Defining integer valued functions in rings of continuous definable functions over a topological field

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October 29, 2018

Abstract

Let $\mathcal{K}$ be an expansion of either an ordered field $(K, \leqslant)$, or a valued field (K, v). Given a definable set $X \subseteq K^m$ let $\mathcal{C}(X)$ be the ring of continuous definable functions from X to K. Under very mild assumptions on the geometry of X and on the structure $\mathcal{K}$, in particular when $\mathcal{K}$ is o-minimal or P-minimal, or an expansion of a local field, we prove that the ring of integers $\mathbb{Z}$ is interpretable in $\mathcal{C}(X)$. If $\mathcal{K}$ is o-minimal and X is definably connected of pure dimension $\geqslant 2$, then $\mathcal{C}(X)$ defines the subring $\mathbb{Z}$. If $\mathcal{K}$ is P-minimal and X has no isolated points, then there is a discrete ring $\mathcal{Z}$ contained in K and naturally isomorphic to $\mathbb{Z}$, such that the ring of functions $f \in \mathcal{C}(X)$ which take values in $\mathcal{Z}$ is definable in $\mathcal{C}(X)$.

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1 Introduction

We give a first order definition of the ring of integers within rings of continuous functions that are first order definable in expansions of ordered and valued fields. Before describing a more technical outline of the contents, we explain the context of the article and the results in a colloquial way.

Rings of continuous functions on topological spaces are central objects in functional analysis, topology and geometry. To name an example: They are rings of sections for the sheaf of continuous (say, real valued) functions on a topological space and as such play the algebraic role in the study of topological (Hausdorff) spaces.

By a ring of *definable* continuous functions we mean the following. Let $\mathcal{K}$ be an expansion of an ordered field $(K, \leqslant)$ (e.g. the real field) or a valued field (K, v) (e.g. the p-adics). In both cases K carries a topology that turns K into a topological field. Let $X \subseteq K^n$ be definable (with parameters) in $\mathcal{K}$ and let $C(X)$ be the set of all functions $X \longrightarrow K$ that are continuous and definable in $\mathcal{K}$. If $\mathcal{K}$ is the real or the p-adic field, definable is the same as semi-algebraic. Then $C(X)$ is a ring and similar to the classical case mentioned above, $C(X)$ carries the algebraic information of the *definable* homeomorphism type of X. This is amply illustrated in the case when $\mathcal{K}$ is a real closed field: Let $\mathtt{S}$ be the category of semi-algebraic subsets $X \subseteq \mathcal{K}^n$, $n \in \mathbb{N}$, with continuous semi-algebraic maps as morphisms. Let $\mathtt{C}$ be the category of all the rings $C(X)$ of continuous $\mathcal{K}$-definable functions and $\mathcal{K}$-algebra homomorphisms as morphisms. Then the functor $C : \mathtt{S} \longrightarrow \mathtt{C}$ that sends X to $C(X)$ and a morphism $f : X \longrightarrow Y$ to the $\mathcal{K}$-algebra homomorphism $C(Y) \longrightarrow C(X);\ g \mapsto g \circ f$, is an anti-equivalence of categories. This can also be seen in analogy with algebraic geometry, where varieties defined over a field K are anti-equivalent to affine K-algebras, leading to the modern language of schemes. For semi-algebraic sets the machinery is developed in this vain in terms of so called semi-algebraic spaces, see [Sch87]. The same connection exists between definable sets X in the p-adic context and the rings $C(X)$ for that context. In each case, one may thus think of rings of continuous definable functions as "coordinate rings" when studying topological properties of definable sets.

Model theoretic studies of rings of continuous (definable) functions may be found in [PS02], [Tre07] for the case of real closed fields and in [Bél91],[Bél95], [GT08] in the p-adic case. Model theory of rings of continuous functions on topological spaces have been studied in [Che80], which serves as a main source for inspiration for us.

In this article we study model theoretic properties of rings of continuous definable functions under fairly mild assumption on the base structure. The principal goal is to show that, in most cases, these rings *interpret* the ring of integers in a uniform way; in particular these rings are not decidable. This is established in Theorem 5.5. Now undecidability was already known in some cases. For example, one can interpret the lattice of closed subsets of $\mathbb{R}^n$ in the ring $C(\mathbb{R}^n)$ of continuous semi-algebraic functions $\mathbb{R}^n \longrightarrow \mathbb{R}$. When $n \geqslant 2$, this lattice itself is undecidable by [Grz51] and indeed interprets the ring of integers, see [Tre17]. However, in this lattice, one cannot interpret $C(\mathbb{R}^n)$ in a uniform way as we will see in Remark 5.8.

In the p-adic case, the first author has shown in [Dar06] that the lattice of closed subsets of $\mathbb{Q}_p^n$ is even decidable and it was unknown whether $C(\mathbb{Q}_p^n)$ has a decidable theory at all.

In section 6 we show that in many cases the rings $C(X)$ actually *define* the subset of constant functions with values in a natural isomorphic copy of $\mathbb{Z}$, see Theorem 6.7 for the precise formulation. When $\mathcal{K}$ is an o-minimal expansion of a field and X is definably connected of local dimension $\geqslant 2$ everywhere, then indeed $C(X)$ defines the ring of constant functions with values in $\mathbb{Z}$. As a consequence, when $\mathcal{K}$ expands the real field we obtain that the real field, seen as constant functions $\mathbb{R}^n \longrightarrow \mathbb{R}$ is definable in $C(X)$. This implies that the projective hierarchy is definable in these rings.

Detailed description of the main results and set up.

We consider an expansion $\mathcal{K}$ of a topological field $(K, \mathcal{O})$ where $\mathcal{O}$ is either the unit interval $[-1, 1]$ of a total order $\leqslant$ on K, or the ring of a non-trivial valuation v on K. We endow K^m with the corresponding topology, for every integer $m \geqslant 0$. We will make almost everywhere the following assumption on $\mathcal{K}$.

(BFin) Every definable subset of K that is closed, bounded and discrete, is finite.

As is well known, every discrete definable subset of K is finite if $\mathcal{K}$ is o-minimal [vdD98], P-minimal [HM97] or C-minimal [HM94], and more generally if it is a dp-minimal ordered or valued field [Sim11], [JSW17]. The same holds true if $\mathcal{K}$ is any Henselian (non-trivially) valued field of characteristic 0, or any algebraically bounded expansion of such [vdD89]. But property (BFin) is also obviously satisfied if $\mathcal{O}$ is compact, hence if $(K, \mathcal{O})$ is *any* expansion of $\mathbb{R}$, the field of real numbers, or any valued local field (that is a finite extension of the field $\mathbb{Q}_p$ of p-adic numbers, or the field of Laurent series $F((t))$ over a finite field F).

Given any two definable sets $X \subseteq K^m$ and $Y \subseteq K^n$ we let $\mathcal{C}(X, Y)$ denote the set of continuous functions $X \to Y$ that are definable with parameters in $\mathcal{K}$. If Y is a subring of K, e.g. when $Y = \mathbb{Z}$, then $\mathcal{C}(X, Y)$ is considered as a ring, where addition and multiplication is given point-wise. When $Y = K$, we just write $\mathcal{C}(X)$. Let $\tau^{\mathbb{Z}} = \{\tau^k \colon k \in \mathbb{Z}\}$ for some non-zero $\tau \in \mathcal{O}$ with $1/\tau \notin \mathcal{O}$. We also furnish $\tau^{\mathbb{Z}}$ with a ring structure so that the bijection $\mathbb{Z} \longrightarrow \tau^{\mathbb{Z}}$, $k \mapsto \tau^k$ is an isomorphism. Again, pointwise addition and multiplication turns $\mathcal{C}(X, \tau^{\mathbb{Z}})$ into a ring.

Our main result is that, under very general conditions on $\mathcal{K}$ and X, the ring $\mathcal{C}(X, \mathbb{Z})$ (resp. $\mathcal{C}(X, \tau^{\mathbb{Z}})$) is definable (resp. interpretable) in the ring structure of $\mathcal{C}(X)$ expanded by the set

$$\mathcal{B} = \{s \in \mathcal{C}(X) \mid \forall x \in X : s(x) \in \mathcal{O}\}.$$

In many cases, in particular when $\mathcal{K}$ is o-minimal case or an expansion of a p-adically closed field, we will see that $\mathcal{B}$ is already definable in $\mathcal{C}(X)$, see Remark 4.4. On the other hand $\mathcal{O}$ will *not* be definable in the ring structure when K is algebraically closed.

A crucial input and starting point of the paper is the definability of the Nullstellensatz in a weak, but surprisingly general form in Theorem 3.4. This says that for almost *any* ring A of functions from some given set S to a given field K, the $n+1$-ary relation $\{f_1 = 0\} \cap \ldots \cap \{f_n = 0\} \subseteq \{g = 0\}$ of A is the Jacobson radical relation of A. For example it suffices to ask that K is not algebraically closed, A contains the constant functions with value in K and that all functions in A without zero are invertible. This is explained in section 3. In the case of o-minimal or P-minimal structures it implies that the lattice of closed definable subsets of X is interpretable in $\mathcal{C}(X)$.

The technical heart of this paper is Section 5 where we prove our first main result (Theorem 5.5 and Corollary 5.6).

Theorem A *Assume that $\mathcal{K}$ satisfies (BFin). Let τ be a non-zero and non-invertible element of $\mathcal{O}$. Let $X \subseteq K^m$ be a definable set which has arbitrarily many germs*[1] *at some point p_0.*

- *If $\mathcal{O} = [-1, 1]$ then the ring of functions $f \in \mathcal{C}(X)$ such that $f(p_0) \in \mathbb{Z}$ is definable in $(\mathcal{C}(X), \mathcal{B})$.*
- *In any case the set of functions $f \in \mathcal{C}(X)$ such that $f(p_0) \in \tau^{\mathbb{Z}}$ is definable in $(\mathcal{C}(X), \mathcal{B})$, and its natural ring structure is interpretable in $(\mathcal{C}(X), \mathcal{B})$.*

As a consequence the ring of integers $\mathbb{Z}$ is interpretable in $(\mathcal{C}(X), \mathcal{B})$.

If $\mathcal{K}' = (K', \dots)$ is an elementary extension of $\mathcal{K}$, and X' is the subset of K'^m defined by the same formula as X, there is a natural embedding of $(\mathcal{C}(X), \mathcal{B})$ into $(\mathcal{C}(X'), \mathcal{B}')$. It follows from the above result that, surprisingly enough, this is not an elementary embedding in general (Corollary 5.7).

Note that the above theorem is fairly general: it only assumes that $\mathcal{K}$ satisfies (BFin) and X has arbitrarily many germs at p_0. In Section 6 we improve it by assuming, in addition to (BFin), that $\mathcal{K}$ has a good dimension theory for definable sets (see the axioms list (Dim) in Section 2). This holds true if $\mathcal{K}$ is *o*-minimal, *P*-minimal or *C*-minimal, and more generally if it is dp-minimal (see Remark 2.4). We can then prove our second main result.

