $h \mapsto ghg^{-1}$, which induces $\mathrm{Ad}_g := d_e i_g : T_eG \to T_eG$ and in turn $\mathrm{ad} := d_e\mathrm{Ad} : T_eG \to \mathrm{End}(T_eG)$. Then $\mathfrak{g} = T_eG$ is a Lie algebra once endowed with the bracket $[x, y] = \mathrm{ad}(x)(y)$.

⊳ More generally, if M is a G-space, then any $p \in M$ induces a map $\sigma_p : G \to M$, with differential $d_e\sigma_p : \mathfrak{g} \to T_pM$. For any $x \in \mathfrak{g}$, we denote by $\vec{x} \in \mathfrak{X}(M)$ the vector field defined by $p \mapsto d_e\sigma_p(x)$. Recall that $\mathfrak{X}(M)$ can be endowed with a Lie algebra structure. The morphism $\mathfrak{g} \to \mathfrak{X}(M)$ is called the *infinitesimal action* map.

⊳ When $M = G$, any $g \in G$ induces multiplications $L_g, R_g : G \to G$ given by $L_g(h) = gh$ and $R_g(h) = hg$. For $x \in \mathfrak{g}$, we denote by $x^L : g \mapsto d_eR_g(x)$ and $x^R : g \mapsto d_eL_g(x)$ the associated vector fields. The differentials $d_gL_{g^{-1}}$ and $d_gR_{g^{-1}} : T_gG \to \mathfrak{g}$ define $\mathfrak{g}$-valued 1-forms that can be viewed in $\mathfrak{g} \otimes \Omega^1(G)$, and called left and right *Maurer–Cartan* forms. A vector field $X : G \to TG$ on G and a $\mathfrak{g}$-valued 1-form $\theta : TG \to \mathfrak{g}$ yield a $\mathfrak{g}$-valued function $\theta(X) : G \to \mathfrak{g}$, that is $\theta(X) \in \mathcal{O}(G) \otimes \mathfrak{g}$. Via the trivialization $TG \cong G \times \mathfrak{g}$ given by $d_gR_{g^{-1}}$, we have $x^L(g) = (g, x)$ and $x^R(g) = (g, \mathrm{Ad}_g(x)))$.

⊳ Consider P a G-principal bundle on a manifold M. The *adjoint bundle* $\mathfrak{g}_P$ of P over M is the quotient of $P \times \mathfrak{g}$ by the action of G given by $(p, x).g = (g.p, \mathrm{Ad}_{g^{-1}}(x))$.

3 The algebraic side

3.1 Linear shifted symplectic structures

Definition 3.1. An n-shifted symplectic structure on a cochain complex V is a cochain map $\omega : \bigwedge^2 V \to \Bbbk[n]$ such that $\omega^\flat$ is a quasi-isomorphism.

Example 3.2 (1-term complex)**.** Consider $V = W[-d]$, where W is a non trivial vector space (a complex concentrated in degree 0). Then V^* is concentrated in degree $-d$. By the non-degeneracy condition, V only admits n-shifted symplectic structures for $n = -2d$.

- If d is odd, then $\bigwedge^2 V = \mathrm{Sym}^2(W)[-2d]$ and an n-shifted symplectic structure on V is a scalar product on W.
- If d is even, then $\bigwedge^2 V = \bigwedge^2 W[-2d]$, and an n-shifted symplectic structure on V is a standard symplectic structure on W.

For a sub-example, consider a Lie group G. Any non-degenerate Ad-invariant scalar product on $\mathfrak{g}$ defines a 2-shifted symplectic structure on $\mathfrak{g}[1]$ as a complex of G-representations. ♢

Example 3.3 (de Rham continued)**.** Consider X an n-dimensional closed oriented manifold and $V = (\Omega^\bullet(X), d_{\mathrm{dR}})[1]$. Set $\omega(\alpha, \beta) = \int_X \alpha \wedge \beta$ in degree $n-2$ for any pair of differential forms. Here we need to be careful and make a difference between the formal $\bigwedge$ and the $\wedge$ of differential forms. We need to check that ω defines a chain map. Take two forms α, β such that $|\alpha| + |\beta| = n-1$. Then $\alpha \bigwedge \beta$ defines an element in $(\bigwedge^2 V)^{n-3}$ and we need to prove that $\omega(d(\alpha \bigwedge \beta)) = 0$ where d is the differential in $\bigwedge^2 V$, so that $d(\alpha \bigwedge \beta) = d_{\mathrm{dR}}\alpha \bigwedge \beta + (-1)^{|\alpha|}\alpha \bigwedge d_{\mathrm{dR}}\beta$ since $\alpha \in V^{|\alpha|-1}$. Therefore, thanks to the Leibniz rule, $\omega(d(\alpha \bigwedge \beta)) = \int_X d_{\mathrm{dR}}(\alpha \wedge \beta) = 0$. We then have

$$\omega^\flat : \Omega^\bullet(X)[1] \to \left(\Omega^\bullet(X)[1]\right)^* [2-n] = \Omega^\bullet(X)^* [1-n]$$

such that $H^{k-1}(\omega^\flat) : H^k(\Omega^\bullet(X)) \to H^{k-n}(\Omega^\bullet(X)^*) = H^{n-k}(\Omega^\bullet(X))^*$ is an isomorphism thanks to Poincaré duality. Therefore V is $(2-n)$-shifted symplectic. ♢

Example 3.4 (connections)**.** Consider X an n-dimensional closed oriented manifold and G a Lie group with Lie algebra $\mathfrak{g}$. For any vector bundle $E \to X$ and any $n \geq 0$, consider the space $\Omega^n(E) = \Gamma(\bigwedge^n T^*X \otimes E)$ of E-valued forms of degree n. A connection ∇ is a map $\Omega^0(E) = \Gamma(E) \to \Omega^1(E)$ satisfying

$$\nabla(fs) = df \otimes s + f\nabla s$$

for all $f \in \mathcal{O}(X)$ and $s \in \Gamma(E)$. A connection can be consistently extended to a degree one endomorphism of $\Omega^\bullet(E)$ by the formula

$$\nabla(\alpha \otimes s) = d_{\mathrm{dR}}(\alpha) \otimes s + (-1)^{|\alpha|}\alpha \wedge \nabla s.$$

We won't recall here the definition of the curvature of a connection, but it vanishes if and only if $\nabla^2 = 0$, *ie* if it is a differential. The connection is then said to be flat.

