Intersection theory on punctual Hilbert schemes and graded Hilbert schemes

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Abstract

The rational Chow ring $A^*(S^{[n]}, \mathbb{Q})$ of the Hilbert scheme $S^{[n]}$ parametrising the length n zero-dimensional subschemes of a toric surface S can be described with the help of equivariant techniques. In this paper, we explain the general method and we illustrate it through many examples. In the last section, we present results on the intersection theory of graded Hilbert schemes.

Introduction

Let S be a smooth projective surface and $S^{[n]}$ the Hilbert scheme parametrising the length n zero-dimensional subschemes of S. How to describe the cohomology ring $H^*(S^{[n]}, \mathbb{Q})$ and the Chow ring $A^*(S^{[n]}, \mathbb{Q})$?

A first approach is based on the work of Nakajima, Grojnowski and Lehn among others [12], [16], [13], [14], [2]. The direct sum $\oplus_{n\in\mathbb{N}} H^*(S^{[n]}, \mathbb{Q})$ is an (infinite dimensional) irreducible representation and carries a Fock space structure [15]. Lehn settles a connection between the Fock space structure and the intersection theory of the Hilbert scheme via the action of the Chern classes of tautological bundles [11].

An other method, independent of the Fock space formalism introduced by Nakajima, has been developed in [5] when S is a toric surface. The point is that the extra structure coming from the torus action brings into the scene an equivariant Chow ring which is easier to compute than the classical Chow ring. The classical Chow ring is a quotient of the equivariant Chow ring.

The computations of this equivariant approach are explicit. They rely on the standard description of the cohomology of the Grassmannians and on a description of the tangent space to the Hilbert scheme at fixed points.

The main goal of this paper is to present this equivariant approach. We follow the general theory and we illustrate it with the case $S = \mathbb{P}^2$ and $n = 3$ as the main example.

In the last section, we bring our attention to graded Hilbert schemes, which played an important role in the equivariant computations. We present results on the set theoretic intersection of Schubert cells, which suggest that intersection theory on graded Hilbert schemes could be described in terms of combinatorics of plane partitions.

Throughout the paper, we use the formalism of Chow rings and work over any algebraically closed field k. When $k = \mathbb{C}$, the Chow ring coïncides with usual cohomology since the action of the two-dimensional torus T on S induces an action of T on $S^{[d]}$ with a finite number of fixed points.

1 Equivariant intersection theory

1.1 General results

In this section, we recall the facts about equivariant Chow rings that we need. To simplify the presentation, we work with rational coefficients and the notation $A^*(X) := A^*(X, \mathbb{Q})$ denotes the rational Chow ring.

The construction of an equivariant Chow ring associated with an algebraic space endowed with an action of a linear algebraic group has been settled by Edidin and Graham [3]. Their construction is modeled after the Borel construction in equivariant cohomology.

Proposition 1. *Let G be an algebraic group, X an equidimensional quasi-projective scheme with a linearized G-action and $i, j \in \mathbb{Z}$, $i \leq dim(X), j \geq 0$. There exists a representation V of G such that*

- *V contains an open set U on which G acts freely,*
- *$U \to U/G$ exists as a scheme and is a principal G bundle,*
- *$codim_V V \setminus U > \dim(X) - i$.*

The quotient $X_G = (X \times U)/G$ under the diagonal action exists as a scheme. The groups $A_i^G(X) := A_{i+\dim(V)-\dim(G)}(X_G)$ and $A_G^j(X) := A^G_{\dim(X)-j}(X)$ are independent of the choice of the couple (U, V).

Definition 2. *The group $A_G^i(X)$ is by definition the equivariant Chow group of X of degree i.*

Example 3. *If $G = T = (k^*)^n$ is a torus, then a possible choice for the couple (U, V) is $V = (k^l)^n$ with $l >> 0$, and $U = (k^l - \{0\})^n$ with T acting on V by $(t_1, \ldots, t_n)(x_1, \ldots, x_n) = (t_1x_1, \ldots, t_nx_n)$. The quotient U/T is isomorphic to $(\mathbb{P}^{l-1})^n$.*

Example 4. *Let p be a point and $T = (k^*)^n$ the torus acting trivially on p. Then $A_T^*(p) \simeq \mathbb{Q}[h_1, \ldots, h_n]$ where h_i has degree 1 for all i.*

Proof. By the above example, $A_T^*(p) = \lim_{l\to\infty} A^*((\mathbb{P}^{l-1})^n) = \lim_{l\to\infty} \mathbb{Q}[h_1, \ldots, h_n]/(h_1^l, \ldots, h_n^l) = \mathbb{Q}[h_1, \ldots, h_n]$, where h_i has degree 1 (according to the definition of the equivariant Chow group, the limit considered is a degreewise stabilisation thus the limit is the polynomial ring and not a power series ring). ∎

If X is smooth, then $(X \times U)/G$ is smooth too and $A_T^*(X)$ is a ring : the intersection of two classes $u, v \in A_T^*(X)$ takes place in the Chow ring $A^*((X \times U)/G)$.

Example 5. *The isomorphism $A_T^*(p) \simeq \mathbb{Q}[h_1, \dots, h_n]$ of the last example is an isomorphism of rings.*

Definition 6. *Let E be a G-equivariant vector bundle on X and $E_G \to X_G$ the vector bundle with total space $E_G = (E \times U)/G$. The equivariant Chern class $c_j^G(E)$ is defined by $c_j^G(E) = c_j(E_G) \in A^j(X_G) = A_G^j(X)$.*

The identification of $A_T^*(p)$ with a ring of polynomials R can be made intrinsic using equivariant Chern classes.