Theorem B *Assume that $\mathcal{K}$ satisfies (Dim) and (BFin). Let τ be a non-zero and non-invertible element of $\mathcal{O}$. Let $X \subseteq K^m$ be a definable set of pure dimension $d \geqslant 2$.*

(1) *If $\mathcal{O} = [-1, 1]$ then $\mathcal{C}(X, \mathbb{Z})$ is definable in $(\mathcal{C}(X), \mathcal{B})$.*

(2) *In any case $\mathcal{C}(X, \tau^{\mathbb{Z}})$ is definable in $(\mathcal{C}(X), \mathcal{B})$.*

Theorem 6.7 actually gives a more precise and more general statement. Let us also mention that in the *P*-minimal case (among others, see condition (Z) in Section 6) the condition on the pure dimension of X can be relaxed: the result holds true whenever X has no isolated point.

2 Model theoretic and topological set up

Let K be a field and let $\mathcal{O}$ be either the unit interval $[-1, 1]$ of a total order $\leqslant$ of K, or the ring of a non-trivial valuation v of K.

In both cases we let $\mathcal{O}^\times$ denote the set of non-zero $a \in \mathcal{O}$ with $a^{-1} \in \mathcal{O}$. This is a multiplicative subgroup of $K^\times = K \setminus \{0\}$. We let $|\ | : K^\times \to K^\times/\mathcal{O}^\times$ be the residue map and extend it by $|0| = 0$. In the ordered case this is just the usual absolute value, in the valued case $|x|$ is a multiplicative notation for the valuation. The set $|K| = \{|x| \colon x \in K\}$ is totally ordered by the relation $|y| \leqslant |x|$ if and only if $y \in x \cdot \mathcal{O}$.

[1] Roughly speaking, X has arbitrarily many germs at p_0 if there exists arbitrarily many disjoint definable subsets of S of $X \setminus \{p_0\}$ such that $p_0 \in \overline{S}$. See Section 2 for a precise definition.

Note that in the valued case, $|y| \leqslant |x|$ if and only if $v(y) \geqslant v(x)$. The multiplication defined on $|K|$ by $|x| \cdot |y| = |xy|$ extends the multiplication of $|K| \setminus \{0\} = K^\times/\mathcal{O}^\times$. The latter is a totally ordered abelian (multiplicative) group. We denote it by $|K^\times|$, or $v(K^\times)$ when additive notation is more appropriate.

For every $x \in K^m$ we set $\|x\| = \max(|x_1|, \ldots, |x_m|)$, and for all $X \subseteq K^m$ we write $\|X\| = \{\|x\| \colon x \in X\}$; if $m = 1$ we simply write $|X|$ (or $v(X)$ in additive notation). Open and closed **balls** in K^m with center $c \in K^m$ and radius $r \in K^\times$ are defined as usually; both are clopen in the valued case. We endow K^m with the topology defined by the open balls, and $|K|$ with the image of this topology (which induces the discrete topology on $|K^\times|$ in the valued case). For any set S, we write $\overline{S}$ for the **topological closure** of S and $\partial S = \overline{S} \setminus S$ for the **frontier** of S.

In this paper, **definable** means "first-order definable with parameters". Let $\mathcal{L}_{\text{og}} = \{e, *, \leqslant\}$ be the language of (additive or multiplicative) ordered groups, let $\mathcal{L}_{\text{ring}} = \{0, 1, +, -, \times\}$ be the language of rings and let $\mathcal{L}$ be an extension of $\mathcal{L}_{\text{ring}}$ containing a unary predicate symbol $\mathcal{O}$. We will be working with expansions $\mathcal{K}$ of an ordered or valued field K to $\mathcal{L}$, where the symbol $\mathcal{O}$ is interpreted as explained above. Definability refers to $\mathcal{L}$ for subsets of K^m and to $\mathcal{L}_{\text{og}}$ for subsets of $|K^\times|$.

Now let $\mathcal{C}(X)$ be the ring of continuous definable functions as explained in the introduction. For all $f, g \in \mathcal{C}(X)$ the sets $\{f = g\}$, $\{f \neq g\}$, $\{|f| \leqslant |g|\}$ and so on, are defined as the subsets of X on which the corresponding relation holds true. For example $\{f = 0\} = \{x \in X \colon f(x) = 0\}$ is the zero-set of f. On $\mathcal{C}(X)$ we work with the relation

$$g \sqsubseteq f \iff \forall x \in X \ \big(g(x) = 0 \Rightarrow f(x) = 0\big).$$

We prove in Theorem 3.4 that this relation is definable in the ring $\mathcal{C}(X)$.

Definition 2.1 Let $X \subseteq K^m$ be a definable set, $p_0 \in X$ and let U be a definable neighborhood of p_0 in X. We say that a function $s \in \mathcal{C}(U)$ **vanishes on a germ** at p_0 if s has a non-isolated zero at p_0. We say that $s_1, \ldots, s_k \in \mathcal{C}(U)$ **vanish on separated germs** at p_0 if there are $\delta_1, \ldots, \delta_k \in \mathcal{C}(U \setminus \{p_0\})$ with the following properties.

(S1) The sets $S_i = \{s_i = 0\} \setminus \{p_0\}$ are pairwise disjoint.

(S2) Each s_i vanishes on a germ at p_0.

(S3) For $i \neq j$, the function δ_i is constantly 1 on S_i and constantly 0 on S_j.

(S4) Each δ_i is bounded on $U \setminus \{p_0\}$.

We call functions $\delta_1, \ldots, \delta_k$ with these properties **separating functions** for $s_1, \ldots, s_k$. Finally we say that X **has arbitrarily many germs** at p_0 if for every positive integer k there is a definable neighborhood U of p_0 in X and k functions in $\mathcal{C}(U)$ that vanish on separated germs at p_0.

Examples 2.2 Intuitively this means that p_0 can be approached in X through arbitrarily many disjoint ways.

(1) For every $r \geqslant 2$, $X = K^r$ has arbitrarily many germs at the origin (see Proposition 6.4).

(2) If $\mathcal{K}$ is o-minimal then $X = K$ does not have arbitrarily many germs at 0, because any definable set $S \subseteq K \setminus \{0\}$ whose closure contains 0 will necessarily meet one of the two intervals $(-\infty, 0)$ or $(0, +\infty)$.

(3) If $\mathcal{K}$ is any expansion of the real field $\mathbb{R}$ including the sin function then $X = K$ has arbitrarily many germs at 0: take $s_i(x) = x \sin((1 - ix)/(kx))$ for $1 \leqslant i \leqslant k$.

(4) If $\mathcal{K}$ is any expansion of a valued field whose value group is a $\mathbb{Z}$-group then K has arbitrarily many germs at 0: take $s_i(x) = x\chi_i(x)$ where χ_i is the characteristic function of the (clopen) set of elements of $\mathcal{O} \setminus \{0\}$ whose valuation is congruent to i modulo k.

A **coordinate projection** $\pi : K^m \longrightarrow K^r$ is a map of the form $\pi(x) = (x_i)_{i \in I}$ for some $I \subseteq \{1, \ldots, m\}$ of size $r \geqslant 0$; we write π_I when necessary. The **dimension** of a non-empty set X is the maximal $r \leqslant m$ such that $\pi(X)$ has non-empty interior for some coordinate projection $\pi : K^m \to K^r$. This is extended to $\dim(\emptyset) = -\infty$. The **local dimension** of X at a point $x \in K^m$ is

$$\dim(X, x) = \min\big\{ \dim B \cap X \colon B \text{ is an open ball centered at } x \big\}.$$

Note that $\dim(X, x) = -\infty$ if and only if $x \notin \overline{X}$, and that $\dim(X, x) = 0$ if and only if x is an isolated point of X.

Definition 2.3 For every integer $d \geqslant 0$ we write

$$\Delta_d(X) = \big\{ x \in X \colon \dim(X, x) = d \big\}.$$

Further we write $W_d(X)$ for the set of all $x \in X$ for which there is an open ball B centered at x and a coordinate projection $\pi : K^m \to K^d$ that induces by restriction a homeomorphism between $B \cap X$ and an open subset of K^d.

Since the open balls are uniformly definable in $\mathcal{L}_{\mathrm{ring}} \cup \{\mathcal{O}\}$ the sets $\Delta_d(X)$ and $W_d(X)$ are definable in $\mathcal{L}$.

We say that $\mathcal{K}$ **satisfies (Dim)** if for every definable set $X \subseteq K^m$ and every definable map $f : X \to K^r$ the following properties hold true.

(Dim1) $\dim(f(X)) \leqslant \dim(X)$.

(Dim2) $\dim(X) = \dim(\overline{X})$ and if $X \neq \emptyset$, then $\dim(\partial X) < \dim(X)$.

(Dim3) If $\dim(X) = d \geqslant 0$ then $\dim(X \setminus W_d(X)) < d$.[2]

Remark 2.4 These properties hold true in every dp-minimal expansion of a field which is not strongly minimal ([SW18]). This implies and generalises known results on o-minimal, C-minimal and P-minimal fields (see [vdD98], [HM94], [HM97] and [CKDL17]).

[2] In the classical cases of o-minimal, C-minimal and P-minimal structures, the set $\Delta_d(X)$ is usually considered instead of $W_d(X)$ in (Dim3). However it is this slightly stronger statement with $W_d(X)$ which we need in Section 6. It appears in Proposition 4.6 of [SW18].

Note that the sets $\Delta_d(X)$ are pairwise disjoint and that $\bigcup_{l\geqslant d}\Delta_l(X)$ is closed in X for each d, while $W_d(X)$ is open in X.

Property 2.5 *Property (Dim1) implies that the dimension is preserved by definable bijections and that $W_d(X)\subseteq\Delta_d(X)$ for every $d\geqslant 0$. In particular the sets $W_d(X)$ are pairwise disjoint.*

Proof: The first assertion is obvious, we prove the second one. If $x\in W_d(X)$, then there is a definable neighborhood U of x in X, a coordinate projection $\pi: K^m\to K^d$ and an open subset V of K^d such that $\pi|_U$ is a homeomorphism onto V. In particular $\dim U = d$ by the first assertion, hence $\dim W_d(X)\geqslant d$. For every sufficiently small open ball B centered at x we have $B\cap X\subseteq U$, hence $\pi|_{B\cap X}$ is a homeomorphism onto a non-empty open subset of K^d and so $\dim(B\cap X)=d$. This proves $\dim(X,x)=d$ hence $W_d(X)\subseteq\Delta_d(X)$. Since the sets $\Delta_d(X)$ are pairwise disjoint, so are the sets $W_d(X)$. $\square$

3 Definability of the poset of zero sets

Let X be a set and let A be a ring of functions $X\longrightarrow K$ for some field K. We show in Theorem 3.4 that for a huge class of examples, the $(n+1)$-ary relation $\{f_1=0\}\cap\ldots\cap\{f_n=0\}\subseteq\{g=0\}$ of A is equivalent to g being in the Jacobson radical of the ideal $(f_1,\ldots,f_n)$. In particular, theses relations are 0-definable in the ring A. The crucial ingredients are contained in Proposition 3.2 and Proposition 3.3.