Consider (P, ∇) a principal G-bundle with a flat connection ∇, so that we get a cochain complex of $\mathfrak{g}_P$-valued forms

$$V = \big(\Omega^\bullet(\mathfrak{g}_P), \nabla\big)[1].$$

For every G-invariant symmetric non-degenerate pairing $\langle -, - \rangle$ on $\mathfrak{g}$, we have a $(2-n)$-shifted symplectic structure on V given by

$$\omega(\alpha, \beta) = \int_X \langle \alpha, \beta \rangle,$$

where the pairing is extended to $\mathfrak{g}_P$-valued forms. ◊

Exercise 4 (the case of 2-term complexes)**.** Let $V = \big(E \xrightarrow{a} F\big)$ be a n-shifted symplectic cochain complex concentrated in degrees d and $d+1$. Assume that V is not acyclic (that is a is not an isomorphism), since otherwise $V \simeq 0$, which is trivially n-shifted symplectic for every n (the n-shifted symplectic structure being zero).

(i) Prove that the case n even boils down to the 1-term complex case.

(ii) Assume from now on that n is odd, that is $n = -1 - 2d$. Prove that a cochain map $\omega : \bigwedge^2 V \to \Bbbk[n]$ is determined by a linear map $\omega_L : E \otimes F \to \Bbbk$ satisfying

$$\omega_L(e_1 \otimes a(e_2)) = (-1)^d \omega_L(e_2 \otimes a(e_1)) \tag{1}$$

for all $e_1, e_2 \in E$.

(iii) Define the linear map $\alpha : E \to F^*$ by $\alpha(e) = \omega_L(e \otimes -)$. Prove that $\omega^b = (\alpha, \alpha^*)$.

(iv) The non-degeneracy condition amounts to requiring that α and α^* induce linear isomorphisms $\ker(a) \cong \ker(a^*)$ and $\mathrm{coker}(a) \cong \mathrm{coker}(a^*)$, which are equivalent conditions. When E, F are finite dimensional, prove that the above is equivalent to

$$\ker(a) \cap \ker(\alpha) = \{0\} \text{ and } \dim E = \dim F. \tag{2}$$

Example 3.5 (2-term complex in Lie theory - I)**.** Let G be a Lie group with Lie algebra $\mathfrak{g}$. Consider the adjoint action on $\mathfrak{g}^*$ and the associated infinitesimal action $\mathfrak{g} \to \mathfrak{X}(\mathfrak{g}^*)$. Denote by A the ring of functions $\mathcal{O}(\mathfrak{g}^*)$, and fix a basis (e_i) of $\mathfrak{g}$. We have $\mathfrak{X}(\mathfrak{g}^*) \cong A \otimes \mathfrak{g}^*$ via

$$\mathfrak{X}(\mathfrak{g}^*) \ni \left(\varphi \in \mathfrak{g}^* \mapsto \sum_i f_i(\varphi) e_i^*\right) \mapsto \left(\sum_i f_i(\varphi) \otimes e_i^*\right) \in A \otimes \mathfrak{g}^*.$$

Through this identification we decompose the infinitesimal action in

$$\vec{e}_j = \sum_i f_{ji} \otimes e_i^* : \varphi \mapsto \varphi([e_j, -])$$

so that $f_{ji}(\varphi) = \varphi([e_j, e_i]) = -f_{ij}(\varphi)$. We view the map

$$A \otimes \mathfrak{g} \xrightarrow{a} A \otimes \mathfrak{g}^* \quad ; \quad f \otimes x \mapsto f\vec{x}$$

as a 2-term complex of A-modules in degrees -1 and 0, and we set

$$\omega_L : (A \otimes \mathfrak{g}) \underset{A}{\otimes} (A \otimes \mathfrak{g}^*) \cong A \otimes \mathfrak{g} \otimes \mathfrak{g}^* \xrightarrow{\mathrm{id} \otimes \mathrm{ev}} A.$$

It satisfies (1) as

$$\omega_L(e_k, ae_j) = \sum_i f_{ji} \otimes e_i^*(e_k) = f_{jk} = -f_{kj} = -\omega_L(e_j, ae_k).$$

Actually we have $a = -a^*$ for the same reason

$$a(e_j)(e_i) = f_{ji} = -f_{ij} = -a(e_j)(e_i) = -a^*(e_j)(e_i),$$

so here we have $\alpha = \mathrm{id}$, and our 2-term complex is 1-shifted symplectic. ◊

Example 3.6 (2-term complex in Lie theory - II)**.** Let G be a reductive Lie group with a choice of a non-degenerate invariant pairing $\langle -,-\rangle : \mathrm{Sym}^2\,\mathfrak{g} \to \Bbbk$. The group G acts on $B := \mathcal{O}(G)$ by conjugation, yielding the infinitesimal action

$$\mathfrak{g} \to \mathfrak{X}(G) \quad ; \quad x \mapsto \big(\vec{x} = x^L - x^R : g \mapsto (d_e R_g - d_e L_g)(x)\big).$$

We view the map

$$B \otimes \mathfrak{g} \xrightarrow{a} \mathfrak{X}(G) \quad ; \quad f \otimes x \mapsto f\vec{x}$$

as a 2-term complex of A-modules in degrees -1 and 0, and we set

$$\omega_L : (B \otimes \mathfrak{g}) \underset{B}{\otimes} \mathfrak{X}(G) \cong (B \otimes \mathfrak{g}) \underset{B}{\otimes} (B \otimes \mathfrak{g}) \cong B \otimes \mathfrak{g} \otimes \mathfrak{g} \xrightarrow{\mathrm{id} \otimes \langle -,-\rangle} B$$

where the first map is obtained applying the average $\mu = \frac{1}{2}\left(d_g L_{g^{-1}} + d_g R_{g^{-1}}\right)$ of the left and right Maurer–Cartan forms.