Proposition 7. *Let $\hat{T}$ be the character group of a torus $T \simeq (k^*)^n$. Any character $\chi \in \hat{T}$ defines a one-dimensional representation of T by $t.k = \chi(t)k$, hence an equivariant bundle over the point and an equivariant Chern class $c_1^T(\chi)$. The map $\chi \to c_1^T(\chi) \in A_T^1(p)$ extends to an isomorphism $R = Sym_{\mathbb{Q}}(\hat{T}) \to A_T^*(p)$, where $Sym_{\mathbb{Q}}(\hat{T})$ is the symmetric algebra over $\mathbb{Q}$ of the group $\hat{T}$.*

Example 8. *Let $T = k^*$ be the one dimensional torus acting on the projective space $\mathbb{P}^r = Proj\ k[x_0, \dots, x_r]$ by $t.(x_0 : \dots : x_r) = (t^{n_0}x_0 : \dots : t^{n_r}x_r)$. Then $A_T^*(\mathbb{P}^r) = \mathbb{Q}[t,h]/p(h,t)$ where $p(h,t) = \sum_{i=0}^r h^{r-i}e_i(n_0t, \dots, n_rt)$, e_i being the i-th elementary symmetric polynomial.*

Proof. X_T is the $\mathbb{P}^r$ bundle $\mathbb{P}(\mathcal{O}(n_0) \oplus \dots \oplus \mathcal{O}(n_r))$ over $\mathbb{P}^{l-1}$. The rational Chow ring of this projective bundle is $\mathbb{Q}[h,t]/(p(h,t), t^l)$. We have the result when l tends to ∞. ■

Example 9. *Let V be a representation of G and $G(k,V)$ the corresponding Grassmannian. Then $A_G^*(G(k,V))$ is generated as an R module by the equivariant Chern classes of the universal quotient bundle.*

Proof. The quotient $(G(k,V) \times U)/G$ is a Grassmann bundle over U/G with fiber isomorphic to $G(k,V)$. Since the Chow rings of Grassmann bundles are generated over the Chow ring of the base by the Chern classes of the universal quotient bundle, the result follows. ■

1.2 Results specific to the action of tori

Brion [1] pushed further the theory of equivariant Chow rings when the group is a torus T acting on a variety X.

Theorem 10. *[1] Let X be a smooth projective T-variety. The restriction morphism $i_T^* : A_T^*X \to A_T^*X^T$ is injective.*

Example 11. *Let $T = k^*$ act on $\mathbb{P}^1$ by $t.(x : y) = (tx : y)$. The inclusion $i_T^* : A_T^*(\mathbb{P}^1) \to A_T^*(\{0, \infty\}) = R^2$ identifies $A_T^*(\mathbb{P}^1)$ with the couples (P,Q) of polynomials $\in R = \mathbb{Q}[t]$ such that $P(0) = Q(0)$.*

Proof. Let $V = Vect(x, y)$ be the 2 dimensional vector space with $\mathbb{P}(V) = \mathbb{P}^1$. By the above $A_T^*(\mathbb{P}^1)$ is generated by the Chern classes of the universal quotient bundle as an R-module. On the point $\infty = ky \in \mathbb{P}(V)$, the quotient bundle Q is isomorphic to kx and T acts with character t. Thus $c_1(Q)_\infty = t$. Similarly, the restriction of Q to the point $0 = kx$ is a trivial equivariant bundle and $c_1(Q)_0 = 0$. Thus $c_1(Q)$ restricted to $\{0, \infty\}$ is $(0, t)$. Obviously, $c_0(Q) = (1, 1)$. Thus $A_T^*(\mathbb{P}^1) = \mathbb{Q}[t](0, t) + \mathbb{Q}[t](1, 1)$ as expected. ∎

If $T' \subset T$ is a one codimensional torus, the localisation morphism i_T^* factorizes: $A_T^*(X) \to A_T^*(X^{T'}) \xrightarrow{i_{T'}^*} A_T^*(X^T) = R^{X^T}$. Brion has shown

Theorem 12. *[1] Let X be a smooth projective variety with an action of T. The image $Im(i_T^*)$ satisfies $Im(i_T^*) = \cap_{T'} Im(i_{T'}^*)$ where the intersection runs over all subtori T' of codimension one in T.*

An important point is that the equivariant Chow groups determine the usual Chow groups. The fibers of $X_T \to U/T$ are isomorphic to X. Let $j : X \to X_T$ be the inclusion of a fiber and $j^* : A_T^*(X) \to A^*(X)$ the corresponding restriction.

Theorem 13. *[1] Let $R^+ = \hat{T}R \subset R$ be the set of polynomials with positive valuation. The morphism j^* is surjective with kernel $R^+ A_T^*(X)$.*

Example 14. $A^*(\mathbb{P}^1) = (\mathbb{Q}[t](t, 0) + \mathbb{Q}[t](1, 1))/(\mathbb{Q}[t]^+(t, 0) + \mathbb{Q}[t]^+(1, 1)) \simeq \mathbb{Q}[t]/(t^2)$. *The isomorphism sends* $(P = \sum p_i t^i, Q = \sum q_i t_i)$ *with* $p_0 = q_0$ *to* $(p_0, p_1 - q_1)$.

Finally, we have an equivariant Kunneth formula for the restriction to fixed points, proved in [5].

Theorem 15. *Let X and Y be smooth projective T-varieties with finite set of fixed points X^T and Y^T. Let $A_T^*(X) \subset R^{X^T}$, $A_T^*(Y) \subset R^{Y^T}$, and $A_T^*(X \times Y) \subset R^{X^T \times Y^T}$ the realisation of their equivariant Chow rings via localisation to fixed points. The canonical isomorphism $R^{X^T} \otimes_R R^{Y^T} \simeq R^{X^T \times Y^T}$ sends $A_T^*(X) \otimes A_T^*(Y)$ to $A_T^*(X \times Y)$.*

Example 16. *Let T be the one dimensional torus acting on $\mathbb{P}^1 \times \mathbb{P}^1$ by $t.([x_1, y_1], [x_2 : y_2]) = ([tx_1, y_1], [tx_2 : y_2])$. For each copy of $\mathbb{P}^1$, $A_T^*(\mathbb{P}^1)$ is generated as an R-module, by the elements $e = 1.\{0\} + 1.\{\infty\} = (1, 1)$ and $f = 0.\{0\} + t.\{\infty\} = (0, t)$. By the Kunneth formula, $A_T^*(\mathbb{P}^1 \times \mathbb{P}^1) \subset \mathbb{Q}[t]^4$ is generated by the elements $(1, 1, 1, 1) = (1, 1) \otimes (1, 1)$, $(0, t, 0, t) = (1, 1) \otimes (0, t)$, $(0, 0, t, t) = (0, t) \otimes (1, 1)$, $(0, 0, 0, t^2) = (0, t) \otimes (0, t)$ where the coordinates are the coefficients with respect to the four points $(a, b) \in \mathbb{P}^1 \times \mathbb{P}^1$ with $a, b \in \{0, \infty\}$.*

The strategy

Let's sum up the situation. The equivariant Chow ring $A_T^*(X)$ satisfies the usual functorial properties of a Chow ring: there is an induced pushforward for a proper morphism , an induced pull-back for a flat morphism, equivariant vector bundles have equivariant Chern classes... When the fixed point set X^T is finite, the computations are identified with calculations in products of polynomial rings.