Let I be an ideal of a ring A. The **Jacobson radical** $\mathrm{Jac}(I)$ of I is defined as the intersection of the maximal ideals of A containing I (*cf.* [Mat89, p. 3]). The Jacobson radical of *the ring* A is defined as $\mathrm{Jac}(0)$.

Remark 3.1 One checks easily that $\mathrm{Jac}(0)=\{a\in A\mid\forall x\in A: 1+ax\in A^\times\}$, where $A^\times$ denotes the units of A. Translating this description for A/I back to A shows that $\mathrm{Jac}(I)=\{a\in A\mid\forall x\in A\;\exists y\in A, z\in I:(1+ax)y=1+z\}$.

Proposition 3.2 *Let K be a field.*

(1) *If K is not algebraically closed, then there are polynomials $u(x,y),v(x,y)$ in two variables over K such that the unique zero of $xu(x,y)+yv(x,y)$ in K^2 is $(0,0)$.*

(2) *Assume that R is a real closed field and $K=R[i]$ is its algebraic closure. We write a^* for the complex conjugation of $a\in K$ with respect to R. Then for the functions $u(x,y)=x^*$ and $v(x,y)=y^*$ defined on K^2, the unique zero of $xu(x,y)+yv(x,y)$ in K^2 is $(0,0)$.*

(3) *Assume that K is a topological field (see [War93]) of characteristic $\neq 2$, where a basis of neighborhoods of $0\in K$ is given by the non-zero ideals of a ring $\mathcal{O}$ with fraction field K and non-zero Jacobson radical[3].*

[3] Our principal example here is a proper valuation ring $\mathcal{O}$ of K.

Then there are τ-continuous functions $u, v : K^2 \longrightarrow K$ that are definable in the expansion $(K, \mathcal{O})$ of K by the set $\mathcal{O}$, such that the unique zero of $xu(x,y)+yv(x,y)$ in K^2 is $(0,0)$.

Proof: (1) Since K is not algebraically closed there is a polynomial $p(x) = x^d + a_{d-1}x^{d-1}+\ldots+a_0 \in K[t]$ without zeroes in K such that $d \geqslant 1$. Then its homogenization

$$q(x,y) = x^d + a_{d-1}x^{d-1}y + \ldots + a_0 y^d$$

has a unique zero in K^2, namely $(0,0)$: If $b = 0$, then $q(a,b) = 0$ just if $a = 0$. If $b \neq 0$, then $q(a,b) = b^d \cdot p(\frac{a}{b}) \neq 0$.

We choose $u(x,y) = x^{d-1}$ and $v(x,y) = a_{d-1}x^{d-1}+\ldots+a_0 y^{d-1}$ and see that $xu(x,y)+yv(x,y) = q(x,y)$ has the required properties.

(2) is clear.

(3) We write $|K| = \{a\mathcal{O} \mid a \in K\}$ for the set of principal fractional ideals of $\mathcal{O}$ and $|a| = a\mathcal{O}$ for $a \in K$. Further we write $|a| \leqslant |b|$ instead of $|a| \subseteq |b|$.

Claim. For every topological space X, each continuous function $f : X \longrightarrow K$ and all $x_0 \in X$ with $f(x_0) \neq 0$ there is an open neighborhood U of x_0 in X such that the restriction of $|f| : X \longrightarrow |K|$ to U is constant.

Proof. Replacing f by $\frac{1}{f(x_0)} \cdot f$ if necessary, we may assume that $f(x_0) = 1$. By the assumption in (iii) there is a non-zero element ε in the Jacobson radical of $\mathcal{O}$ such that $1 + \varepsilon\mathcal{O}$ is an open neighborhood of 1 in K. By continuity of f at x_0 there is a neighborhood U of x_0 in X with $f(U) \subseteq 1 + \varepsilon\mathcal{O}$. Since $\varepsilon\mathcal{O} \subseteq \mathrm{Jac}(0)$, no element in $1 + \varepsilon\mathcal{O}$ can be in any maximal ideal of $\mathcal{O}$. Hence $f(U) \subseteq 1 + \varepsilon\mathcal{O} \subseteq A^\times$, which shows that $|f(x)| = |1|$ for all $x \in U$. $\diamond$

Returning to the proof of (3), we define

$$U = \big\{|x-y| < |x+y|\big\}, \quad V = \big\{0 \neq |x-y| \not< |x+y|\big\},$$

$$Z = \big\{x - y = x + y = 0\big\}.$$

This is a partition of K^2 in $(K, \mathcal{O})$-definable sets. Note that $x+y \neq 0$ in U and $x-y \neq 0$ in V. Using the claim with $X = K^2$ it is easy to see that U and V are open in K^2. Further, the functions $u, v : K^2 \longrightarrow K$ defined by

$$u(x,y) = \begin{cases} x+y & \text{if } (x,y) \in U, \\ x-y & \text{if } (x,y) \in K^2 \setminus U \end{cases} \quad \text{and} \quad v(x,y) = \begin{cases} x+y & \text{if } (x,y) \in U, \\ y-x & \text{if } (x,y) \in K^2 \setminus U, \end{cases}$$

are continuous on U and on V. Moreover, since $x - y = x + y = 0$ on Z, both u and v tend to 0 at every point of Z. Thus u and v are continuous on X. By construction they are also definable in $(K, \mathcal{O})$. On U, $xu(x,y) + yv(x,y) = (x+y)^2$ has no zero. On V, $xu(x,y) + yv(x,y) = (x-y)^2$ has no zeroes.

Since K has characteristic $\neq 2$, the set Z is $\{(0,0)\}$, which establishes the assertion. $\square$

Proposition 3.3 *Let K be a field and let $u, v : K^2 \longrightarrow K$ be functions such that the unique zero of $xu(x,y) + yv(x,y)$ in K^2 is $(0,0)$.*

Let X be a set and let A be a ring of functions $X \longrightarrow K$ such that A is closed under composition with u and v, i.e., $u \circ (f,g), v \circ (f,g) \in A$ for all $f, g \in A$.

(1) *For all $f_1, \ldots, f_n \in A$ there are $g_1, \ldots, g_n \in A$ with*

$$\{f_1 = 0\} \cap \ldots \cap \{f_n = 0\} = \{g_1 f_1 + \ldots + g_n f_n = 0\}.$$

(2) *If every $f \in A$ without zeroes in X is a unit in A then for all $f_1, \ldots, f_n, g \in A$ we have*

$$\{f_1 = 0\} \cap \ldots \cap \{f_n = 0\} \subseteq \{g = 0\} \implies g \in \mathrm{Jac}(f_1, \ldots, f_n).$$

(3) *If A contains all constant functions $X \longrightarrow K$, then for all $f_1, \ldots, f_n, g \in A$ we have*

$$g \in \mathrm{Jac}(f_1, \ldots, f_n) \implies \{f_1 = 0\} \cap \ldots \cap \{f_n = 0\} \subseteq \{g = 0\}.$$

Proof: (1). By induction on n, where $n = 1$ is trivial. The induction step readily reduces to the claim in the case $n = 2$. By assumption on A we know that $h := f_1 \cdot (u \circ (f_1, f_2)) + f_2 \cdot (v \circ (f_1, f_2)) \in A$. By assumption on u, v we see that the zero set of h in X is $\{f_1 = 0\} \cap \{f_2 = 0\}$. Hence we may take $g_1 = u \circ (f_1, f_2)$ and $g_2 = v \circ (f_1, f_2)$.
(2). Assume $\bigcap_{i=1}^n \{f_i = 0\} \subseteq \{g = 0\}$ and let $\mathfrak{m}$ be a maximal ideal of A containing $f_1, \ldots, f_n$. We need to show that $g \in \mathfrak{m}$. By (1), there is some h in the ideal $(f_1, \ldots, f_n)$ generated by $f_1, \ldots, f_n$ in A with zero set $\bigcap_{i=1}^n \{f_i = 0\}$. In particular $h \in \mathfrak{m}$ and $\{h = 0\} \subseteq \{g = 0\}$.

Suppose $g \notin \mathfrak{m}$. Then there is some $a \in A$ and some $m \in \mathfrak{m}$ with $ag + m = 1$, in particular $\{g = 0\} \cap \{m = 0\} = \emptyset$. Since $\{h = 0\} \subseteq \{g = 0\}$ we get $\{h = 0\} \cap \{m = 0\} = \emptyset$. By (1) again, there are $b_1, b_2 \in A$ such that $b_1 h + b_2 m$ has no zeroes. But then by assumption on A, $b_1 h + b_2 m$ is a unit of A, contradicting $h, m \in \mathfrak{m}$.
(3). Let $x \in \bigcap_{i=1}^n \{f_i = 0\}$ and let $e : A \longrightarrow K$ be the evaluation map at x. Since A contains the constant functions we know that e is surjective, hence $\mathfrak{m} = \ker(e)$ is a maximal ideal of A. Then $f_1, \ldots, f_n \in \mathfrak{m}$ and so by assumption $g \in \mathfrak{m}$, i.e. $g(x) = 0$. □

Theorem 3.4 *Let X be a set and let A be a ring of functions $X \longrightarrow K$ containing the constant functions such that every $f \in A$ without zeroes in X is a unit in A. Suppose one of the following conditions hold:*

(a) K is not algebraically closed, or,

(b) K is the algebraic closure of a real closed field R and A is closed under conjugation of $K = R[i]$, or,

(c) K is a topological field of characteristic $\neq 2$, where a basis of neighborhoods of $0 \in K$ is given by the non-zero ideals of a ring $\mathcal{O}$ with fraction field K and non-zero Jacobson radical. Further assume that $w \circ (f,g) \in A$ for every $(K, \mathcal{O})$-definable continuous function $w : K^2 \longrightarrow K$.

The following conditions are equivalent for all $f_1, \ldots, f_n, g \in A$.

(1) $\{f_1 = 0\} \cap \ldots \cap \{f_n = 0\} \subseteq \{g = 0\} \iff g \in \mathrm{Jac}(f_1, \ldots, f_n)$.

(2) $g \in \mathrm{Jac}(f_1, \ldots, f_n)$.

(3) *For all $h \in A$, the element $1 + hg$ is a unit modulo the ideal $(f_1, \ldots, f_n)$.*

Consequently, the $(n+1)$-ary relation $\{f_1 = 0\} \cap \ldots \cap \{f_n = 0\} \subseteq \{g = 0\}$ of A is definable in the ring A by the $\mathcal{L}_{\mathrm{ring}}$-formula

$$\forall x \, \exists y_1, \ldots, y_n, z \ (1 + x \cdot g) \cdot z = 1 + y_1 f_1 + \ldots + y_n f_n.$$

Of particular interest for us is the case $n = 1$. Hence the binary relation $f \sqsubseteq g$ defined as $\{f = 0\} \subseteq \{g = 0\}$, is 0-definable in A.

Proof: The assumptions in Proposition 3.3 hold by Proposition 3.2. Hence Proposition 3.3(2),(3) imply the equivalence of (1) and (2). The equivalence of (2) and (3) holds by Remark 3.1. □

Examples 3.5 Let X be a topological space. Then Theorem 3.4 applies to the following rings A of functions $X \longrightarrow K$.

