A finiteness result for subalgebras of polynomials and power series

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Abstract

Let $\mathbf{R} = \mathbb{K}[x_1, \ldots, x_n]$ be the algebra of polynomials in $\underline{x} = (x_1, \ldots, x_n)$ over a field $\mathbb{K}$, and let $\alpha_1, \ldots, \alpha_m$ be m vectors of $\mathbb{N}^n$. Let C be the cone in $\mathbb{R}^n_+$ generated by $\alpha_1, \ldots, \alpha_m$. We prove the following: there does not exist infinite sequences $\mathbf{A} = \mathbb{K}[\underline{x}^{\alpha_1}, \ldots, \underline{x}^{\alpha_m}] \subset \mathbf{A}[\underline{x}^{\beta_1}] \subset \mathbf{A}[\underline{x}^{\beta_1}, \underline{x}^{\beta_2}] \subset \ldots$ where for all $i \geq 1, \beta_i \in C$. Then we give one application.

1 The main result

Let $\mathbf{R} = \mathbb{K}[x_1, \ldots, x_n]$ be the algebra of polynomials in $\underline{x} = (x_1, \ldots, x_n)$ over a field $\mathbb{K}$, and let $\alpha_1, \ldots, \alpha_m$ be m vectors of $\mathbb{N}^n$. Let

$$C = \{\sum_{i=1}^{m} \lambda_i \alpha_i \mid \lambda_1, \ldots, \lambda_m \in \mathbb{R}_+\}$$

be the cone generated by $\alpha_1, \ldots, \alpha_m$. Let $\mathbf{A} = \mathbb{K}[\underline{x}^{\alpha_1}, \ldots, \underline{x}^{\alpha_m}]$. The aim of the paper is to prove the following.

Theorem 1 *There does not exist infinite sequences of the form* $\mathbf{A} \subset \mathbf{A}[\underline{x}^{\beta_1}] \subset \mathbf{A}[\underline{x}^{\beta_1}, \underline{x}^{\beta_2}] \subset \ldots$ *where for all* $i \geq 1, \beta_i \in C$.

The result of the theorem above is false in general. Let $\alpha_1 = (1, 0)$ and $\alpha_2 = (1, 1)$. The sequence

$$\mathbf{A} = \mathbb{K}[x_1, x_1x_2] \subset \mathbf{A}[x_1x_2^2] \subset \mathbf{A}[x_1x_2^2, x_1x_2^3] \subset \cdots \subset \mathbf{A}[x_1x_2^2, x_1x_2^3, \ldots, x_1x_2^k] \subset \ldots$$

is infinite. It is also worth noticing that if $n = 1$ then the result of the theorem is always true (see [2] and [3] for the properties of subsemigroups of $\mathbb{N}$).

Let $S_0 = \langle \alpha_1, \ldots, \alpha_m \rangle$ be the affine semigroup generated by $\{\alpha_1, \ldots, \alpha_m\}$, and for all $i \geq 1$, let $S_i = \langle \alpha_1, \ldots, \alpha_m, \beta_1, \ldots, \beta_i \rangle$, where we recall that for all $i \geq 1, \beta_i \in C$. Let finally S be the semigroup generated by $\{\alpha_1, \ldots, \alpha_m, \beta_1, \ldots, \beta_k, \ldots\}$. Theorem 1 is equivalent to the following.

Theorem 2 *Consider the sequence* $S_0 \subseteq S_1 \subseteq \cdots \subseteq S_k \subseteq \ldots$. *There exists* $l \geq 1$ *such that* $S_{k+l} = S_l$ *for all* $k \geq 0$. *Equivalently, the semigroup* S *is finitely generated.*

Recall that a subset $E \subseteq \mathbb{N}^m$ is said to be an E-set of $\mathbb{N}^m$ if for all $\alpha \in E$ and for all $\beta \in \mathbb{N}^m, \alpha + \beta \in E$. In order to prove the theorems above we shall need the following technical result.