Since it is possible to recover the usual Chow ring from the equivariant Chow ring, the point is to compute the equivariant Chow ring and its restriction to fixed points. The strategy that will be followed in the case of the Hilbert schemes is to use theorem 12 above. It is not obvious a priori that the geometry and the equivariant Chow rings of $(S^{[n]})^{T'}$ and their restriction to fixed points are easier to describe than the equivariant Chow ring of the original variety $S^{[n]}$. This is precisely the work to be done.

2 Iarrobino varieties and graded Hilbert scheme

In our computations of the Chow ring of the Hilbert scheme, a central role will be played by the Iarrobino varieties or graded Hilbert schemes that we introduce now.

Fix a set of dimensions $H = (H_d)_{d\in\mathbb{N}}$ such that $H_d = 0$ for $d >> 0$.

Definition-Proposition 17. *The Iarrobino variety $\mathbb{H}_{hom,H}$ is the set of homogeneous ideals $I = \oplus I_d \subset k[x,y]$ of colength $\sum_d H_d$ such that $codim(I_d, k[x,y]_d) = H_d$. This is a subvariety of $\mathbb{G} = \prod_{d \text{ s.t. } H_d\neq 0} Grass(H_d, k[x,y]_d)$. Moreover, $\mathbb{H}_{hom,H}$ is empty or irreducible.*

Proof. Each vector space I_d corresponds to a point in the Grassmannian $Grass(H_d, k[x,y]_d)$ and I corresponds to a point in the product $\mathbb{G} = \prod Grass(H_d, k[x,y]_d)$. Accordingly, $\mathbb{H}_{hom,H}$ is a subvariety of $\mathbb{G}$. The irreducibility of $\mathbb{H}_{hom,H}$ is shown in [9]. ■

The Chow ring $A_*(\mathbb{H}_{hom,H})$ is related to the Chow ring of $\mathbb{G}$, as shown by King and Walter [10].

Theorem 18. *Let $i : \mathbb{H}_{hom,H} \hookrightarrow \mathbb{G}$ denote the inclusion. The pull back $i^* : A^*(\mathbb{G}) \to A^*(\mathbb{H}_{hom,H})$ is surjective.*

There are natural generalisations of the Iarrobino varieties, introduced by Haiman and Sturmfels [8] and called graded Hilbert schemes. In our case, the graded Hilbert schemes we are interested in are the quasi-homogeneous Hilbert schemes.

Definition-Proposition 19. *Let $weight(x) = a \in \mathbb{N}$,$weight(y) = b \in \mathbb{N}$ with $(a,b) \neq (0,0)$. Consider the set $\mathbb{H}_{ab,H}$ of quasi-homogeneous ideals $I = \oplus_{d\in\mathbb{N}} I_d \subset k[x,y]$ with $codim(I_d, k[x,y]_d) = H_d$. There is a closed embedding $i : \mathbb{H}_{ab,H} \hookrightarrow \mathbb{G} = \Pi_{d\in\mathbb{N}, H_d\neq 0} Grass(H(d), k[x,y]_d)$.*

Remark 20. *We could also consider the case $a \in \mathbb{Z}$ and/or $b \in \mathbb{Z}$. However when $ab < 0$, $\mathbb{H}_{ab,H}$ would be empty or a point. Moreover, changing (a,b) with*

$(-a,-b)$ *gives an isomorphic graduation. Consequently, any non trivial variety* $\mathbb{H}_{ab,H}$ *can be realized with* $a \geq 0$ *and* $b \geq 0$*. We thus consider* $a \in \mathbb{N}$ *and* $b \in \mathbb{N}$ *without loss of generality.*

One wants to extend in this context the results by Iarrobino and King-Walter.

Extending Iarrobino's irreducibility result is possible, but not immediate, as Iarrobino's argument does not extend.

Theorem 21. *[7] The graded Hilbert scheme* $\mathbb{H}_{ab,H}$ *is empty or irreducible.*

Idea of the proof. Since $\mathbb{H}_{ab,H}$ is smooth as the fixed locus of $(\mathbb{A}^2)^{[\sum H_d]}$ under the action of a one dimensional torus, irreducibility is equivalent to connectedness. To prove connectedness, $\mathbb{H}_{ab,H}$ admits a stratification where the cells are the inverse images of the product of Schubert cells on $\mathbb{G}$ by the immersion $\mathbb{H}_{ab,H} \to \mathbb{G}$. Each cell is an affine space. The cells being connected spaces, it suffices to connect together the different cells to prove the connectedness of $\mathbb{H}_{ab,H}$. To this aim, one writes down explicit flat families over $\mathbb{P}^1$. These flat families correspond to curves drawn on $\mathbb{H}_{ab,H}$ that give the link between the different cells. ■

The arguments of King and Walter generalise easily to the quasi-homogeneous case. Moreover, the affine plane $\mathbb{A}^2$ is a toric variety. The action of $T = k^* \times k^*$ on $\mathbb{A}^2$ induces an action of T on $\mathbb{H}_{ab,H}$. It is possible to generalise the results of King-Walter to the equivariant setting. With minor modifications of their method, one gets the following theorem.