(a) K is an ordered field or a p-valued field and

- A is the ring of continuous functions $X \longrightarrow K$, or,
- $X \subseteq K^n$ is definable in K and A is the ring of definable continuous functions $X \longrightarrow K$, or,
- $X \subseteq K^n$ is open (definable) and A is the ring of (definable) k-times differentiable functions $X \longrightarrow K$, or,
- $X \subseteq K^n$ is a variety and A is the ring of rational functions $X \longrightarrow K$ without zeroes on X (sometimes referred to as *regular functions* in the literature).

In each case, condition (a) of Theorem 3.4 applies.

(b) A is the ring of continuous functions $X \longrightarrow \mathbb{C}$, or, $X \subseteq \mathbb{C}^n$ and A is the ring of continuous semi-algebraic functions $X \longrightarrow \mathbb{C}$. In both cases, condition (b) of Theorem 3.4 applies.

(c) K is a valued field of characteristic $\neq 2$, furnished with the valuation topology and A is the ring of continuous functions $X \longrightarrow K$, or, $X \subseteq K^n$ is definable and A is the ring of definable continuous functions $X \longrightarrow K$. In both cases, condition (c) of Theorem 3.4 applies, where $\mathcal{O}$ is the valuation ring of K.

4 Basic properties of zero sets

We collect a few basic facts, which will be used in the rest of the paper. We continue to work with the set up of the introduction and section 2.

Lemma 4.1 (1) *There is $\nu_m \in \mathcal{C}(K^m)$ such that $|\nu_m(x)| = \|x\|$ for every $x \in K^m$. Now let $X \subseteq K^m$ be a definable set, $a \in X$ and let B be a closed ball (resp. B_0 an open ball) with radius $r \in K^\times$ (resp. $r_0 \in K^\times$) and center a.*

(2) *Each of the sets $\{a\}$, B and $B_0^c = K^m \setminus B_0$ are zero sets of functions from $C(X)$. We pick such functions and denote them by δ_a, δ_B and $\delta_{B_0^c}$ respectively.*

(3) *If $B \subseteq B_0$, then there is a function $\delta_{B,B_0} \in \mathcal{C}(X)$ with values in $\mathcal{O}$ such that δ_{B,B_0} vanishes on B_0^c and $\{\delta_{B,B_0} = 1\} = B$.*

Proof: (1). For every $x = (x_1, \dots, x_m)$ let $\nu_m(x) = x_1$ if $\|x\| = |x_1|$, $\nu_m(x) = x_2$ if $\|x\| = |x_2| > |x_1|$, and so on. It is obviously definable, and easily seen to be continuous on K^m.

(2) and (3). We may take $\delta_a(x) = \nu_m(x - a)$ restricted to X, using (1). For the other functions, in the valued case we may take the indicator function of $X \setminus B$ for δ_B, and the indicator function of B_0 for $\delta_{B_0^c}$ as well as for δ_{B,B_0} (they are continuous because B and B_0 are clopen). In the ordered case we have $0 < r_0 < r$ and $|K|$ identifies with the set of non-negative elements of K so we may take $\delta_B(x) = \max\{0, \|x - a\| - |r|\}$, $\delta_{B_0^c}(x) = \max\{0, r - \|x - a\|\}$ and $\delta_{B,B_0}(x) = \max\{0, \min\{1, u(x)\}\}$ where $u(x) = (r - \|x - a\|)/(r - r_0)$. □

Lemma 4.2 *There are formulas* $\mathrm{Point}(f)$ *and* $\mathrm{Inter}(f, g, h)$ *in* $\mathcal{L}_{\mathrm{ring}}$*, and a formula* $\mathrm{Isol}(s, p)$ *in* $\mathcal{L}_{\mathrm{ring}} \cup \{\mathcal{B}\}$ *such that for every definable set $X \subseteq K^m$:*

(1) $\mathcal{C}(X) \models \mathrm{Point}(f) \iff$ *f has a single zero in X.*

(2) $\mathcal{C}(X) \models \mathrm{Inter}(f, g, h) \iff \{f = 0\} \cap \{g = 0\} = \{h = 0\}$.

(3) $(\mathcal{C}(X), \mathcal{B}) \models \mathrm{Isol}(s, p) \iff \{p = 0\}$ *is an isolated point of* $\{s = 0\}$.

Proof: Let $\mathcal{V}(X) = \big\{\{s = 0\} \colon s \in \mathcal{C}(X)\big\}$ ordered by inclusion. It is a bounded distributive lattice. Indeed for every $f, g \in \mathcal{C}(X)$ we have

$$\begin{aligned}\{f = 0\} \cup \{g = 0\} &= \{fg = 0\} \\ \{f = 0\} \cap \{g = 0\} &= \{\nu_2(f, g) = 0\},\end{aligned}$$

where $\nu_2 \in \mathcal{C}(K^2)$ is the function given by Lemma 4.1 (1). For every $a \in K^m$, $\{a\} = \{\delta_a = 0\}$ where δ_a is given by Lemma 4.1 (2), so the atoms of $(\mathcal{V}(X), \subseteq)$ are exactly the singletons. The two first points then follow from the fact that $(\mathcal{V}(X), \subseteq)$ is uniformly interpretable in $\mathcal{C}(X)$ as the quotient ordered set of the preorder $\sqsubseteq$, see Theorem 3.4.

For item (3), assume that $\{p = 0\} = \{p_0\}$. If p_0 is an isolated point of $\{s = 0\}$, let B_0 be an open ball with center p_0 which is disjoint from $S = \{s = 0\} \setminus \{p_0\}$. By Lemma 4.1 (2) there is a function $\delta_{B_0^c} \in \mathcal{C}(X)$ such that $\delta_{B_0^c}(x) = 0$ if and only if $x \notin B_0$. Finally let $u = \nu_2(s, \delta_{B_0^c})$ where ν_2 is given by Lemma 4.1 (1). Then $\{u = 0\} = \{s = 0\} \cap B_0^c = S$, hence $s \sqsubseteq pu$ and $s \not\sqsubseteq u$. Conversely if there is $u \in \mathcal{C}(X)$ such that $s \sqsubseteq pu$ and $s \not\sqsubseteq u$ then

$\{u \neq 0\}$ is a neighborhood of p_0 disjoint from S hence p_0 is an isolated point $\{s = 0\}$. So this property is axiomatized by the formula

$$\mathrm{Isol}(s,p) \equiv \mathrm{Point}(p) \wedge p \sqsubseteq s \wedge \exists u \big(s \sqsubseteq pu \wedge s \not\sqsubseteq u \big).$$

$\square$

Proposition 4.3 *For every parameter-free formula $\varphi(y)$ in $\mathcal{L}_{\mathrm{ring}}$ (resp. $\mathcal{L}_{\mathrm{ring}} \cup \{\mathcal{O}\}$) in k free variables, there is a parameter-free formula $[\varphi]$ in $\mathcal{L}_{\mathrm{ring}}$ (resp. $\mathcal{L}_{\mathrm{ring}} \cup \{\mathcal{B}\}$) in $k+1$ variables such that for every definable set $X \subseteq K^m$, every $h \in \mathcal{C}(X)^k$ and every $s \in \mathcal{C}(X)$ we have*

$$(\mathcal{C}(X), \mathcal{B}) \models [\varphi](h,s) \iff \forall x \in \{s = 0\},\ K \models \varphi(h(x)).$$

Proof: If $\varphi(y)$ is a polynomial equation $P(y) = 0$ (y a single variable), we may take $[\varphi](y,s)$ as $s \sqsubseteq P(y)$. If $\varphi(y)$ is the formula $P(y) \in \mathcal{O}$, we may take $[\varphi](y,s)$ as

$$\forall p \left(\mathrm{Point}(p) \wedge p \sqsubseteq s \ \longrightarrow\ \exists f \left(f \in \mathcal{B} \wedge p \sqsubseteq P(y) - f \right) \right),$$

expressing the fact that a function $P(y)$ has values in $\mathcal{O}$ on the zero set of s just if for all $x \in \{s = 0\}$, $P(y)$ agrees with a function $f \in \mathcal{B}$ at x.

This proves the result for atomic formulas. For arbitrary formulas, $[\varphi]$ is defined by induction as follows.

- $[\varphi \wedge \psi]$ is $[\varphi] \wedge [\psi]$.
- $[\neg\varphi](y,s)$ is $\forall p \big((\mathrm{Point}(p) \wedge p \sqsubseteq s) \to \neg[\varphi](y,p) \big)$.
- $[\exists w\, \varphi](y,s)$ is $\forall p \big((\mathrm{Point}(p) \wedge p \sqsubseteq s) \to \exists w\, [\varphi](w,y,p) \big)$.

We only show the right to left implication of the claimed equivalence in the case of an existential quantifier, all other implications are straightforward. Take $h \in \mathcal{C}(X)^k$, $s \in \mathcal{C}(X)$ and assume that $K \models \exists w\, \varphi(w, h(x))$ for every $x \in \{s = 0\}$. We need to show that $(\mathcal{C}(X), \mathcal{B}) \models \forall p \big((\mathrm{Point}(p) \wedge p \sqsubseteq s) \to \exists w\, [\varphi](w,y,p) \big)$. So take $p \in \mathcal{C}(X)$ whose unique zero p_0 satisfies $s(p_0) = 0$. By assumption there is some $c \in K$ with $K \models \varphi(c, h(p_0))$. Let $h_0 \in \mathcal{C}(X)$ be the constant function with value c. Now, for every $z \in X$ and every $q \in \mathcal{C}(X)$ that has a unique zero z, if $p(z) = 0$ then $K \models \varphi(h_0(z), h(z))$. By induction, this means $(\mathcal{C}(X), \mathcal{B}) \models [\varphi](h_0, h, p)$, as required.

Finally, by choice of the formulas $[\varphi]$ we see that $[\varphi]$ is an $\mathcal{L}_{\mathrm{ring}}$-formula, if φ is an $\mathcal{L}_{\mathrm{ring}}$-formula. $\square$

Remark 4.4 If $\mathcal{O}$ is definable in K by a formula $\varphi(x,a)$ in $\mathcal{L}_{\mathrm{ring}}$, where a is an n-tuple of parameters from K, then Proposition 4.3 applied to $\varphi(x,y)$ implies that $\mathcal{B}$ is definable in $\mathcal{C}(X)$ by the $\mathcal{L}_{\mathrm{ring}}$-formula $[\varphi](x, a^*, 0)$ with parameters a^*, the constant function of $\mathcal{C}(X)^n$ with value a. There are plenty of such fields with definable orders or valuations (including all the non-algebraically closed local fields):

- real-closed and p-adically closed fields;
- the field of rational numbers (by Lagrange's Four Squares Theorem);
- one-variable functions fields over number fields [MS17];
- dp-minimal valued fields that are not algebraically closed [Joh15];
- the field of Laurent series $F((t))$ with the natural valuation [Ax65];

and numerous others: see [FJ17] for a summary on Henselian valuation rings definable in their fraction field.

Having arbitrarily many germs at a point is a local property, but it can be made a bit more global as follows.