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Lemma 3 *Given an E-set of $\mathbb{N}^m$, there exists a unique minimal finite subset $\{\gamma_1, \ldots, \gamma_s\} \in E$ such that $E = \cup_{i=1}^s (\gamma_i + \mathbb{N}^m)$. We set $B(E) = \{\gamma_1, \ldots, \gamma_s\}$ and we call it the boundary of E.*

Proof. The ideal $I = (\underline{x}^\gamma \mid \alpha \in E)$ of $\mathbf{R}$ is finitely generated. Let $\underline{x}^{\gamma_1}, \ldots, \underline{x}^{\gamma_s}$ be the unique minimal generating system of I, then $\{\gamma_1, \ldots, \gamma_s\}$ satisfies the equality $E = \cup_{i=1}^s (\gamma_i + \mathbb{N}^n)$.■

Lemma 4 *(see [4], for example) There exists a subset $\{\xi_1, \ldots, \xi_r\}$ of $C \cap \mathbb{N}^n$ such that for all $\alpha \in C \cap \mathbb{N}^n$, there exist $\lambda_1, \ldots, \lambda_m \in \mathbb{N}$, and $j \in \{1, \ldots, r\}$, such that $\beta = \sum_{i=1}^m \lambda_i \alpha_i + \xi_j$.*

Proof. Let $P = \{\sum_{i=1}^m \lambda_i \alpha_i \mid 0 \leq \lambda_i < 1 \text{ for all } 1 \leq i \leq m\}$, and let $\{\xi_1, \ldots, \xi_r\} = P \cap \mathbb{N}^n$. Let $\beta \in C \cap \mathbb{N}^n$. We have

$$\beta - \sum_{i=1}^m \lfloor \lambda_i \rfloor \alpha_i = \sum_{i=1}^m \{\lambda_i\} \alpha_i \in P$$

where $\lfloor x \rfloor$ denotes the integer part of x, and $\{x\}$ its fractional part. As $\sum_{i=1}^m \{\lambda_i\} \alpha_i \in P \cap \mathbb{N}^n$, this finishes the proof. ■

Let the notations be as in Lemma 4. Fix $j \in \{1, \ldots, r\}$, and let

$$E_j = \{(\lambda_1, \ldots, \lambda_m) \in \mathbb{N}^n \mid \sum_{i=1}^m \lambda_i \alpha_i + \xi_j \in S\}.$$

If $(\lambda_1, \ldots, \lambda_m) \in E_j$ and $(\theta_1, \ldots, \theta_m) \in \mathbb{N}^m$, then

$$\sum_{i=1}^m (\lambda_i + \theta_i) \alpha_i + \xi_j = \sum_{i=1}^m \lambda_i \alpha_i + \xi_j + \sum_{i=1}^m \theta_i \alpha_i \in S$$

In particular $(\lambda_1, \ldots, \lambda_m) + (\theta_1, \ldots, \theta_m) \in E_j$. This proves that E_j is an E-set.

Given $\gamma \in B(E_j)$, we set

$$\beta(\gamma) = \sum_{i=1}^m \gamma_i \alpha_i + \xi_j$$

and we recall that $\beta(\gamma)$ is an element of S. Let $\{\beta_1, \ldots, \beta_t\}$ be the set of $\beta(\gamma), \gamma \in \cup_{i=1}^s B(E_j)$. We have the following.

Lemma 5 *The subsemigroup S is an affine semigroup generated by $\{\alpha_1, \ldots, \alpha_m, \beta_1, \ldots, \beta_t\}$.*

Proof. Let the notations be as above, and let $\alpha \in S$. Write $\alpha = \sum_{i=1}^m \lambda_i \alpha_i + \xi_j$, where $\lambda_1, \ldots, \lambda_m \in \mathbb{N}$ and $j \in \{1, \ldots, r\}$. By hypothesis, $(\lambda_1, \ldots, \lambda_m) \in E_j$, whence there exists $\gamma \in B(E_j)$ and $\theta \in \mathbb{N}^m$ such that

$$(\lambda_1, \ldots, \lambda_m) = \gamma + \theta$$

in particular

$$\alpha = \sum_{i=1}^m \lambda_i \alpha_i + \xi_j = \sum_{i=1}^m \gamma_i \alpha_i + \xi_j + \sum_{i=1}^m \theta_i \alpha_i = \sum_{i=1}^m \theta_i \alpha_i + \beta(\gamma)$$

As $\beta(\gamma) \in \{\beta_1, \ldots, \beta_t\}$, our assertion is proved.■

Proof of Theorem 1. With the notations above, we have $S = \langle \alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_t \rangle$. This proves Theorem 2, hence Theorem 1.