Theorem 22. *The natural restriction morphisms* $i^* : A^*(\mathbb{G}) \to A^*(\mathbb{H}_{ab,H})$ *and* $i_T^* : A_T^*(\mathbb{G}) \to A_T^*(\mathbb{H}_{ab,H})$ *are surjective.*

Corollary 23. *The above theorem induces a description of* $A_T^*(\mathbb{H}_{ab,H})$*. Indeed, if* $j : (\mathbb{H}_{ab,H})^T \to \mathbb{H}_{ab,H}$ *is the inclusion of the* T*-fixed points, then* $A^*(\mathbb{H}_{ab,H}) = im(j_T^*) = im(ij)_T^*$ *and the computation of* $(ij)_T^* : A_T^*(\mathbb{G}) \to A_T^*(\mathbb{H}_{ab,H}^T)$ *follows easily from the description of* $A_T^*(\mathbb{G})$ *using equivariant Chern classes.*

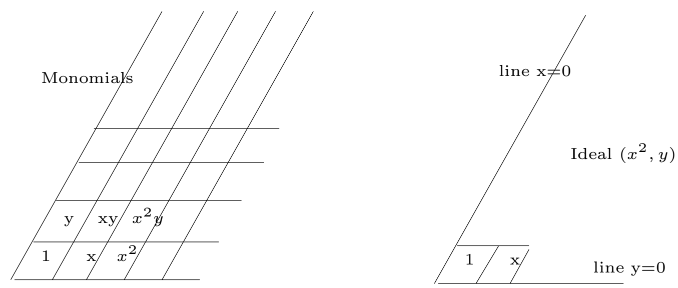
Example 24. *Let* $\mathbb{H}_{hom,H}$ *be the Iarrobino variety parametrising the homogeneous ideals in* $k[x,y]$ *with Hilbert function* $H = (1,1,0,0,\dots)$*. The torus* $T = k^*$ *acts by* $t(x,y) = (tx,y)$*. The two fixed points are the ideals* (x^2,y) *and* (x,y^2)*. The Iarrobino variety* $\mathbb{H}_{hom,H}$ *embeds in the Grassmannian* $Grass(1,k[x,y]_1)$ *of one dimensional subspace of linear forms. The universal quotient* Q *over the Grassmannian is a* T*-bundle. Its restriction over the points* (x^2,y) *and* (x,y^2) *is a* T*-representation corresponding to the characters* $t \mapsto t$ *and* $t \mapsto 1$*. Thus the first Chern class of the universal quotient is* $(t,0) \in R^{\mathbb{H}^T_{hom,H}} = R^2$*. Finally* $A_T^*\mathbb{H}_{hom,H} = (c_1^T(Q), c_0^T(Q)) = R(t,0) + R(1,1)$.

3 Geometry of the fixed locus

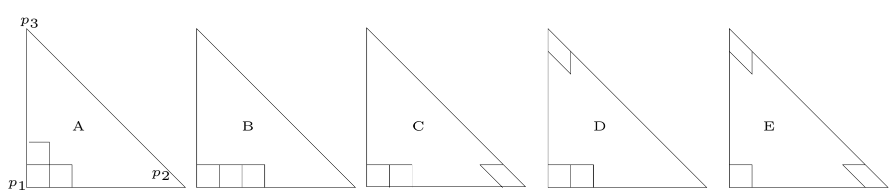
3.1 Geometry of $(S^{[d]})^T$

Let S be a smooth projective toric surface. The 2-dimensional torus T which acts on S acts naturally on $S^{[d]}$. According to theorems 10 and 12, one main ingredient to describe the equivariant Chow ring $A_T^*(S^{[d]})$ is to describe the geometry of the fixed loci $(S^{[d]})^T$ and $(S^{[d]})^{T'}$ of the Hilbert scheme $S^{[d]}$ under the action of the two dimensional torus T and under the action of any one-dimensional torus $T' \subset T$.

Consider the example $S = \mathbb{P}^2 = Proj\ k[X,Y,Z]$ and $(\mathbb{P}^2)^{[3]}$ the associated Hilbert scheme. The action of $T = k^* \times k^*$ on $\mathbb{P}^2$ is $(t_1,t_2).X^aY^bZ^c = (t_1X)^a(t_2Y)^bZ^c$. First, we describe the finite set $((\mathbb{P}^2)^{[3]})^T$. Obviously, a subscheme $Z \in ((\mathbb{P}^2)^{[3]})^T$ has a support included in $\{p_1,p_2,p_3\}$ where the p_i's are the toric points of $\mathbb{P}^2$ fixed under T. Through each toric point, there are two toric lines with local equations $x = 0$ and $y = 0$. Since Z is T-invariant, the ideal $I(Z)$ is locally generated by monomials $x^\alpha y^\beta$.



Using the two lines, we can represent graphically the monomials $x^\alpha y^\beta$ as shown. An ideal $I \subset k[x,y]$ generated by monomials is represented by the set of monomials which are not in the ideal. For instance, the ideal (x^2,y) which does not contain the monomials $1, x$ is drawn in the above figure.



Proposition 25. *In $(\mathbb{P}^2)^{[3]}$, there are a finite number of subschemes invariant under the action of the two-dimensional torus. Up to permutation of the projective variables X,Y,Z of $\mathbb{P}^2$, there are five such subschemes A,B,C,D,E.*

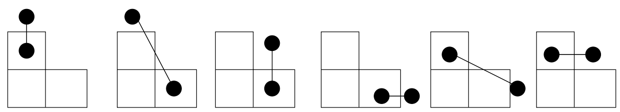
Proof. By the above, the invariant subschemes are represented by monomials around each toric point of $\mathbb{P}^2$. The number of monomials is the degree of the subscheme, ie. 3 in our situation. Up to permutation of the axes, all the possible cases A, B, C, D, E are given in the picture. ■

In general, we have:

Proposition 26. *The points of* $(S^{[d]})^T$ *are in one-to-one correspondence with the tuples of staircases* $(E_i)_{i \in S^T}$ *such that* $\sum_{i \in S^T} cardinal(E_i) = d$.