Property 4.5 *Let $X \subseteq K^m$ be a definable set and let $p_0 \in X$. Then X has arbitrarily many germs at p_0 (see Definition 2.1) if and only if for every positive integer k there exist k functions in $\mathcal{C}(X)$ that vanish on separated germs at p_0 with separating functions in $\mathcal{C}(X \setminus \{p_0\})$.*

Proof: One implication is obvious. For the converse, assume that a definable neighborhood U of p_0 in X is given together with $v_1, \dots, v_k \in \mathcal{C}(U)$ that vanish on separated germs at p_0, and with corresponding separating functions $d_1, \dots, d_k \in \mathcal{C}(U \setminus \{p_0\})$. Restricting U if necessary, by continuity at p_0, we may assume that each v_i is bounded on U and that $U = B_0 \cap X$ for some open ball B_0 with center p_0. Let B be a closed ball with center p_0 contained in B_0. By Lemma 4.1 (3) we have a bounded function $h = \delta_{B,B_0} \in \mathcal{C}(X)$ with $h(x) = 0$ on $X \setminus B_0$ and $h(x) = 1$ on $B \cap X$.

For each $i \leqslant k$ let $u_i(x) = 0$ on $X \setminus B_0$ and $u_i(x) = h(x)v_i(x)$ on $B_0 \cap X$. Similarly let $\delta_i(x) = 0$ on $X \setminus B_0$ and $\delta_i(x) = h(x)d_i(x)$ on $B_0 \cap X \setminus \{p_0\}$. Each u_i (resp. δ_i) is continuous on $B_0 \cap X$ and tends to 0 at every point in $\partial(B_0 \cap X)$ (because v_i and δ_i are bounded) hence $u_i \in \mathcal{C}(X)$ and $\delta_i \in \mathcal{C}(X \setminus \{p_0\})$. Now let $s_i = \nu_2(\delta_B, u_i)$, where $\nu_2, \delta_B \in \mathcal{C}(K^2)$ are given by Lemma 4.1 (1),(3). By construction δ_i is bounded on $X \setminus \{p_0\}$, $\delta_i(x) = d_i(x)$ on $B \cap X \setminus \{p_0\}$ and $\{s_i = 0\}$ equals $\{v_i = 0\} \cap B$. So the functions s_i, δ_i inherit from v_i, d_i all the properties (S1)–(S4) of functions vanishing on separated germs, *cf.* Definition 2.1. $\square$

5 Constructing integers using limit values and chunks

The present section is devoted to our first main results of interpretability and definability. Our construction is based on the following subset of K. Let $X \subseteq K^m$ be a definable set. Let $s, p \in \mathcal{C}(X)$ be such that the zero-set of p is a single point p_0 and s vanishes on a germ at p_0. Then for all $f, g \in \mathcal{C}(X)$ for which g has no zeroes in the set $S = \{s = 0\} \setminus \{0\}$, we consider the continuous function f/g on S and write Γ for its graph. We define

$$L_{s,p}(f/g) = \big\{ l \in K \colon (p_0, l) \in \overline{\Gamma} \big\}.$$

Informally, $L_{s,p}$ is the set of "limit values" at p_0 of f/g restricted to S. Note that this is always a closed definable subset of K. We will also consider the following relations on $\mathcal{C}(X)$. Note that they are definable in $\mathcal{L}_{\text{ring}} \cup \{\mathcal{B}\}$ using Proposition 4.3.

$$|f| \leqslant_s |g| \iff \{s=0\} \subseteq \{|f| \leqslant |g|\}$$

$$|f| <_s |g| \iff \{s=0\} \subseteq \{|f| < |g|\}.$$

Lemma 5.1 *There is a parameter-free formula* $\text{Limit}(f,g,h,s,p)$ *in* $\mathcal{L}_{\text{ring}}$ *such that for any definable set* $X \subseteq K^m$*, we have* $\mathcal{C}(X) \models \text{Limit}(f,g,h,s,p)$ *if and only if*

- *the zero-set of* p *is a single point* p_0*,*
- s *vanishes on a germ at* p_0*,*
- g *has no zeroes in the set* $S = \{s=0\} \setminus \{p_0\}$ *and*
- $h(p_0) \in L_{s,p}(f/g)$*.*

Proof: The first three properties are defined by the conjunction $\chi(g,s,p)$ of the formulas in Lemma 4.2. Hence we define $\chi(g,s,p)$ as

$$\text{Point}(p) \wedge \neg\text{Isol}(s,p) \wedge \big(\text{Inter}(g,s,p) \vee \text{Inter}(g,s,1)\big).$$

Let $f,g,h,s,p \in \mathcal{C}(X)$ be such that $\mathcal{C}(X) \models \chi(g,s,p)$ and let p_0 be the zero of p. Pick $\tau \in K^\times$ with $|\tau| < 1$. For every $\varepsilon, v \in \mathcal{C}(X)$ with $\varepsilon(p_0) \neq 0$ and $v(p_0) \neq 0$, by continuity, there is an open ball B_0 centered at p_0 such that $B_0 \cap X \subseteq \{v \neq 0\}$ and for every $x \in B_0 \cap X$, $|\varepsilon(x)| > |\tau\varepsilon(p_0)|$ and $|h(x) - h(p_0)| < |\tau\varepsilon(p_0)|$. So if $h(p_0) \in L_{s,p}(f/g)$, then there is a point $q_0 \in B_0 \cap X \setminus \{p_0\}$ with

$$s(q_0) = 0 \text{ and } \left|\frac{f(q_0)}{g(q_0)} - h(q_0)\right| \leqslant |\varepsilon(q_0)|.$$

Note that $q_0 \in B_0 \setminus \{p_0\}$ implies that $q \not\sqsubseteq pv$, and that $s(q_0) = 0$ implies $g(q_0) \neq 0$, so the second condition above is equivalent to $|f(q_0) - g(q_0)h(q_0)| \leqslant |g(q_0)\varepsilon(q_0)|$. Conversely, if there are such points q_0 for any $\varepsilon, v \in \mathcal{C}(X)$ with $\varepsilon(p_0) \neq 0$ and $v(p_0) \neq 0$, then setting $v = \delta_{B_0^c}$ for any open ball B_0 centered at p_0 (with $\delta_{B_0^c}$ given by Lemma 4.1(2)), we get that $h(p_0) \in L_{s,p}(f/g)$. Thus we can take for $\text{Limit}(f,g,h,s,p)$ the conjunction of $\chi(g,s,p)$ and the formula

$$\forall \varepsilon,\, v \left(p \not\sqsubseteq v\varepsilon \to \exists q \left[\text{Point}(q) \wedge q \not\sqsubseteq pv \wedge q \sqsubseteq s \wedge |f - gh| \leqslant_q |g\varepsilon|\right]\right).$$

□

Lemma 5.2 *Let* $l_1, \ldots, l_k \in K$ *and let* $X \subseteq K^m$ *be a definable set. Let* $p \in X$ *be such that* p *has a single zero* p_0 *at which* X *has arbitrarily many germs. Let* $s_1, \ldots, s_k \in \mathcal{C}(X)$ *be vanishing on separated germs at* p_0 *and let* $S_i = \{s_i = 0\} \setminus \{p_0\}$*. Then there are* $f, g \in \mathcal{C}(X)$ *with* $g(x) \neq 0$ *on* $S = S_1 \cup \cdots \cup S_k$ *such that the restriction of* f/g *to each* S_i *has constant value* l_i*. In particular:*

(1) $\{l_1, \ldots, l_k\} = L_{s,p}(f/g)$, *with* $s = s_1 s_2 \cdots s_k$.

(2) $\{l_1, \ldots, l_k\} = \{l(p_0)\colon l \in \mathcal{C}(X)$ *and* $p_0 \in \overline{S \cap \{f = gl\}}\}$.

Proof: Let $\delta_1, \ldots, \delta_s \in \mathcal{C}(X \setminus \{p_0\})$ be separating functions for $s_1, \ldots, s_k$. Clearly $f = \sum_{i \leqslant k} l_i \delta_i p$ (extended by 0 at p_0) and $g = p$ have the required properties, from which items (1) and (2) follow immediately. For the second item, note that $p_0 \in \overline{S \cap \{f = gl\}}$ simply means that the function l takes the same values as f/g on a subset of S having points arbitrarily close to p_0, hence $l(p_0) \in L_{s,p}(f/g)$ by continuity of l. □

Definition 5.3 Let $(G, +, \leqslant)$ be totally pre-ordered abelian group, hence $\leqslant$ is a total pre-order satisfying $x \leqslant y \Rightarrow x+z \leqslant y+z$. For $\tau \in G$ with $\tau > 0$ we write $\tau\mathbb{Z} = \{\tau k\colon k \in \mathbb{Z}\}$.

We call a subset T of G a **$\tau\mathbb{Z}$-chunk** of G if $\tau \in T$ and for all $\alpha, \beta, \gamma \in T$:

(1) $-\alpha \in T$;

(2) $\alpha + \beta \in [-\gamma, \gamma] \Rightarrow \alpha + \beta \in T$;

(3) $\forall u \in [-\gamma, \gamma],\ \exists!\, \xi \in T,\ \xi \leqslant u < \xi + \tau$.

This should be seen as a finitary, and hence definable, version of integer parts as studied by [MR93] in the case of real closed fields.

Remark 5.4 For every finite $\tau\mathbb{Z}$-chunk of G one checks easily that there is some integer n with $G = \{-n\tau, \ldots, -\tau, 0, \tau, 2\tau, \ldots, n\tau\}$.

In the ordered case (and more generally when there is a total pre-order on K, definable in $\mathcal{L}_{\mathrm{ring}} \cup \{\mathcal{O}\}$ and compatible with the group structure of $(K, +)$), we can consider chunks of $(K, +, \leqslant)$. In the valued case however we have to lift chunks from $|K^\times|$ to $K^\times$. So we will consider chunks of the totally pre-ordered multiplicative group $(K^\times, \times, \leqslant_{\mathcal{O}})$ where $\leqslant_{\mathcal{O}}$ is the inverse image of the order of $|K^\times|$, i.e., $x \leqslant_{\mathcal{O}} y$ is just $|x| \leqslant |y|$. In this case we adapt the terminology to multiplicative language. For example, for $\tau \in K^\times$ with $|\tau| < 1$ we write $\tau^{\mathbb{Z}} = \{\tau^k\colon k \in \mathbb{Z}\}$ instead of $\tau\mathbb{Z}$.