Exercise 5. (i) Show that (1) is satisfied.

(ii) Consider then

$$\alpha = \omega_L^\flat : B \otimes \mathfrak{g} \to \Omega^1(G) \cong B \otimes \mathfrak{g}^* \overset{\langle -,-\rangle}{\cong} B \otimes \mathfrak{g} \cong \mathfrak{X}(G).$$

Prove that through these identifications we have $\alpha(x) = \frac{1}{2}(x^L + x^R)$.

(iii) Conclude that a carries a 1-shifted symplectic structure. ◊

Remark 3.7. The above examples fit in the framework of Lie groupoids and algebroids.

Example 3.8 (lagrangian teaser)**.** Consider a standard symplectic vector space (V, ω) (that is a complex concentrated in degree 0 which is 0-shifted symplectic), along with two lagrangian subspaces L_1, L_2 (half dimensional subspaces which are isotropic: $\omega_{|\wedge^2 L_i} = 0$). Consider the 2-term complex

$$L_1 \oplus L_2 \to V \quad ; \quad (\ell_1, \ell_2) \mapsto \ell_1 - \ell_2,$$

with $L_1 \oplus L_2$ in degree 0 and V in degree 1. Define

$$\omega_L : (L_1 \oplus L_2) \otimes V \to k \quad ; \quad (\ell_1, \ell_2) \otimes v \mapsto \omega(\ell_1 + \ell_2, v).$$

Exercise 6. Check that it defines a (-1)-shifted symplectic structure. ◊

3.2 Lagrangian structures

Definition 3.9. Consider a complex V along with a 2-form $\omega : \bigwedge^2 V \to \Bbbk[n]$, and a cochain map $\varphi : L \to V$. An *isotropic* structure on L (with respect to ω) is a null-homotopy $\eta : \omega|_L := \omega \circ \varphi^{\wedge 2} \sim 0$.

If η is an isotropic structure on φ, we get a null-homotopy $\eta^\flat : \varphi^* \omega^\flat \varphi \sim 0$, and thus, from Exercise 3, a cochain map $(\varphi, \eta^\flat) : L \to \mathrm{hofib}\big(\varphi^* \omega^\flat : V \to L^*[n]\big)$.

Definition 3.10. In the previous setting, the isotropic structure η on φ is said to be *non-degenerate* if $(\varphi, \eta^\flat)$ is a quasi-isomorphism. A non-degenerate isotropic structure is called *lagrangian*.

Remark 3.11. (i) A lagrangian structure yields a long exact sequence in cohomology

$$\cdots \to H^{k-1}(L^*[n]) \to H^k(L) \to H^k(V) \to H^k(L^*[n]) \to H^{k+1}(L) \to \cdots$$

(ii) Note that the null-homotopic compositions

$$L \xrightarrow{\varphi} V \xrightarrow{\varphi^*\omega^\flat} L^*[n] \quad \text{and} \quad L \xrightarrow{\omega^\flat \varphi} V^*[n] \xrightarrow{\varphi^*} L^*[n]$$

are shifted dual, therefore, since $\mathrm{hofib}(\varphi)^* = \mathrm{hofib}(\varphi^*)$,

$$L \to \mathrm{hofib}(\varphi^*\omega^\flat) \text{ quasi-iso} \overset{1.4}{\iff} L^*[n-1] \to \mathrm{hofib}(\varphi) \text{ quasi-iso} \iff L \to \mathrm{hofib}(\varphi^*) \text{ quasi-iso}$$

and we get a morphism of long exact sequences

$$\begin{array}{ccccccccc}
\cdots \longrightarrow & H^{k-1}(L^*[n]) & \longrightarrow & H^k(L) & \longrightarrow & H^k(V) & \longrightarrow & H^k(L^*[n]) & \longrightarrow \cdots \\
& \| & & \| & & \downarrow & & \| & \\
\cdots \longrightarrow & H^{k-1}(L^*[n]) & \longrightarrow & H^k(L) & \longrightarrow & H^k(V^*[n]) & \longrightarrow & H^k(L^*[n]) & \longrightarrow \cdots
\end{array}$$

hence ω is non-degenerate and V is actually n-shifted symplectic.

Example 3.12 (de Rham, again)**.** Let Y be an $(n+1)$-dimensional compact oriented manifold with boundary $\partial Y = X$. Consider the cochain complex $V = (\Omega^\bullet(X), d_{\mathrm{dR}})[1]$ equipped with the $(2-n)$-shifted symplectic structure $\omega(\alpha,\beta) = \int_X \alpha\wedge\beta$ from Example 3.3. The restriction of a form on Y to a form on the boundary X gives a cochain map

$$\varphi : L := (\Omega^\bullet(Y), d_{\mathrm{dR}})[1] \longrightarrow (\Omega^\bullet(X), d_{\mathrm{dR}})[1] \quad ; \quad \alpha \longmapsto \alpha|_X .$$

Define $\eta : \bigwedge^2 L \to \mathbb{R}[n-1]$ by $\eta(\alpha,\beta) = \int_Y \alpha\wedge\beta$. It is isotropic by Stokes' theorem:

$$\eta(d(\alpha \bigwedge \beta)) \overset{3.3}{=} \int_Y d_{\mathrm{dR}}(\alpha\wedge\beta) = \int_X (\alpha\wedge\beta)|_X = \omega(\varphi(\alpha),\varphi(\beta)).$$

It is also non-degenerate as

$$H^\bullet(\mathrm{hofib}(\varphi)) = H^{\bullet+1}(Y,X) \cong H^{n-\bullet}(Y)^* = H^{n-1-\bullet}(L)^* = H^{\bullet-n+1}(L^*) = H^\bullet(L^*[1-n])$$

thanks to relative Poincaré duality. ♢

Exercise 7 (*everything is lagrangian* - Alan Weinstein)**.** Consider $V = 0$ equipped with the trivial n-shifted symplectic structure. Prove that a $(n-1)$-shifted symplectic structure on L is the same as a lagrangian structure on $L \to 0$.