3.2 Tangent space at $p \in (S^{[d]})^T$

We recall the description of the tangent space at a point $p \in (S^{[d]})^T$ ([7], but see also [4] for an other description) .



The six cleft couples for the 6-dimensional tangent space

For simplicity, we suppose that the subscheme p is supported by a single point. Recall that we have associated to p the staircase F of monomials $x^a y^b$ not in $I(p)$ where x, y are the toric coordinates around the support of p. A cleft for F is a monomial $m = x^a y^b \notin F$ with ($a = 0$ or $x^{a-1}y^b \in F$) and ($b = 0$ or $x^a y^{b-1} \in F$). We order the clefts of F according to their x-coordinates: $c_1 = y^{b_1}, c_2 = x^{a_2}y^{b_2}, \ldots, c_p = x^{a_p}$ with $a_1 = 0 < a_2 < \cdots < a_p$. An x-cleft couple for F is a couple $C = (c_k, m)$, where c_k is a cleft ($k \neq p$), $m \in F$, and $mx^{a_{k+1}-a_k} \notin F$. By symmetry, there is a notion of y-cleft couple for F. The set of cleft couples is by definition the union of the (x or y)-cleft couples.

Theorem 27. *The vector space* $T_p S^{[d]}$ *is in bijection with the formal sums* $\sum \lambda_i C_i$*, where* C_i *is a cleft couple for* p.

Example 28. $(\mathbb{P}^2)^{[3]}$ *is a 6 dimensional variety. A basis for the tangent space at a point with local equation* (x^2, xy, y^2) *is the set of cleft couples shown in the figure.*

With equivariant techniques, it is desirable to describe the tangent space as a representation. The torus T acts on the monomials c_k and m with characters χ_k and χ_m. We let $\chi_C = \chi_m - \chi_k$.

Proposition 29. *Under the correspondence of the above theorem, the cleft couple* C *is an eigenvector for the action of* T *with character* χ_C.

3.3 Geometry of $(S^{[d]})^{T'}$

We come now to the description of $(S^{[d]})^{T'}$ where T' is a one-dimensional subtorus of T. We start with an example.

Example 30. *If $S = \mathbb{P}^2$ and $T' \simeq k^*$ acts on $\mathbb{A}^2 \subset \mathbb{P}^2$ via $t.(x,y) = (tx, ty)$ the irreducible components of $((\mathbb{P}^2)^{[3]})^{T'}$ through the points A, B, C, D, E are isomorphic to an isolated point, $\mathbb{P}^1, \mathbb{P}^1 \times \mathbb{P}^1, \mathbb{P}^1 \times \mathbb{P}^1, \mathbb{P}^2$.*

Proof. The tangent space to $((\mathbb{P}^2)^{[3]})^{T'}$ at a point $p \in ((\mathbb{P}^2)^{[3]})^T$ is the tangent space to $(\mathbb{P}^2)^{[3]}$ invariant under T'. Using the description of the tangent space to $(\mathbb{P}^2)^{[3]}$ as a representation in the previous section, one computes the dimension of the tangent space of $((\mathbb{P}^2)^{[3]})^{T'}$ at each of the points A, B, C, D, E. The corresponding dimensions are $0, 1, 2, 2, 2$.

In particular, an irreducible variety through A (resp. B, C, D, E) invariant under T' with dimension 0 (resp. $1, 2, 2, 2$) is the irreducible component of $((\mathbb{P}^2)^{[3]})^{T'}$ through A (resp. through B, C, D, E). It thus suffices to exhibit such irreducible varieties.

A is an isolated point and there is nothing to do.

The component $\mathbb{P}^1$ passing through B can be described geometrically. The set of lines through the origin of $\mathbb{A}^2$ is a $\mathbb{P}^1$. To each such line D, we consider the subscheme Z of degree 3 supported by the origin and included in D. The set of such subschemes Z moves in a $\mathbb{P}^1$. It is the component of $((\mathbb{P}^2)^{[3]})^{T'}$ through B. This component can be identified with the Iarrobino variety with Hilbert function $H = (1, 1, 1, 0, 0, \dots)$.

With the same set of lines through the origin, one can consider the subschemes Z of degree 2 supported by the origin and included in a line D. Since Z moves in a $\mathbb{P}^1$, $Z \cup p$ where p is a point on the line at infinity moves in a $\mathbb{P}^1 \times \mathbb{P}^1$. It is a component through C and D.

Finally, a subscheme Z of degree 2 included in the line at infinity moves in a $\mathbb{P}^2$. Thus the union of Z and the origin moves in a $\mathbb{P}^2$ which is the component of $(S^{[d]})^{T'}$ through E. ■

The following proposition says that all but a finite number of representations of T' give a trivial result.

Proposition 31. *Let a and b be coprime with $|a| \geq 3$ or $|b| \geq 3$. Suppose that $T' = k^*$ acts on $\mathbb{A}^2 \subset \mathbb{P}^2$ via $t.(x,y) = (t^a x, t^b y)$. Then the irreducible components of $((\mathbb{P}^2)^{[3]})^{T'}$ through the points A, B, C, D, E are isolated points.*

Proof. The tangent space at these points is trivial. ■

We see in the examples that the irreducible components of $(\mathbb{P}^2)^{T'}$ are the three toric points of $\mathbb{P}^2$ for a general T'. For some special T', there are two components, namely a toric point and the line joining the remaining two toric points.

For a general toric surface S, the situation is similar.

Proposition 32. *For any T' one codimensional subtorus of T, $S^{T'}$ is made up of isolated toric points (w_i) and of toric lines (y_j) joining pairs of the remaining toric points.*

Definition 33. *We denote by $PFix(T') = \{w_i\}$ the set of isolated toric points in $S^{T'}$ and by $LFix(T') = \{y_j\}$ the set of lines in $S^{T'}$.*

In the $(\mathbb{P}^2)^{[3]}$ example, we identified the irreducible components of $((\mathbb{P}^2)^{[3]})^{T'}$ with products $B_1 \times \ldots \times B_r$ of projective spaces. Some of the projective spaces were identified with a graded Hilbert scheme. For instance, the component of $((\mathbb{P}^2)^{[3]})^{T'}$ through B has been identified with the Iarrobino variety with Hilbert function $H = (1, 1, 1, 0, 0, \ldots)$.