Theorem 5.5 *Assume that $\mathcal{K}$ satisfies (BFin). Let $I = \{x \in K\colon |0| < |x| < |1|\}$. There is a parameter-free formula* $\mathrm{Int}^\times(p, h, \tau^*)$ *in* $\mathcal{L}_{\mathrm{ring}} \cup \{\mathcal{B}\}$ *such that for every definable set $X \subseteq K^m$, if p has a unique zero $p_0 \in X$, if X has arbitrarily many germs at p_0 and if the value $\tau = \tau^*(p_0)$ is in I, then*

$$(\mathcal{C}(X), \mathcal{B}) \models \mathrm{Int}^\times(h, p, \tau^*) \iff h(p_0) \in \tau^{\mathbb{Z}}.$$

If K has a total order $\preccurlyeq$ compatible with $+$ and definable[4] *in $\mathcal{L}_{\mathrm{ring}} \cup \{\mathcal{O}\}$, then there is similarly a formula $\mathrm{Int}^+_{\preccurlyeq}(h, p, \tau^*)$ such that, with the same assumptions on p, if $\tau = \tau^*(p_0) \succ 0$ then*

$$(\mathcal{C}(X), \mathcal{B}) \models \mathrm{Int}^+_{\preccurlyeq}(h, p, \tau^*) \iff h(p_0) \in \tau\mathbb{Z}.$$

[4]This obviously happens in the ordered case, where K is an ordered field and $\mathcal{O}$ is the interval $[-1, 1]$. But Theorem 5.5 does not restrict to this case, as it does not assume any relation at all between $\mathcal{O}$ and the order $\preccurlyeq$.

In other words, Theorem 5.5 says essentially that, given any $\tau \in K^\times$ with $|\tau| < 1$ and any definable set $X \subseteq K^m$ with arbitrarily many germs at some point p_0, then the set of all $h \in \mathcal{C}(X)$ taking values in $\tau^{\mathbb{Z}}$ at p_0, is definable in $(\mathcal{C}(X), \mathcal{B})$; the definition is independent of X, K and m and is in the two definable parameters p_0, τ.

Also notice that by Remark 4.4, the set $\mathcal{B}$ is definable in $\mathcal{L}_{\text{ring}}$ in many cases, hence Theorem 5.5 applies to the pure ring $\mathcal{C}(X)$ in all these cases.

Proof: In order to ease the notation, in this proof we write $\mathcal{C}(X) \models \chi(\dots)$ instead of $(\mathcal{C}(X), \mathcal{B}) \models \chi(\dots)$, for every formula in $\mathcal{L}_{\text{ring}} \cup \{\mathcal{B}\}$.

We first construct a formula which axiomatizes (uniformly) the property that $\tau = \tau^*(p_0) \in \mathcal{O} \setminus \{0\}$ and $L_{s,p}(f/g)$ is a bounded $\tau^{\mathbb{Z}}$-chunk. We use the formula $\text{Limit}(f, g, h, s, p)$ stated in Lemma 5.1. It will be convenient to abbreviate it as $\varphi^\sigma(h)$ where $\sigma = (f, g, s, p)$. Thus for every $f, g, v, s, p \in \mathcal{C}(X)$ we have

$$v(p_0) \in L_{s,p}(f/g) \iff \mathcal{C}(X) \models \varphi^\sigma(v).$$

Let $\psi^\sigma(\alpha, \alpha', \beta, \gamma, p, \tau^*)$ be the conjunction of the following formulas.

1. $p \sqsubseteq \alpha\alpha' - 1 \to \varphi^\sigma(\alpha')$.
2. $|1| \leqslant_p |\alpha\beta\gamma| \leqslant_p |\gamma^2|$.
3. $\forall u \left(|1| \leqslant_p |u\gamma| \leqslant_p |\gamma^2| \to \exists \xi \left[\varphi^\sigma(\xi) \wedge |\xi\tau^*| <_p |u| \leqslant_p |\xi|\right]\right)$.
4. $\forall u, \xi, \xi' \left(\left[A(u, \xi, p, \tau^*) \wedge A(u, \xi', p, \tau^*)\right] \to p \sqsubseteq \xi - \xi'\right)$,
   where $A(u, \xi, p, \tau^*)$ stands for $\varphi^\sigma(\xi) \wedge |\xi\tau^*| <_p |u| \leqslant_p |\xi|$.

Clearly $\tau = \tau^*(p_0) \in I$ and $L_{s,p}(f/g)$ is a $\tau^{\mathbb{Z}}$-chunk if and only if $\mathcal{C}(X)$ satisfies the conjunction, which we will denote $\text{LimCh}^\times(f, g, s, p, \tau^*)$, of $|0| <_p |\tau^*| <_p |1|$, of $\varphi^\sigma(\tau^*)$ and of

$$\forall \alpha, \alpha', \beta, \gamma \left[\varphi^\sigma(\alpha) \wedge \varphi^\sigma(\beta) \wedge \varphi^\sigma(\gamma) \to \psi^\sigma(\alpha, \alpha', \beta, \gamma, p, \tau^*)\right].$$

The formula $\text{LimBCh}^\times(f, g, s, p, \tau^*)$ is then defined as

$$\text{LimCh}^\times(f, g, s, p, \tau^*) \wedge \exists \delta \forall \alpha \big(\varphi^\sigma(\alpha) \to |\alpha| \leqslant_p |\delta|\big).$$

Clearly it holds true if and only if $\tau = \tau^*(p_0) \in I$ and $L_{s,p}(f/g)$ is a bounded $\tau^{\mathbb{Z}}$-chunk. Finally, let $\text{Int}^\times(h, p, \tau^*)$ be the formula

$$\exists f, g, s,\ \text{Limit}(f, g, h, s, p) \wedge \text{LimBCh}^\times(f, g, s, p, \tau^*).$$

If $\mathcal{C}(X) \models \text{Int}^\times(h, p, \tau^*)$ then p has a single zero p_0, s vanishes on a germ at p_0 and $h(p_0) \in L_{s,p}(f/g)$; further, by the above, $\tau = \tau^*(p_0) \in I$, and $L_{s,p}(f/g)$ is a bounded $\tau^{\mathbb{Z}}$-chunk (in particular it is a bounded discrete subset of K). Since $L_{s,p}(f/g)$ is is a closed definable subset of K it is then finite by (BFin). Thus by Remark 5.4 (in multiplicative notation) there is a non-negative integer n such that $L_{s,p}(f/g) = \{\tau^i \colon -n \leqslant i \leqslant n\}$. We obtain $L_{s,p}(f/g) \subseteq \tau^{\mathbb{Z}}$ and so $h(p_0) \in \tau^{\mathbb{Z}}$.

Conversely, assume that p has a single zero p_0 and for this zero we have $\tau = \tau^*(p_0) \in I$ and $h(p_0) = \tau^k$ for some $k \in \mathbb{Z}$. Let $T_h = \{\tau^i \colon -|k| \leqslant i \leqslant |k|\}$. Since X has arbitrarily many germs at p_0, we may invoke Property 4.5 to obtain functions $s_i \in \mathcal{C}(X)$ that vanish at separated germs at p_0, for $-|k| \leqslant i \leqslant |k|$[5]. Let s be the product of the s_i's

[5]Here $|k|$ denotes the ordinary absolute value in $\mathbb{Z}$, not the image of k in $|K|$.

and $S = \{s = 0\}$. By Lemma 5.2 there are $f, g \in \mathcal{C}(X)$ such that $g(x) \neq 0$ on S and

$$T_h = L_{s,p}(f/g) = \big\{l(p_0) \colon l \in \mathcal{C}(X) \text{ and } p_0 \in \overline{S \cap \{f = gl\}}\big\}.$$

Thus $\mathcal{C}(X) \models \mathrm{Limit}(f, g, h, s, p) \wedge \mathrm{LimBCh}^{\times}(f, g, s, p, \tau^*)$, hence $\mathcal{C}(X) \models \mathrm{Int}^{\times}(h, p, \tau^*)$. This finishes the proof of the first equivalence of the theorem.

If K has a total order $\preccurlyeq$ that is definable in $\mathcal{L}_{\mathrm{ring}} \cup \{\mathcal{O}\}$ and compatible with $+$, then by Proposition 4.3 the following relations are definable in $\mathcal{L}_{\mathrm{ring}} \cup \{\mathcal{O}\}$.

$$g \preccurlyeq_s f \iff \{s = 0\} \subseteq \{g \preccurlyeq f\}$$

$$g \prec_s f \iff \{s = 0\} \subseteq \{g \prec f\}$$

Let $\Phi^{\sigma}(\alpha, \beta, \gamma, p, \tau^*)$ be the conjunction of the following formulas.

1. $\varphi^{\sigma}(-\alpha')$.
2. $-\gamma \preccurlyeq_p \alpha + \beta \preccurlyeq_p \gamma \to \varphi^{\sigma}(\alpha + \beta)$.
3. $\forall u \big[-\gamma \preccurlyeq_p u \preccurlyeq_p \gamma \to \exists \xi \, \big(\varphi^{\sigma}(\xi) \wedge \xi \preccurlyeq_p u \prec_p \xi + \tau^*\big)\big]$.
4. $\forall u, \xi, \xi' \, \big(B(u, \xi, p, \tau^*) \wedge B(u, \xi', p, \tau^*)\big] \to p \sqsubseteq \xi - \xi'\big)$,
   where $B(u, \xi, p, \tau^*)$ stands for $\varphi^{\sigma}(\xi) \wedge \xi \preccurlyeq_p u \prec_p \xi + \tau^*$.

We then let $\mathrm{LimCh}^{+}_{\preccurlyeq}(f, g, s, p, \tau^*)$ be the conjunction of $0 \prec_p \tau^*$ with $\varphi^{\sigma}(\tau^*)$ and

$$\forall \alpha, \beta, \gamma \left[\varphi^{\sigma}(\alpha) \wedge \varphi^{\sigma}(\beta) \wedge \varphi^{\sigma}(\gamma) \to \Phi^{\sigma}(\alpha, \beta, \gamma, p, \tau^*)\right],$$

and let $\mathrm{LimBCh}^{+}_{\preccurlyeq}(f, g, s, p, \tau^*)$ be

$$\mathrm{LimCh}^{+}_{\preccurlyeq}(f, g, s, p, \tau^*) \wedge \exists \delta \forall \alpha \big(\varphi^{\sigma}(\alpha) \to \alpha \preccurlyeq_p \delta\big).$$

Finally we take $\mathrm{Int}^{+}_{\preccurlyeq}(h, p, \tau^*)$ as $\exists f, g, s\big(\mathrm{Limit}(f, g, h, s, p) \wedge \mathrm{LimBCh}^{\times}(f, g, s, p, \tau^*)\big)$. The proof that it satisfies the second equivalence in the Theorem is analogous to the multiplicative case and left to the reader. $\square$

Corollary 5.6 *Assume that $\mathcal{K}$ satisfies (BFin) and take $\tau \in K^{\times}$ with $|\tau| < |1|$. Let $X \subseteq K^m$ be a definable set having arbitrarily many germs at a given point p_0. Let $p, \tau^* \in \mathcal{C}(X)$ be such that p_0 is the unique zero of p and $\tau^*(p_0) = \tau$. Then the ring of integers $(\mathbb{Z}, +, \times)$ is interpretable in $(\mathcal{C}(X), \mathcal{B})$ with parameters (p, τ^*).*