Definition 3.13. A *lagrangian correspondence*

$$\begin{array}{ccc} & L & \\ \swarrow & & \searrow \\ V_1 & & V_2 \end{array}$$

is the data of two n-shifted symplectic complexes (V_i, ω_i) along with a lagrangian mophism $L \to (V_1 \oplus V_2, \omega_1 \oplus -\omega_2)$.

The point of this definition is that correspondences do "compose" well *via* homotopy fiber products. Precisely, consider the following diagram

$$\begin{array}{ccccccc}
 & & & L_{13} & & & \\
 & & {\scriptstyle p}\swarrow & & \searrow{\scriptstyle q} & & \\
 & L_{12} & & & & L_{23} & \\
{\scriptstyle a_1}\swarrow & & \searrow{\scriptstyle a_2} & & {\scriptstyle b_2}\swarrow & & \searrow{\scriptstyle b_3} \\
V_1 & & & V_2 & & & V_3
\end{array}$$

where the bottom lagrangian correspondences are given and we aim at building L_{13} such that $L_{13} \to (V_2 \oplus V_3, \omega_2 \oplus -\omega_3)$ is lagrangian.

Definition 3.14. We define the *composition* of L_{12} and L_{23} by

$$L_{13} = L_{12} \circ L_{23} := \mathrm{hofib}\Big(L_{12} \oplus L_{23} \xrightarrow{a_2 - b_2} V_2\Big).$$

Note that the isotropic structures on L_{12} and L_{23} are given by homotpies

$$\eta_{12} : \omega_1 a_1^{\wedge 2} \sim \omega_2 a_2^{\wedge 2} \quad \text{and} \quad \eta_{23} : \omega_2 b_2^{\wedge 2} \sim \omega_3 b_3^{\wedge 2}$$

that induce, by Exercise 2(i), homotopies

$$\eta_{12} p^{\wedge 2} : \omega_1 (a_1 p)^{\wedge 2} \sim \omega_2 (a_2 p)^{\wedge 2} \quad \text{and} \quad \eta_{23} q^{\wedge 2} : \omega_2 (b_2 q)^{\wedge 2} \sim \omega_3 (b_3 q)^{\wedge 2}.$$

Exercise 8. The projection $L_{13} \to V_2[-1]$ defines a homotopy $h : a_2 p \sim b_2 q$.

Thanks to Exercise 2(ii,iii), we get a homotopy

$$\bar{h} : \omega_2(a_2 p)^{\wedge 2} \sim \omega_2(b_2 q)^{\wedge 2},$$

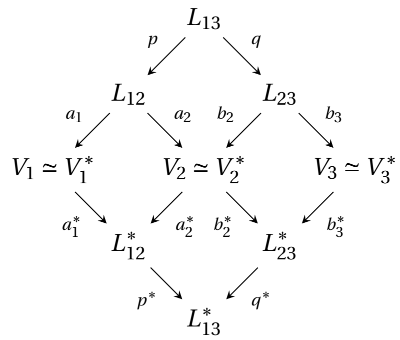
and finally

$$\eta_{12} p^{\wedge 2} + \bar{h} + \eta_{23} q^{\wedge 2} : \omega_1(a_1 p)^{\wedge 2} \sim \omega_3(b_3 q)^{\wedge 2}$$

endows $L_{13} \to V_1 \oplus V_3$ with an isotropic structure.

Proposition 3.15. *This isotropic structure is non-degenerate.*

Proof. Consider the diagram where we omit the shifts by n



and where all inner squares are homotopy pullbacks by hypothesis. Applying Lemma 1.3 we first get that all inner rectangles consisting in two adjacent inner squares are homotopy pullbacks. Applying it again to a pair of adjacent such rectangles we then get that the outer square is a homotopy pullback as wished. □

Example 3.16. Set $V_1 = V_3 = 0$ and consider $V_2 = V$ a 0-shifted symplectic vector space, along with two genuine lagrangian subspaces L_{12} and L_{23}. Then L_{13} is the complex $(L_{12} \oplus L_{23} \to V)$ in degrees $-1, 0$ with differential $(\ell_1, \ell_2) \mapsto \ell_1 - \ell_2$, and is endowed with a lagrangian structure $L_{13} \to 0$, meaning that L_{13} is (-1)-shifted symplectic thanks to 7.

Exercise 9. Check that it matches the one from 3.8. ◊

4 The geometric side

4.1 Affine derived geometry

▷ Denote by $\mathtt{cdga}^{\leq 0}$ the category of commutative differential non-positively graded $\Bbbk$-algebras. These are cochain complexes endowed with a grading preserving multiplication such that the differential satisfies the graded Leibniz rule.

▷ A dg-module over $A \in \mathtt{cdga}^{\leq 0}$ is a standard A-bimodule M which is a cochain complex such that

- $A^i \cdot M^j \subseteq M^{i+j}$;
- $\forall (a, m) \in A \times M$, $am = (-1)^{|a||m|} ma$;
- $d_M(am) = (d_A a)m + (-1)^{|a|} a(d_M m)$.

▷ A cochain map $f : M \to N$ between A-dg-modules is a morphism of A-modules if $f(am) = af(m)$ for all $(a, m) \in A \times M$.

▷ The tensor product $M \otimes_A N$ is defined as the quotient of the tensor product $M \otimes N$ of cochain complexes by the submodule generated by all elements of the form $ma \otimes n - m \otimes an$, where $(a, m, n) \in A \times M \times N$.