It is worth noticing that the result of Theorem 1 holds in the formal case, i.e. when $\mathbf{R} = \mathbb{K}[\![x_1, \ldots, x_n]\!]$ is the algebra of formal power series in $x_1, \ldots, x_n$ over $\mathbb{K}$.

Application. Let $<$ be a well ordering on $\mathbb{N}^n$. Given a non zero element $f = \sum_\alpha c_\alpha \underline{x}^\alpha$ in $\mathbf{R}$ we set $\mathrm{Supp}(f) = \{\alpha \mid c_\alpha \neq 0\}$ and we call it the support of f. We set $\exp(f)$ to be the maximum element of $\mathrm{S}upp(f)$ with respect to $<$, and we call it the leading exponent of f. We finally set $\mathrm{M}(f) = c_{\exp(f)} \underline{x}^{\exp(f)}$ and we call it the leading monomial of f. Let $f_1, \ldots, f_r$ be nonzero elements of $\mathbf{R}$, and let $\mathbf{B} = \mathbb{K}[f_1, \ldots, f_r]$. We set $\exp(\mathbf{B}) = \{\exp(f) \mid f \in \mathbf{B} \setminus \{0\}\}$, and $\mathrm{M}(\mathbf{B}) = \mathbb{K}[\mathrm{M}(f) \mid f \in \mathbf{B} \setminus \{0\}\}$. Clearly $\exp(\mathbf{B})$ is a subsemigroup of $\mathbb{N}^n$, and $\mathrm{M}(\mathbf{B})$ is a monomial algebra. We say that a set $F \in \mathbf{B}$ is a canonical basis of $\mathbf{B}$ if $\mathrm{M}(\mathbf{B}) = \mathbb{K}[\mathrm{M}(f) \mid f \in F]$. Clearly F is a canonical basis of $\mathbf{B}$ if and only if $\exp(\mathbf{B})$ is generated by $\{\exp(f) \mid f \in F\}$ (see [1] and [4] for the calculation and different properties of canonical bases).

Let $\mathbf{B} = \mathbb{K}[\![x_1 + x_2, x_1 x_2, x_1 x_2^2]\!]$ and let $<$ be a well ordering on N^2 for which $(0,1) < (1,0)$. Then we can prove (see [1], rs) that $x_1 x_2^k \in \mathrm{M}(\mathbf{B}$ for all $k > 0$. In particular, $\mathrm{M}(\mathbf{B})$ is not finitely generated. In general, it is not easy to have necessarily and sufficient conditions for a canonical basis to be finite, however, thinking of Theorem 1, we can have the following sufficient condition.

Proposition 6 *Let the notations be as above, and let* $\alpha_i = \exp(f_i)$ *for all* $i \in \{1, \ldots, r\}$. *Let* $C = \{\sum_{i=1}^r \lambda_i \alpha_i \mid \lambda_1, \ldots, \lambda_r \in \mathbb{R}_+\}$. *If* $\mathrm{Supp}(f_i) \in C$ *for all* $i \in \{1, \ldots, r\}$, *then* $\exp(\mathbf{B})$ *is finitely generated.*

Proof. Let $f \in \mathbf{B}$, and write $f = \sum_\theta c_\theta f_1^{\theta_1} \ldots f_r^{\theta_r}$. As $\mathrm{Supp}(f_1^{\theta_1} \ldots f_r^{\theta_r}) \subseteq C$ for all θ such that $c_\theta \neq 0$, we have $\mathrm{Supp}(f) \subseteq C$, and consequently $\exp(f) \in C$. Hence the result is a consequence of Lemma 5.

The result above holds also in the formal case, if we consider a well ordering on $\mathbb{N}^n$ which is compatible with the formal structure of $\mathbb{K}[\![x_1, \ldots, x_n]\!]$ (see [1] for more details).

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