Proof: For every $k, l \in \mathbb{Z}$ let $l|k$ denote the divisibility relation. It is well known (see for example [Ric89]) that multiplication is 0-definable in $(\mathbb{Z}, +, |, \leqslant)$, hence it suffices to interpret the latter structure in $(\mathcal{C}(X), \mathcal{B})$. Let $\mathcal{Z} = \{f \in \mathcal{C}(X) \colon f(p_0) \in \tau^{\mathbb{Z}}\}$, and for every $f \in \mathcal{Z}$ let $\sigma(f)$ be the unique $k \in \mathbb{Z}$ such that $f(p_0) = \tau^k$. Then

$$f(p_0) \in \tau^{\mathbb{Z}} \iff (\mathcal{C}(X), \mathcal{B}) \models \mathrm{Int}^{\times}(f, p, \tau^*),$$

by Theorem 5.5 and therefore $\mathcal{Z}$ is definable in $(\mathcal{C}(X),\mathcal{B})$. The equivalence relation $\sigma(f)=\sigma(g)$ on $\mathcal{Z}$ is also definable, because

$$\sigma(f)=\sigma(g) \iff f(p_0)=g(p_0) \iff (\mathcal{C}(X),\mathcal{B})\models p\sqsubseteq f-g.$$

For every $f,g\in\mathcal{Z}$ we have obviously $\sigma(f)+\sigma(g)=\sigma(fg)$ and

$$\sigma(g)\geqslant\sigma(f) \iff |g(p_0)|\leqslant|f(p_0)| \iff (\mathcal{C}(X),\mathcal{B})\models |g|\leqslant_p|f|.$$

This gives an interpretation of $(\mathbb{Z},+,\leqslant)$ in $(\mathcal{C}(X),\mathcal{B})$. Moreover $\sigma(g)$ divides $\sigma(f)$ if and only if $f(p_0)\in(\tau^{\sigma(g)})^{\mathbb{Z}}=(g(p_0))^{\mathbb{Z}}$. We obtain

$$\sigma(f)|\sigma(g) \iff (\mathcal{C}(X),\mathcal{B})\models \mathrm{Int}^{\times}(f,p,g)$$

in the terminology of Theorem 5.5. Consequently $(\mathbb{Z},+,|,\leqslant)$ is interpretable in $(\mathcal{C}(X),\mathcal{B})$, from which the result follows. $\square$

We say that $|K^{\times}|$ is **τ-archimedean** for some $\tau\in K$ if $\tau^{\mathbb{Z}}$ is coinitial in $|K^{\times}|$, that is for every $x\in K^{\times}$ there is an integer k such that $|\tau^k|\leqslant|x|$. In the ordered case, $\mathcal{K}$ is 2-archimedean if and only if it is archimedean in the sense that $\mathbb{Z}$ is cofinal in K.

Corollary 5.7 *Let $\mathcal{K}'=(K',\dots)$ be any elementary extension of $\mathcal{K}$ and assume that $\mathcal{K}$ and $\mathcal{K}'$ satisfy (BFin). Let $X\subseteq K^m$ be a definable set having arbitrarily many germs at a given point p_0. Let $X'\subseteq K'^m$ be defined by a formula defining X. If $|K^{\times}|$ is τ-archimedean for some $\tau\in K$ and $|K'^{\times}|$ is not τ-archimedean, then the natural $\mathcal{L}_{\mathrm{ring}}\cup\{\mathcal{B}\}$-embedding of $\mathcal{C}(X)$ into $\mathcal{C}(X')$ is* not *an elementary embedding.*

Proof: Let $p\in\mathcal{C}(X)$ be such that $\{p=0\}=\{p_0\}$, and τ^* be the constant function on X with value τ. The assumption that $|K^{\times}|$ is τ-archimedean implies that $|0|<|\tau|<|1|$. For every $f\in\mathcal{C}(X)$, there exists $k\in\mathbb{Z}$ such that $|f(p_0)|\leqslant|\tau^k|$, so there exists $g\in\mathcal{C}(X)$ such that $|f(p_0)|\leqslant|g(p_0)|$ and $g(p_0)\in\tau^{\mathbb{Z}}$ (it suffices to take $g=(\tau^*)^k$). Thus by Theorem 5.5

$$(*)\qquad (\mathcal{C}(X),\mathcal{B})\models\forall f\,\exists g,\ |f|\leqslant_p|g|\wedge\mathrm{Int}^{\times}(g,p,\tau^*).$$

On the other hand, since K' is not τ-archimedean, there exists an element $a'\in K'$ such that $|a'|\not\leqslant|\tau^k|$ for every $k\in\mathbb{Z}$. Let f' be the constant function on X with value a'. For every $g'\in\mathcal{C}(X')$ such that $g'(p_0)\in\tau^{\mathbb{Z}}$, $|f'(p_0)|\not\leqslant|g'(p_0)|$, hence the formula in $(*)$ is not satisfied in $(\mathcal{C}(X'),\sqsubseteq,\mathcal{B})$. $\square$

Remark 5.8 Let $R\subseteq S$ be real closed fields. Let $L(R^n)$ be the lattice of closed and semi-algebraic subsets of R^n. In [Ast13] it is shown that the natural embedding $L(R^n)\to L(S^n)$ is an elementary map. However, if $n\geqslant 2$ and $R=\mathbb{R}\subsetneq S$, then we know from 5.7 that the natural embedding $C(R^n)\to C(S^n)$ is not elementary. Consequently for real closed fields R, there is no interpretation of the ring $C(R^n)$ in the lattice $L(R^n)$ that is independent of R.

6 Defining integers using local dimension

In this section, assuming (Dim), we show that the set of functions in $\mathcal{C}(X)$ that take values in $\tau^{\mathbb{Z}}$ or $\tau\mathbb{Z}$ for some $\tau \in K^\times$ at some point p_0 as in Theorem 5.5, is definable in $(\mathcal{C}(X), \mathcal{B})$, provided p_0 is not of small local dimension. In order to do so we first prove that the points at which X has arbitrarily many germs are dense among those at which X has local dimension $\geqslant 2$.

Recall from Definition 2.3 that $\Delta_k(X)$ denotes the set of $x \in X$ of local dimension k and $W_k(X)$ denotes the set of $x \in X$ such that there is an open ball B centered at x and a coordinate projection $\pi : K^m \to K^k$ which induces by restriction a homeomorphism between $B \cap X$ and an open subset of K^k.

Lemma 6.1 *Assume that $\mathcal{K}$ satisfies (Dim). For every definable set $X \subseteq K^m$ and every integer $k \geqslant 0$, $W_k(X)$ is a dense subset of $\Delta_k(X)$. If non-empty, both of them have dimension k.*

Proof: We already now that $W_k(X) \subseteq \Delta_k(X)$ by Property 2.5. Pick $x \in \Delta_k(X)$ and an open ball $B \subseteq K^m$ centered at x. By shrinking B if necessary we may assume that $\dim(B \cap X) = k$. From (Dim3) we know $W_k(B \cap X) \neq \emptyset$. On the other hand, $W_k(B\cap X) \subseteq W_k(X)$ because $B\cap X$ is open in X. Consequently $W_k(B\cap X) \subseteq B\cap W_k(X)$ and so $B \cap W_k(X) \neq \emptyset$. This proves density.

By (Dim2) it only remains to prove that $W_k(X)$ has dimension k, provided it is not empty. Clearly $\dim W_k(X) \geqslant k$. If $\dim W_k(X) = l > k$ then by (Dim3) $W_l(W_k(X))$ is non-empty. But $W_k(X)$ is open in X, hence $W_l(W_k(X))$ is contained in $W_l(X)$. So $W_l(W_k(X))$ is contained both in $W_l(X)$ and in $W_k(X)$, a contradiction since $W_l(X)$ and $W_k(X)$ are disjoint by Property 2.5. $\square$

For any two subsets A, B of a topological space X, the **strong order** $\Subset$ is defined by

$$B \Subset A \iff B \subseteq \overline{A \setminus B}.$$

If $X \subseteq K^m$ is definable and $f, g \in \mathcal{C}(X)$ we define

$$g \Subset f \iff \{g = 0\} \Subset \{f = 0\}.$$

Lemma 6.2 *For every definable set $X \subseteq K^m$ and every $f, g \in \mathcal{C}(X)$*

$$g \Subset f \iff \mathcal{C}(X) \models \forall h\, (f \sqsubseteq gh \to g \sqsubseteq h).$$

In particular $\Subset$ is definable in $\mathcal{L}_{\text{ring}}$.

Proof: For every definable set $U \subseteq X$ that is open in X and each $p_0 \in U$ there is a function $h \in \mathcal{C}(X)$ with $h(p_0) \neq 0$ and $h(x) = 0$ on $X \setminus U$ (using Lemma 4.1 it suffices to take $h = \delta_{B_0^c}$ for any open ball B_0 with center p_0 such that $B \cap X$ is contained in U), hence $p_0 \in \{h \neq 0\} \subseteq U$. Hence the sets $\{h = 0\}$ with $h \in \mathcal{C}(X)$ form a basis of closed sets of the topology of X. Since $g \Subset f$ just if every closed definable set containing $\{f=0\} \setminus \{g=0\}$ also contains $\{g=0\}$, we see that $g \Subset f$ if and only if for all $h \in \mathcal{C}(X)$

$$\{f=0\} \setminus \{g=0\} \subseteq \{h=0\} \Rightarrow \{g=0\} \subseteq \{h=0\}.$$

This is equivalent to $\{f=0\} \subseteq \{g=0\} \cup \{h=0\} \implies \{g=0\} \subseteq \{h=0\}$, in other words, $f \sqsubseteq gh \Longrightarrow g \sqsubseteq h$. □

Proposition 6.3 *Assume that $\mathcal{K}$ satisfies (Dim). For every integer $k \geqslant 0$ and every $p_0 \in X$ the following are equivalent.*

(1) $\dim(X, p_0) \geqslant k$.

(2) $\forall v \in \mathcal{C}(X)$, $v(p_0) \neq 0 \Rightarrow \exists f_0 \Subset f_1 \Subset \cdots \Subset f_k \in \mathcal{C}(X)$ *such that* $f_0 \notin \mathcal{C}(X)^\times$ *and* $\{f_k = 0\}$ *is disjoint from* $\{v=0\}$.

Furthermore, there is a parameter-free formula $\chi_k(p)$ in $\mathcal{L}_{\text{ring}}$ such that $\mathcal{C}(X) \models \chi_k(p)$ if and only if p has a single zero $p_0 \in X$ and $\dim(X, p_0) = k$.

Proof: (1)⇒(2). Assume that $\dim(X, p_0) = l \geqslant k$ and take $v \in \mathcal{C}(X)$ with $v(p_0) \neq 0$. Then $\{v \neq 0\}$ is a neighborhood of p_0 in X hence by Lemma 6.1 there is a point $q_0 \in \{v \neq 0\} \cap W_l(X)$. Let V be a definable neighborhood of q_0 in X, contained in $\{v \neq 0\} \cap W_l(X)$ and homeomorphic to an open subset W of K^l via the restriction of a coordinate projection $\pi : K^m \to K^l$.