▷ The internal $\mathrm{Hom}^\bullet_A(M,N)$ is defined by

$$\mathrm{Hom}^n_A(M,N) = \{\, f \in \mathrm{Hom}^n(M,N) \mid \forall a,m \in A\times M,\ f(ma) = f(m)a \,\}.$$

▷ Consider a morphism $B \to A$ in $\mathtt{cdga}^{\leq 0}$. For a given A-dg-module M one can consider B-linear *derivations*

$$\mathrm{Der}^n_B(A,M) = \{\, D \in \mathrm{Hom}^n_B(A,M) \mid \forall a,a' \in A,\ D(aa') = (Da)a' + (-1)^{n|a|} aDa' \,\},$$

yielding a cochain complex $\mathrm{Der}^\bullet_B(A,M)$ with differential $dD = d_M D - (-1)^n D d_A$. One can check that for all A-dg-modules M,N, the cochain complexes $\mathrm{Hom}^\bullet_B(N,M)$ and $\mathrm{Der}^\bullet_B(A,M)$ inherit a structure of A-module given by left multiplication.

▷ The A-dg-module of B-linear *Kähler differentials* is defined by $\Omega^1_{A/B} = I/I^2$ where I is the kernel of the multiplication $A \otimes_B A \to A$. There is a distinguished map $d : A \to \Omega^1_{A/B}$ given by $da = a \otimes 1 - 1 \otimes a$.

Exercise 10. Check that $d \in \mathrm{Der}^0(A, \Omega^1_{A/B})$, that it is a cochain map, and that it is universal: precomposition by d gives an isomorphism of A-dg-modules

$$\mathrm{Hom}^\bullet_A(\Omega^1_{A/B}, M) \simeq \mathrm{Der}^\bullet_B(A,M)$$

for all A-dg-module M.

In particular $\Omega^1_{A/B}$ and $\mathrm{Der}_B(A) := \mathrm{Der}^\bullet_B(A,A)$ are dual A-dg-modules. When $B = \Bbbk$ we usually drop the index B.

Definition 4.1. Let $f : C \to A$ be a morphism in $\mathrm{cdga}^{\leq 0}_k$. A *semi-free resolution* of f is a factorisation

$$\begin{array}{ccc} C & \xrightarrow{\quad f \quad} & A \\ & {\scriptstyle \tilde f}\searrow \quad \nearrow & \\ & \tilde A & \end{array}$$

such that:

(i) the morphism $\widetilde{A} \to A$ is a quasi-isomorphism;

(ii) denoting by $(-)^\natural$ the "underlying commutative graded algebra" functor that forgets the differential, we have that
$$A^\natural \cong \mathrm{Sym}_{C^\natural}(V)$$
as a commutative $C^\natural$-algebra, for some non-positively graded vector space V.

Remark 4.2.

- A semi-free resolution of A is a semi-free resolution of the unit $\Bbbk \to A$.
- The categories A-Mod and $\tilde A$-Mod of dg-modules are quasi-equivalent.
- Semi-free resolutions always exist.

Example 4.3. Consider the algebra $A = \Bbbk[x]/x^2$ concentrated in degree 0. We get a semi-free resolution of A by adding a generator ξ in degree -1:

$$\tilde A = \Bbbk[x,\xi] = \Bbbk[x]\xi \oplus \Bbbk[x] = \left(\Bbbk[x]\xi \xrightarrow{\delta} \Bbbk[x]\right)$$

such that $\xi^2 = 0$ and $\delta\xi = x^2$. ◊

Definition 4.4. Consider $C \to A \in \mathtt{cdga}^{\leq 0}$ along with a semi-free resolution $C \to \tilde A$. The *relative tangent and cotangent* complexes are defined as $\mathbb{T}_{A/C} = T_{\tilde A/C} := \mathrm{Der}_C(\tilde A)$ and $\mathbb{L}_{A/C} = \Omega^1_{\tilde A/C}$. We drop the term *relative* and C when $C = \Bbbk$.

Exercise 11. Prove that $\mathbb{L}_{A/C}[-1] = \mathrm{hofib}\,(A \otimes_C \mathbb{L}_C \to \mathbb{L}_A)$.

Example 4.5 (4.3 followed)**.** We have $\mathbb{L}_A = \Bbbk[x,\xi]d\xi \oplus \Bbbk[x,\xi]dx$ with $\delta(d\xi) = d(\delta\xi) = 2xdx$ [exercise], and

$$\mathbb{T}_A = \Bbbk[x,\xi]\partial_x \oplus \Bbbk[x,\xi]\partial_\xi$$

with $|\partial_\xi| = 1$ and $\langle \delta(\partial_x), d\xi\rangle = \langle \partial_x, \delta(d\xi)\rangle = \langle \partial_x, 2xdx\rangle = 2x$, therefore $\delta(\partial_x) = 2x\partial_\xi$. Observe [exercise] that $\mathbb{T}_A$ is a (-1)-shifted symplectic $\tilde{A}$-dg-module with respect to ω defined by $\omega(\partial_x, \partial_\xi) = 1$. ♢

Definition 4.6. Consider $f : C \to A$ and $g : C \to B$ two morphisms in $\mathtt{cdga}^{\leq 0}$. The relative (left) *derived tensor product* of A and B over C is

$$A \otimes_C^{\mathbb{L}} B := \tilde{A} \otimes_C B$$

where $\tilde{f} : C \to \tilde{A}$ is a semi-free resolution of f.

Exercise 12. Consider the above situation, and assume that $\tilde{A} = A$. Consider a semi-free resolution $\tilde{g} : C \to \tilde{B}$ of g.

(i) Prove that $A \to A \otimes_C \tilde{B} =: \tilde{D}$ is semi-free, and that it resolves $A \to A \otimes_C B = D$.