In general, the irreducible components of $(S^{[d]})^{T'}$ are products $B_1 \times \ldots \times B_r$ where the components B_i are projective spaces or graded Hilbert schemes.

Let's look at the situation more closely to describe these components. If $Z \in (S^{[d]})^{T'}$, the support of Z is invariant under T'. The invariant locus in S is a union of isolated points (w_i) and lines (y_i). We denote by $W_i(Z)$ and $Y_i(Z)$ the subscheme of Z supported respectively by the point w_i and by the line y_i. By construction, we have:

Proposition 34. *A subscheme $Z \in (S^{[d]})^{T'}$ admits a decomposition $Z = \cup_{w_i \in PFix(T')} W_i(Z) \cup_{y_i \in LFix(T')} Y_i(Z)$.*

Obviously, $(S^{[d]})^{T'}$ is not irreducible : when Z moves in in a connected component, the degree of $W_i(Z)$ and $Y_i(Z)$ should be constant. But fixing the degree of $W_i(Z)$ and $Y_i(Z)$ is not sufficient to characterize the irreducible components of $(S^{[d]})^{T'}$ as shown by the components identified in example 30.

Example 35. *The components of $((\mathbb{P}^2)^{[3]})^{T'}$ through A and B are 2 distinct Iarrobino varieties corresponding to the same degrees* 3 *on the point* $(0, 0)$ *in* $\mathbb{A}^2$ *and 0 on the line at infinity.*

The finer invariant which distinguishes the irreducible components is similar to the one used for the Iarrobino varieties. It is a Hilbert function taking into account the graduation provided by the action.

Example 36. *Let $O_A = k[x, y]/(x^2, xy, y^2)$ and $O_B = k[x, y]/(y, x^3)$ be the ring functions corresponding to the points A and B in example 30. The action of T' on O_A is diagonalizable with characters $1, t, t$ whereas the action of T' on O_B acts with characters $1, t, t^2$.*

Let $Z = Spec\ O_Z \in (S^{[d]})^{T'}$. The torus T' acts on O_Z with a diagonalizable action. In symbols, $O_Z = \oplus V_\chi$, where $V_\chi \subset O_Z$ is the locus where T' acts through the character $\chi \in \hat{T}'$.

Definition 37. *If $Z \in (S^{[d]})^{T'}$, we define $\underline{H}_Z : \hat{T}' \to \mathbb{N}$, $\chi \to \dim V_\chi$ and we let $Z = \cup W_i(Z) \cup Y_i(Z)$ the decomposition of Z introduced above. The tuple of functions $H_Z = (\underline{H}_{W_i(Z)}, \underline{H}_{Y_i(Z)})$ indexed by the connected components $\{w_i, y_j\}$ of $S^{T'}$ is by definition the Hilbert function associated to Z.*

Theorem 38. *[7] The set of Hilbert functions $H_Z = (\underline{H}_{W_i(Z)}, \underline{H}_{Y_i(Z)})$ corresponding to the subschemes $Z \in (S^{[d]})^{T'}$ is a finite set. Moreover, the irreducible components of $(S^{[d]})^{T'}$ are in one-to-one correspondence with this set of Hilbert functions H_Z.*

The decomposition of $(S^{[d]})^{T'}$ as a product follows easily from the description of the Hilbert functions. Let $B_{w_i}(H)$ be the set of subschemes $W_i \subset S$, T' fixed, supported by the fixed point w_i with $\underline{H}_{W_i} = H$. Similarly, let $B_{y_i}(H)$ be the set of subschemes $Y_i \subset S$, T' fixed, with support in the fixed line y_i and $\underline{H}_{Y_i} = H$. A reformulation of the above is thus:

Theorem 39. $S^{[d],T'} = \cup B_{w_1}(H_{w_1}) \times \ldots B_{w_r}(H_{w_r}) \times B_{y_1}(H_{y_1}) \ldots \times B_{y_s}(H_{y_s})$ *where* $\{w_1, \ldots, w_r\} = PFix(T')$ *are the isolated fixed points,* $\{y_1, \ldots, y_s\} = LFix(T')$ *are the fixed lines, and the union runs through all the possible tuples of Hilbert functions* $(H_{w_1}, \ldots, H_{w_r}, H_{y_1}, \ldots, H_{y_s})$.

The next two propositions identify geometrically the factors $B_{w_i}(H_{w_i})$ and $B_{y_j}(H_{y_j})$ of the above product.

By the very definition, we have:

Proposition 40. *For every isolated fixed point* $w_i \in PFix(T')$ *and every function* $H_{w_i} : \hat{T}' \to \mathbb{N}$, *the variety* $B_{w_i}(H_{w_i})$ *is a graded Hilbert scheme.*

Example 41. *In example 30, take* $w = (0,0) \in \mathbb{A}^2$ *and Hilbert function* $H(\chi) = 1$ *for the three characters* $\chi = 1, t, t^2$ *and* $H(\chi) = 0$ *otherwise. Then* $B_w(H)$ *is the component of* $((\mathbb{P}^2)^{[3]})^{T'}$ *passing through* B, *which has been identified with the homogeneous Hilbert scheme* $H_{hom,H}$.