(ii) [base change] Prove that the push-out square

$$\begin{array}{ccc} C & \longrightarrow & A \\ \downarrow & & \downarrow \\ \tilde{B} & \longrightarrow & \tilde{D} \end{array}$$

induces an isomorphism of $\tilde{D}$-dg-modules

$$\mathbb{L}_{D/A} = \Omega^1_{\tilde{D}/A} \simeq \tilde{D} \otimes_{\tilde{B}} \Omega^1_{\tilde{B}/C} \simeq D \otimes_B \mathbb{L}_{B/C}.$$

(iii) Exercise 11 can be generalized in $D \otimes_B \mathbb{L}_{B/C} = \operatorname{hofib}(\mathbb{L}_{D/C} \to \mathbb{L}_{D/B})$. Prove that it implies that

$$D \otimes_A \mathbb{L}_{B/C}[-1] = \operatorname{hofib}\big(D \otimes_C \mathbb{L}_C \to D \otimes_B \mathbb{L}_B\big).$$

(iv) Using Lemma 1.3 and the square

$$\begin{array}{ccc} D \otimes_C \mathbb{L}_C & \longrightarrow & D \otimes_A \mathbb{L}_A \\ \downarrow & & \downarrow \\ D \otimes_B \mathbb{L}_B & \longrightarrow & \mathbb{L}_D \end{array}$$

deduce from the above that

$$D \otimes_C \mathbb{L}_C \simeq \operatorname{hofib}\big((D \otimes_A \mathbb{L}_A) \oplus (D \otimes_B \mathbb{L}_B) \to \mathbb{L}_D\big).$$

By duality, this implies

$$\mathbb{T}_D = D \otimes_A \mathbb{T}_A \underset{D \otimes_C \mathbb{T}_C}{\overset{h}{\times}} D \otimes_B \mathbb{T}_B := \operatorname{hofib}\big((D \otimes_A \mathbb{T}_A) \oplus (D \otimes_B \mathbb{T}_B) \to (D \otimes_C \mathbb{T}_C)\big). \tag{3}$$

Example 4.7. Consider $X = \mathbb{A}^1 \hookrightarrow \mathbb{A}^2 = Z$ the affine line embedded into the affine plane as $\{y = 0\}$. We would like to compute the derived self-intersection of X into Z. Algebraically, on functions, we have

$$A := \mathscr{O}(X) = k[x] \longleftarrow k[x,y] = \mathscr{O}(Z) := C \qquad 0 \leftarrow\!\shortmid y.$$

The cdga of functions on the derived self-intersection of $\mathbb{A}^1$ in $\mathbb{A}^2$ is computed as the derived tensor product $A \otimes_C^{\mathbb{L}} A$. We first resolve $C \to A$ by $C \longrightarrow \widetilde{A} = k[\xi, x, y] \longrightarrow A$, with $|\xi| = -1$, and $\delta\xi = y$. Then we get $A \otimes_C^{\mathbb{L}} A = \widetilde{A} \otimes_C A \cong k[x,\xi] = k[\xi] \otimes k[x]$ (as $\delta(\xi) = 0$ now), which corresponds to the space $\mathbb{A}^1 \times \mathbb{A}^1[-1]$, where $\mathbb{A}^1[-1]$ is the so-called odd affine line. The tangent complex of the derived self-intersection is therefore

$$\mathbb{T}_{k[x,\xi]} = k[x,\xi]\partial_x \oplus k[x,\xi]\partial_\xi, \tag{4}$$

with $|\partial_\xi| = 1$ and zero differential. As suggested by (3), we want to compare (4) with

$$\Xi := \operatorname{hofib}\Big(\big(k[x,\xi] \otimes_{\widetilde{A}} T_{\widetilde{A}}\big)^{\oplus 2} \xrightarrow{(\theta,-\theta)} k[x,\xi] \otimes_C T_C\Big)$$

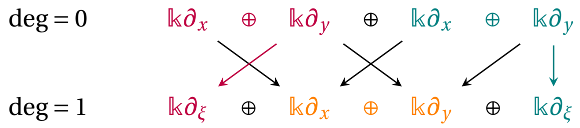
where:

(i) the $k[x,\xi]$-dg-module $k[x,\xi]\otimes_{\tilde{A}} T_{\tilde{A}}$ is freely generated by ∂_x and ∂_y in degree 0, and ∂_ξ in degree 1 with $\delta(\partial_y) = \partial_\xi$;

(ii) the $k[x,\xi]$-dg-module $k[x,\xi]\otimes_C T_C$ is freely generated in degree 0 by ∂_x and ∂_y;

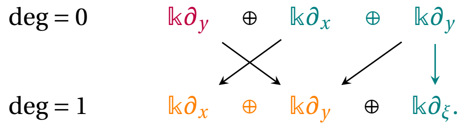
(iii) the morphism of $k[x,\xi]$-dg-modules θ sends ∂_ξ to 0 and is the identity on the other generators.

We thus get that Ξ is the free $k[x,\xi]$-dg-module generated by the following two-term complex:



where the purple part corresponds to the first copy of $T_{\tilde{A}}$, the teal part to the second copy, and the orange one to T_C. The remaining black arrows describe $(\theta,-\theta)$.

Exercise 13. Prove that there exists a projection from the above 2-term complex to $\Bbbk\partial_x \oplus \Bbbk\partial_\xi$, with kernel



Prove that this kernel has no cohomology and conclude. ◊

4.2 Affine derived symplectic geometry: definitions

What follows was introduced in the seminal paper [4].