As for the other components, we have:

Proposition 42. *For every fixed line* $y_i \in LFix(T')$ *and every function* $H_{y_i} : \hat{T}' \to \mathbb{N}$, $B_{y_i}(H_{y_i})$ *is a product of projective spaces.*

Illustration of the last proposition on an example. Consider $T' = k^*$ acting on $\mathbb{A}^2 \subset \mathbb{P}^2$ via $t.(x,y) = (tx, y)$. The line $x = 0$ is T'-fixed. We say that a scheme Z is horizontal of length n if it is in the affine plane and $I(Z) = (y - a, x^n)$, or if it is a limit of such schemes when the support $(0, a)$ moves to infinity. Two horizontal schemes Z_1 and Z_2 of respective length $n_1 \neq n_2$ move in a $\mathbb{P}^1 \times \mathbb{P}^1$. When the supports of Z_1 and Z_2 are distinct, $Z_1 \cup Z_2$ is a scheme of length $n_1 + n_2$. We thus obtain a rational function $\mathbb{P}^1 \times \mathbb{P}^1 \cdots > (\mathbb{P}^2)^{[n_1+n_2]}$. The schemes being horizontal, the limit of $Z_1 \cup Z_2$ is completely determined by the support when the schemes Z_1 and Z_2 collide. More formally, the rational function extends to a well defined morphism $\varphi : \mathbb{P}^1 \times \mathbb{P}^1 \to (\mathbb{P}^2)^{[n_1+n_2]}$. This morphism is an embedding and gives an isomorphism between $\mathbb{P}^1 \times \mathbb{P}^1$ and one of the components $B_{y_i}(H_{y_i})$ introduced above.

If $n_1 = n_2$ in the above paragraph, φ is not an embedding any more because of the action of the symmetric group which exchanges the roles of Z_1 and Z_2. But taking the quotient, we obtain an embedding $\mathbb{P}^2 = (\mathbb{P}^1 \times \mathbb{P}^1)/\sigma_2 \to (\mathbb{P}^2)^{[n_1+n_2]}$ which is an isomorphism on a component. ■

4 Description of the Chow ring

4.1 Description using generators

Let $T' \subset T$ be a one dimensional subtorus. Recall that each irreducible component C of $(S^{[d]})^{T'}$ is a product $B_{w_1}(H_{w_1}) \times \dots B_{w_r}(H_{w_r}) \times B_{y_1}(H_{y_1}) \dots \times B_{y_s}(H_{y_s})$ where $\{w_1, \dots, w_r\} = PFix(T')$ and $\{y_1, \dots, y_s\} = LFix(T')$.

The factors $B_{w_i}(H_{w_i})$ are graded Hilbert schemes. We have seen in example 24 the computation of generators for their equivariant Chow ring. We denote by $M_{w_i,T',H_{w_i}}$ this equivariant Chow ring.

The factors $B_{y_i}(H_{y_i})$ are product of projective spaces. We have seen in examples 8, 9 or 16 the computation of generators for their equivariant Chow ring. We denote by $N_{y_i,T',H_{y_i}}$ this equivariant Chow ring.

Then the equivariant Chow ring of the component C is given by the Kunneth formula of theorem 15:

$$A_T^*(C) = \bigotimes_{w_i \in PFix(T')} M_{w_i,T',H_{w_i}} \bigotimes_{y_i \in Lfix(T')} N_{y_i,T',H_{y_i}}$$

When $H = (H_{w_1}, \dots, H_{w_r}, H_{y_1}, \dots, H_{y_s})$ runs through the possible Hilbert functions to describe all the irreducible components C of $(S^{[d]})^{T'}$ and using theorem 12, we finally get:

Theorem 43. *[5]*

$$A_T^*(S^{[d]}) = \bigcap_{T' \subset T} \bigoplus_{\#H=d} (\bigotimes_{w_i \in PFix(T')} M_{w_i,T',H_{w_i}} \bigotimes_{y_i \in LFix(T')} N_{y_i,T',H_{y_i}})$$

4.2 Second description of the Chow ring: From generators to relations

In the last formula, the modules $M_{w_i,T',H_{w_i}}$ and $N_{y_i,T',H_{y_i}}$ were described with explicit generators. It is possible to adopt the relations point of view rather than the generators point of view. This is an algebraic trick which relies on Bott's integration formula, proved by Edidin and Graham in an algebraic context. The equivariant Chow ring is then described as a set of tuples of polynomials satisfying congruence relations.

The proposition below that makes the transition from generators to relations involves equivariant Chern classes of the restrictions $T_{S^{[d]},p}$ of the tangent bundle $T_{S^{[d]}}$ at fixed points $p \in (S^{[d]})^T$. Since we have described the fiber of the tangent bundle at these points as a T-representation, computing the equivariant Chern classes is straightforward and the set of relations can be computed.

Proposition 44. *[5] Let $\beta_i = (\beta_{ip})_{p \in (S^{[d]})^T}$ be a set of generators of the $\mathbb{Q}[t_1, t_2]$-module $i_T^* A_T^*(S^{[d]}) \subset \mathbb{Q}[t_1, t_2]^{(S^{[d]})^T}$. Let $\alpha = (\alpha_p) \in \mathbb{Q}[t_1, t_2]^{(S^{[d]})^T}$. Then the following conditions are equivalent.*

- $\alpha \in i_T^* A_T^*(S^{[d]})$

- *$\forall i$, the congruence*

$$\sum_{p\in(S^{[d]})^T}(\alpha_p\beta_{ip}\prod_{q\neq p}c^T_{\dim S^{[d]}}(T_{S^{[d]},q}))\equiv 0\ (mod \prod_{p\in(S^{[d]})^T}c^T_{\dim S^{[d]}}(T_{S^{[d]},p}))$$

holds.

We apply the method to $(\mathbb{P}^2)^{[3]}$. There are 22 fixed points thus the equivariant Chow ring is a subring of $\mathbb{Q}[t_1,t_2]^{22}$. Five of the fixed points $A,\dots,E$ have been introduced in the examples. The other fixed points are obtained from these five by a symmetry. For instance, $A_{12}=\sigma(A)$ where σ is the toric automorphism of $\mathbb{P}^2$ exchanging the points p_1 and p_2.