▷ Consider $(A,\delta)\in \mathtt{cdga}^{\leq 0}$, and M an A-dg-module. Recall the tensor algebra $T_A(M) = \oplus_{n\geq 0} M^{\otimes_A n}$ with differential

$$\delta(m_1\otimes\cdots\otimes m_n) = \sum_{1\leq k\leq n} (-1)^{|m_1|+\cdots+|m_{k-1}|} m_1\otimes\cdots\otimes m_{k-1}\otimes \delta_M(m_k)\otimes m_{k+1}\otimes\cdots\otimes m_n.$$

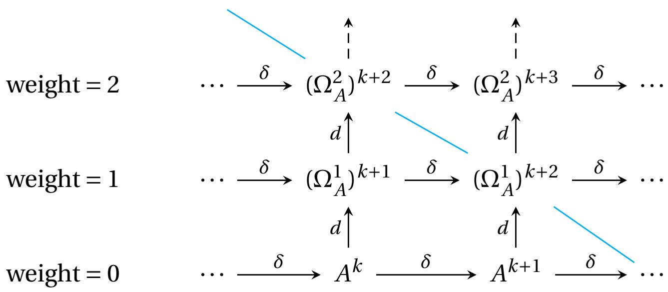
Define the *de Rham complex* of A as

$$\mathrm{DR}^\bullet_A = \mathrm{Sym}^\bullet_A\left(\Omega^1_A[-1]\right).$$

It is a *mixed* complex as it has two differentials: the *internal* one δ, that increases the degree by 1 but preserves the symmetric weight, and the de Rham one $d = d_{\mathrm{dR}}$, that increases this weight by 1. The latter is defined as in Example 1.2: it extends $d : A = \Omega^0_A \to \Omega^1_A$ and satisfies the Leibniz rule. Note that for all $a_0,\dots,a_n\in A$,

$$d_{\mathrm{dR}}(a_0(da_1)\cdots(da_n)) = (da_0)(da_1)\cdots(da_n)$$

so that one can check [exercise] that the two differentials on $\mathrm{DR}^\bullet(A)$ commute because $d : A\to\Omega^1_A$ is a cochain complex. Also note that the de Rham differential also increases the degree by 1. To summarize, if we set $\Omega^p_A = \mathrm{DR}^p_A[p]$:



where the degree is constant along parallels to the cyan line.

▷ A 2-*form of degree n* on A is an $(n+2)$-cocycle in

$$\mathrm{Sym}^2\left(\Omega^1_A[-1]\right) =: \wedge^2_A\left(\Omega^1_A\right)[-2]$$

with respect to δ. By definition it is a cochain map

$$A[-n-2] \to \wedge^2_A\left(\Omega^1_A\right)[-2] \iff \wedge^2_A \mathbb{T}_A \to A[n]$$

that we may require to be non-degenerate.

▷ A *closed* 2-*form of degree n* on A is an $(n+2)$-cocycle ω in the *total complex*

$$\left(\prod_{k\geq 0} \mathrm{Sym}^{2+k}\left(\Omega^1_A[-1]\right), \delta + d\right).$$

By definition it is a series of forms $(\omega_k)_{k\geq 0}$, with ω_k of weight $2+k$, satisfying

$$\delta\omega_0 = 0 \quad \text{and} \quad d\omega_k + \delta\omega_{k+1} = 0 \quad \forall k \geq 0.$$

In particular ω_0 is 2-form of degree n and ω_1 is a null-homotopy $d\omega_0 \sim 0$.

Definition 4.8. An *n-shifted symplectic structure* on A or $X = \mathrm{Spec}(A)$ is a closed 2-form ω of degree n such that ω_0 is non-degenerate.

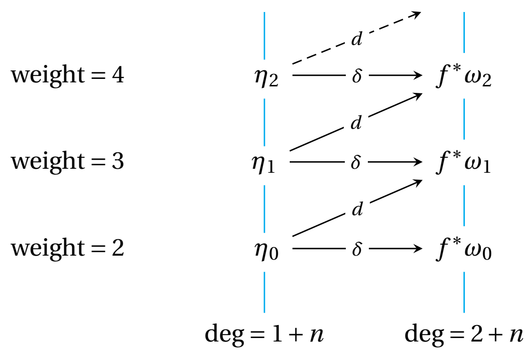
Through Spec, we define the category of affine derived schemes as the opposite of the homotopy category of $\mathtt{cdga}^{\leq 0}$, where quasi-isomorphisms are formally inverted. Consider $f: Y \to X$ a morphism of affine derived schemes corresponding to $f^*: A \to B$ in $\mathtt{cdga}^{\leq 0}$. Recall that it induces $f^*: B \otimes_A \Omega^1_A \to \Omega^1_B$. Assume that $X = \mathrm{Spec}(A)$ is endowed with an n-shifted symplectic structure ω. By construction we have a quasi-isomorphism

$$\omega_0^\flat : \mathbb{T}_X \to \mathbb{L}_X[n].$$

▷ An *isotropic structure* on f with respect to ω is a null-homotopy $\eta: f^*\omega \sim 0$. It is given by a series $(\eta_k)_{k\geq 0}$ of forms η_k of weight $2+k$ and degree $1+n$ such that

$$f^*\omega_0 = \delta\eta_0 \quad \text{and} \quad f^*\omega_{k+1} = \delta\eta_{k+1} + d\eta_k \quad \forall k \geq 0.$$

Graphically:



Recall that we have a morphism $\mathbb{T}_f : \mathbb{T}_B \to B \otimes_A \mathbb{T}_A$. The equality $f^*\omega_0 = \delta\eta_0$ says that η_0 is a homotopy between 0 and

$$f^*\omega_0 : \wedge^2_B \mathbb{T}_B \xrightarrow{\wedge^2 \mathbb{T}_f} B \otimes_A \wedge^2_A \mathbb{T}_A \xrightarrow{\mathrm{id}\otimes\omega_0} B \otimes_A A[n] = B[n].$$

Definition 4.9. A *lagrangian structure* on f is an isotropic structure η such that η_0 is non-degenerate in the sense of Definition 3.10.