Theorem 45. *[5] The equivariant Chow ring $A^*_T((\mathbb{P}^2)^{[3]})\subset\mathbb{Q}[t_1,t_2]^{\{A,A_{12},\dots,E\}}$ is the set of linear combinations $aA+a_{12}A_{12}+\dots+eE$ satisfying the relations*

- $a+a_{13}-d-d_{13}\equiv 0\ (mod\ t_2^2)$
- $d-d_{13}\equiv 0\ (mod\ t_2)$
- $a-a_{13}\equiv 0\ (mod\ t_2)$
- $a-b\equiv 0\ (mod\ 2t_1-t_2)$
- $b-b_{13}\equiv 0\ (mod\ t_2)$
- $-b+3c-3c_{12}+b_{12}\equiv 0\ (mod\ t_1^3)$
- $-b+c+c_{12}-b_{12}\equiv 0\ (mod\ t_1^2)$
- $3b-c+c_{12}+-3b_{12}\equiv 0\ (mod\ t_1)$
- $b-b_{23}\equiv 0\ (mod\ t_2-t_1)$
- $c-d+c_{23}-d_{23}\equiv 0\ (mod\ (t_1-t_2)^2)$
- $c+d-c_{23}-d_{23}\equiv 0\ (mod\ (t_1-t_2))$
- $c_{23}-d_{23}\equiv 0\ (mod\ (t_1-t_2))$
- $c-c_{13}\equiv 0\ (mod\ t_2)$
- $d-2e+d_{12}\equiv 0\ (mod\ t_1^2)$
- $d-d_{12}\equiv 0\ (mod\ t_1)$
- *all relations deduced from the above by the action of the symmetric group S_3.*

The Chow ring $A^*((\mathbb{P}^2)^{[3]})$ is the quotient of $A^*_T((\mathbb{P}^2)^{[3]})$ by the ideal generated by the elements $fA+\dots+fE$, $f\in\mathbb{Q}[t_1,t_2]^+$.

5 Graded Hilbert schemes revisited

The quasi-homogeneous Hilbert schemes played a central role in the computation of $A^*_T(S^{[d]})$ and their Chow ring was computed using equivariant techniques. In this section, we present a result suggesting that their Chow ring could admit an alternative description in terms of combinatorics of partitions.

To simplify the notations, we restrict from now on our attention to the homogeneous case of Iarrobino varieties, but the statements below can be formulated in the quasi-homogeneous case [6].

Recall that the graded Hilbert scheme $\mathbb{H}_{hom,H}$ embeds in a product of Grassmannians: $\mathbb{H}_{hom,H} \hookrightarrow \mathbb{G} = \prod_{d\in\mathbb{N},H_d\neq 0} Grass(H_d, k[x,y]_d)$. The Grassmannians are stratified by their Schubert cells, constructed with respect to the flag $F_1 = Vect(x^d) \subset F_2 = Vect(x^d, x^{d-1}y) \subset \dots F_{d+1} = k[x,y]_d$. The product $\mathbb{G}$ is stratified by the product of Schubert cells, and $\mathbb{H}_{hom,H}$ is stratified by the restrictions of these products of Schubert cells. We still call these restrictions Schubert cells on $\mathbb{H}_{hom,H}$.

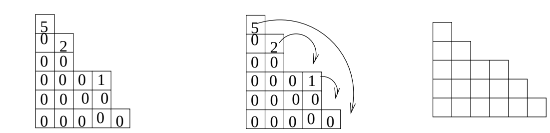
Example 46. *A Schubert cell of a Iarrobino variety $\mathbb{H}_{hom,H}$ contains a unique monomial ideal $I \subset k[x,y]$ that we represent as usual by the set of monomials $E(I) = \{x^a y^b \notin I\}$.*

The Grassmannians in the product $\mathbb{G}$ are trivial when $H_d = \dim k[x,y]_d$. When the numbers of non trivial Grassmannians in $\mathbb{G}$ is one, the inclusion $\mathbb{H}_{hom,H} \subset \mathbb{G}$ is an isomorphism. In this Grassmannian case, the closures of the Schubert cells form a base of $A^*(\mathbb{H}_{hom,H})$ and the intersection is classically described in terms of combinatorics involving the partitions associated to the cells.

Question: *Is it possible to describe the intersection theory in terms of partitions when $\mathbb{H}_{hom,H}$ is not a Grassmannian ?*

The intersection theory when $\mathbb{H}_{hom,H}$ is not a Grassmannian is more complicated than in the Grassmannian case. On the set theoretical level, the intersection $\overline{C} \cap \overline{D}$ between the closures of two cells C and D is difficult to determine. The closure $\overline{C}$ of a cell C is not a union of cells any more.

However, a necessary condition for the incidence $\overline{C} \cap D \neq \emptyset$ is known and expressed in terms of reverse plane partitions with shape $E(J)$, where J is the unique monomial ideal in D.



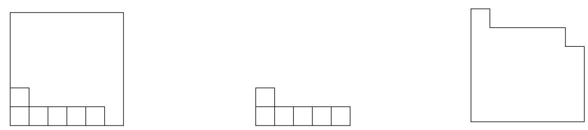
E(J) and the plane partition E(I)

Definition 47. *A reverse plane partition with shape $E \subset \mathbb{N}^2$ is a two-dimensional array of integers n_{ij}, $(i,j) \in E$ such that $n_{i,j} \leq n_{i,j+1}$, $n_{i,j} \leq n_{i+1,j}$*

Definition 48. *A monomial ideal I is linked to a monomial ideal J by a reverse plane partition n_{ij} with support $E(J)$ if $E(I) = \{x^{a+n_{a,b}} y^{b-n_{ab}}, (a,b) \in E(J)\}$*

Example 49. *In the above figure, $E(I)$ is linked to $E(J)$.*

Definition 50. *If $I \subset k[x, y]$ is a monomial ideal of colength n, the complement of I is the ideal $C(I)$ such that $E(C(I))$ contains the monomials $x^a y^b$, $a < n$, $b < n$ with $x^{n-1-a} y^{n-1-b} \in I$ (see the figure below).*



E(I) E(C(I))

Theorem 51. *[6] Let C and $C' \subset \mathbb{H}_{hom,H}$ be two cells containing the monomial ideals I and I'. If $\overline{C} \cap C' \neq \emptyset$, then*

- *I is linked to I' by a reverse plane partition.*
- *$C(I)$ is linked to $C(I')$ by a reverse plane partition.*

By analogy with the Grassmannian case, we are led to the following open question: Can we describe the intersection theory on the Graded Hilbert schemes in terms of combinatorics of the plane partitions ?

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