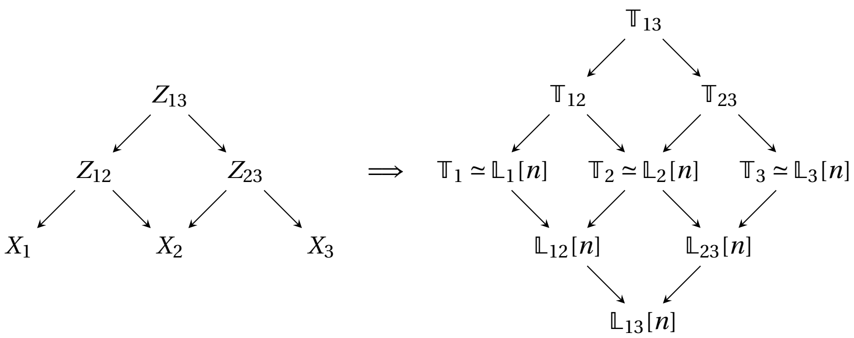
Lagrangian structures for derived affine schemes enjoy the same (well, opposite) features as the linear ones, that is:

(i) [1] a lagrangian structure on a morphism $X \to \mathrm{pt}$ to the point endowed with the tricial n-shifted symplectic structure is the same thing as a $(n-1)$-shifted structure on X;

(ii) [1] lagrangian correspondences compose well by derived fiber product. More precisely, consider n-shifted symplectic affine derived schemes $X_k = \operatorname{Spec}(A_k)$, $k \in \{1,2,3\}$, and lagrangian morphisms $\operatorname{Spec}(C_{k\ell}) = Z_{k\ell} \to X_k \times \overline{X}_\ell$ (where $\overline{X}$ means that we consider the opposite symplectic structure) for $k\ell \in \{12, 23\}$. The derived fiber product here is

$$Z_{13} := Z_{12} \overset{h}{\underset{X_2}{\times}} Z_{23} = \operatorname{Spec}\left(C_{12} \overset{\mathbb{L}}{\underset{A_2}{\otimes}} C_{23}\right) =: \operatorname{Spec}(C_{13})$$

and thanks to (3) we have the following diagrams, the second one being made of homotopy pullbacks of C_{13}-dg-modules:



where $\mathbb{T}_k = C_{13} \otimes_{A_k} \mathbb{T}_{A_k} = C_{13} \otimes_{A_k} \mathbb{T}_{X_k}$, $\mathbb{T}_{k\ell} = C_{13} \otimes_{C_{k\ell}} \mathbb{T}_{C_{k\ell}} = C_{13} \otimes_{C_{k\ell}} \mathbb{T}_{Z_{k\ell}}$, same for $\mathbb{L}_?$.

(iii) [4] genuine smooth lagrangian subschemes are examples of lagrangian morphisms.

A blend of these three properties yields a very important class of examples : critical loci. The derived approach of those recently led to great improvements for instance in obstruction and Donaldson–Thomas theories. Concretely, consider an honest differentiable function $f : X \to \Bbbk$ and set $X_1 = X_3 = \operatorname{Spec}(\Bbbk) = \{\mathrm{pt}\}$, $X_2 = T^*X$ the à-shifted symplectic usual cotangent bundle, $df : Z_{12} = X \to T^*X$, $0 : Z_{23} = X \to T^*X$ the zero section. Then we get that the *derived critical locus*

$$\operatorname{dCrit}(f) := X \,{}_0\!\overset{h}{\underset{T^*X}{\times}}{}_{df}\, X$$

is (-1)-shifted symplectic. This should be thougth as a derived enhancement of the intersection $\{(x,0)\} \cap \{(x, df_x)\}$ of two lagrangian subvaieties of T^*X, which might in tun be identified with $\{x \mid df_x = 0\} \subseteq X$.

4.3 Two examples of derived zero loci

We slightly generalize what is above by considering any closed 1-form $\lambda : X \to T^*X$ and then consider the *derived zero locus*

$$\operatorname{dZ}(\lambda) := X \,{}_0\!\overset{h}{\underset{T^*X}{\times}}{}_{\lambda}\, X$$

which is (-1)-shifted symplectic.

▷ Consider $X = \operatorname{Spec}(\Bbbk[x])$ and $\lambda = \alpha(x)dx$. In this case,

$$T^*X \cong \mathbb{A}^2 = \operatorname{Spec}\big(\Bbbk[x,y]\big) \quad \text{with} \quad \omega = dx \wedge dy = \omega_0.$$

The 0 section corresponds to Example 4.7 where we used the semi-free resolution $\Bbbk[x,y] \to k[x,y,\xi] \simeq k[x]$, with $\delta(\xi) = y$. We get

$$\mathcal{O}(\operatorname{dZ}(\lambda)) = \Bbbk[x,y,\xi] \underset{\Bbbk[x,y]}{\otimes}{}_{\alpha} \Bbbk[x] \simeq \big(\, \Bbbk[x,\xi] : \delta(\xi) = \alpha(x)\,\big).$$

The differential can thus be written as $\delta = \alpha\partial_\xi$. In order to compute the (-1)-shifted symplectic structure on the derived intersection, we first have to understand what is the null-homotopy $\omega \sim 0$ in the resolution $k[x,y,\xi]$. We have $\eta_0 = \eta = dx \wedge d\xi$:

$$d\eta = 0 \quad \text{and} \quad \delta\eta_0 = \delta dx \wedge d\xi + dx \wedge \delta d\xi = dx \wedge dy = \omega.$$

Therefore, the (-1)-shifted symplectic structure on $\mathscr{O}(\mathrm{dZ}(\lambda))$ is $dx \wedge d\xi$.

▷ If $\lambda = df = x^2 dx$, that is, $f = \frac{1}{3}x^3$, then

$$\mathscr{O}(\mathrm{dZ}(\lambda)) = \mathscr{O}\left(\mathrm{dCrit}(\tfrac{1}{3}x^3)\right) \simeq \left(\Bbbk[x,\xi] : \delta = x^2\partial_\xi\right) \simeq k[x]/x^2.$$

This cdga carries a (-1)-shifted symplectic structure and we recover a fact already noticed in Example 4.3: the tangent complex of $k[x]/x^2$ carries a linear (-1)-shifted symplectic structure.

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