Let $\varphi = (\varphi_1, \ldots, \varphi_l) : K^l \longrightarrow K^l$ be defined by $\varphi(y) = y - q_0$. For $1 \leqslant i \leqslant l$ let $g_i = \nu_i(\varphi_1, \ldots, \varphi_i)$ where $\nu_i \in \mathcal{C}(K^i)$ is the map defined in Lemma 4.1 (1). Each g_i is in $\mathcal{C}(K^l)$ and $\{g_i = 0\}$ is the affine subspace of K^m defined by $\varphi_1 = \cdots = \varphi_i = 0$, hence $g_0 \Subset \cdots \Subset g_l$. Let B be a closed ball in K^m centered at q_0 and contained in V. Take $\delta_B \in \mathcal{C}(K^m)$ with $\{\delta_B = 0\} = B$ as given by Lemma 4.1 (2). Finally let f_i be the restriction of $\nu_2(g_i \circ \pi, \delta_B)$ to X, where ν_2 is defined in Lemma 4.1 (1). Clearly $f_i \in \mathcal{C}(X)$, $f_0(q_0) = 0$ hence $f_0 \notin \mathcal{C}(X)^\times$, $\{f_k = 0\} \subseteq B$ is disjoint from $\{v = 0\}$, and $f_0 \Subset \cdots \Subset f_k$ because π maps the zero-set of each f_i homeomorphically to $\{g_i = 0\} \cap \pi(B)$.
(2)⇒(1). Let B_0 be an open ball in K^m centered at p_0. We have to prove that $\dim(B_0 \cap X) \geqslant k$. Let $v \in \mathcal{C}(X)$ be such that $v(x) = 0$ on $X \setminus B_0$ and $v(p_0) \neq 0$ (e.g. one can choose $v = \delta_{B_0^c}$ as in Lemma 4.1 (2)). By assumption there are $f_0 \Subset \cdots \Subset f_k \in \mathcal{C}(X)$ such that $f_0 \notin \mathcal{C}(X)^\times$ and $\{f_k = 0\}$ is disjoint from $\{v = 0\}$, hence contained in $B_0 \cap X$. It then suffices to prove that each set $Z_i = \{f_i = 0\}$ has dimension $\geqslant i$. This holds for Z_0 because $f_0 \notin \mathcal{C}(X)^\times$ implies $Z_0 \neq \emptyset$. When $i > 0$, the set Z_i is non-empty since it contains Z_0; further $Z_{i-1} \subseteq \partial(Z_i \setminus Z_{i-1})$ because $Z_{i-1} \Subset Z_i$; consequently $\dim Z_{i-1} < \dim Z_i$ by (Dim2). Item (1) now follows by induction on i.

Using Lemma 6.2 and Lemma 4.2, the equivalence of (1) and (2) implies the existence of a parameter-free formula $\chi_{\geqslant k}(p)$ in $\mathcal{L}_{\text{ring}}$, such that for every $p \in \mathcal{C}(X)$ we have $\mathcal{C}(X) \models \chi_{\geqslant k}(p)$ just if p has a single zero $p_0 \in X$ and $\dim(X, p_0) \geqslant k$. We then take χ_k as $\chi_{\geqslant k} \wedge \neg \chi_{\geqslant k+1}$. □

Proposition 6.4 *Every definable open subset U of K^r with $r \geqslant 2$ has arbitrarily many germs at every point $p_0 \in U$.*

Proof: For each $k \geqslant 1$ we need to define s_i and δ_i verifying conditions (S1)-(S4) of Definition 2.1. It suffices to do the case of K^r (and then take restrictions to U of the functions s_i and δ_i built for K^r). Let $V_k = K^r$, take a hyperplane $H \subset K^r$ containing p_0 and k distinct lines $L_1, \dots, L_k$ passing through p_0 and not contained in H. Let $\sigma_i : K^r \to L_i$ be the projection onto L_i along H. Now let $s_i(x) = \nu_r(x - \sigma_i(x))$ where ν_r is given in Lemma 4.1 (1). Conditions (S1) and (S2) are fulfilled since $\{s_i = 0\} = L_i$ for each i. Finally let $\delta_i = \prod_{j \neq j} \delta_{i,j}$ where $\delta_{i,j}(x) = s_j(x)/s_j(\sigma_i(x))$. Clearly $\delta_{i,j}(x) \in \mathcal{C}(K^r \setminus \{p_0\})$ and we have $\delta_{i,j}(x) = 1$ on $L_i \setminus \{p_0\}$ and $\delta_{i,j}(x) = 0$ on $L_j \setminus \{p_0\}$, hence δ_i has properties (S3) and (S4). □

The intuition coming from real geometry suggests that having local dimension $\geqslant 2$ should be necessary (and sufficient) for a definable set to have arbitrarily many germs at a point. In contrast, the next result shows that local dimension $\geqslant 1$ (which is obviously necessary) is sufficient at least if $\mathcal{K}$ has the following property, which holds true for example in P-minimal structures like the p-adics.

(Z) $v(K^\times)$ is elementarily equivalent to $(\mathbb{Z}, +, \leqslant)$ (i.e., it is a $\mathbb{Z}$-group) and for every definable set $X \subseteq K^\times$, $v(X)$ is definable in $v(K^\times)$.

Proposition 6.5 *If $\mathcal{K}$ satisfies (Z), then every definable open subset U of K has arbitrarily many germs at every point of U.*

Proof: For any $p_0 \in U$, the set $\{v(u - p_0) \colon u \in U \setminus \{p_0\}\}$ is a Presburger set by (Z), and it is not bounded above in $v(K^\times)$. Hence it contains some set

$$A = \big\{\xi \in v(K^\times) \colon \alpha \leqslant \xi \text{ and } \xi \equiv a\,[N]\big\}$$

for some $\alpha \in v(K^\times)$ and some integers a and $N \geqslant 1$ (where $\xi \equiv a\,[N]$ denotes the usual congruence relation $\xi - a \in Nv(K^\times)$). Given an integer $k \geqslant 1$, for $1 \leqslant i \leqslant k$ let

$$A_i = \big\{\xi \in A \colon \alpha \leqslant \xi \text{ and } \xi \equiv a + iN\,[kN]\big\}$$

and let $S_i = v^{-1}(A_i) \cap U$. The sets S_i are pairwise disjoint, clopen in $U \setminus \{p_0\}$, and p_0 belongs to the closure of each of them. So the functions defined by $s_i(x) = 0$ if $x \in S_i$ and $s_i(x) = \delta_{p_0}(x)$ otherwise (where $\{\delta_{p_0} = 0\} = \{p_0\}$, see Lemma 4.1) are continuous on U. Clearly each of them vanishes on a germ at p_0 since $\{s_i = 0\} = S_i \cup \{p_0\}$. □

Remark 6.6 For all definable sets $X \subseteq K^m$, $Y \subseteq K^n$ and every point $p_0 \in X$, if there is a definable homeomorphism $\varphi : U \to V$ such that U (resp. V) is a definable neighborhood of p_0 in X (resp. of $\varphi(p_0)$ in Y) then X has arbitrarily many germs at p_0 if and only if Y has arbitrarily many germs at $\varphi(p_0)$, because this property is local. It then follows from Proposition 6.4 that for every definable set $X \subseteq K^m$ and every integer $k \geqslant 2$, X has arbitrarily many germs at every point of $W_k(X)$ for every $k \geqslant 2$ (and even for $k = 1$ if $\mathcal{K}$ satisfies (Z) by Proposition 6.5).

For every integer $k \geqslant 0$, every definable set $X \subseteq K^m$ and every $Y \subseteq K$ let $\mathcal{C}_k(X, Y)$ (resp. $\mathcal{C}_{\geqslant k}(X, Y)$) be the set of functions $f \in \mathcal{C}(X)$ such that $f(x) \in Y$ for every $x \in \Delta_k(X)$ (resp. $x \in \bigcup_{l \geqslant k} \Delta_l(X)$), and $f(x) = 0$ otherwise.

Theorem 6.7 *Assume that $\mathcal{K}$ satisfies (Dim) and (BFin). Let $X \subseteq K^m$ be a definable set and let τ be a non-zero element of K.*

(1) *If $|\tau| \neq |1|$ then the set $\mathcal{C}_k(X, \tau^{\mathbb{Z}})$ is definable in $\mathcal{L}_{\mathrm{ring}} \cup \{\mathcal{B}\}$ for every integer $k \geqslant 2$, hence so is $\mathcal{C}_{\geqslant 2}(X, \tau^{\mathbb{Z}})$. If moreover $\mathcal{K}$ satisfies (Z) the same holds true for $k \geqslant 1$.*

(2) *If K has a total order $\preccurlyeq$ compatible with $+$ and definable in $\mathcal{L}_{\mathrm{ring}} \cup \{\mathcal{O}\}$ then the set $\mathcal{C}_k(X, \tau\mathbb{Z})$ is definable in $\mathcal{L}_{\mathrm{ring}} \cup \{\mathcal{B}\}$ for every integer $k \geqslant 2$, hence so is $\mathcal{C}_{\geqslant 2}(X, \tau\mathbb{Z})$.*

By Remark 4.4, Theorem 6.7 leads to definability in $\mathcal{L}_{\mathrm{ring}}$ in many cases (in particular if $\mathcal{K}$ is an expansion of a local field or if the theory of $\mathcal{K}$ is dp-minimal) provided K is not algebraically closed.

Remark 6.8 The formulas given by the proof of Theorem 6.7 involve only one parameter: the function with constant value τ. Thus if $\tau \in \mathbb{Z}$ these definitions of $\mathcal{C}_k(X, \tau^{\mathbb{Z}})$ and $\mathcal{C}_k(X, \tau\mathbb{Z})$ are parameter-free.

Proof: Replacing τ by $1/\tau$ if necessary we may assume that $|\tau| < |1|$. Let $\chi_k(p)$ and $\mathrm{Int}^{\times}(h, p, \tau^*)$ be the formulas given by Proposition 6.3 and Theorem 5.5 respectively. Consider the formula $\mathrm{Z}_k(h, \tau^*)$ defined as

$$\forall p \left[\chi_k(p) \to \forall v \left(p \not\sqsubseteq v \to \exists q \left[q \not\sqsubseteq v \wedge \mathrm{Int}^{\times}(h, q, \tau^*)\right]\right)\right].$$

Given $\tau \in K^{\times}$ with $|0| < |\tau| < |1|$ let $\tau^* \in \mathcal{C}(X)$ be the function with constant value τ. Let h be any function in $\mathcal{C}(X)$ and $D_h = \{x \in X : h(x) \in \tau^{\mathbb{Z}}\}$.

Assume that $\mathcal{C}(X) \models \mathrm{Z}_k(h, \tau^*)$. For any point $p_0 \in \Delta_k(X)$ and any open ball $B_0 \subseteq K^m$ centered at p_0 let $p = \delta_{p_0}$ and $v = \delta_{B_0^c}$ be the functions from Lemma 4.1. Then $\mathcal{C}(X)$ satisfies $\chi_k(p)$ and $p \not\sqsubseteq v$; hence $\mathrm{Z}_k(h, \tau^*)$ gives $q \in \mathcal{C}(X)$ having a single zero q_0 with $q_0 \in B$ and $h(q_0) \in \tau^{\mathbb{Z}}$. In other words p_0 belongs to the closure of D_h. By continuity we obtain $h(p_0) \in \tau^{\mathbb{Z}}$, hence $h \in \mathcal{C}_k(X, \tau^{\mathbb{Z}})$.

Conversely assume that $h \in \mathcal{C}_k(X, \tau^{\mathbb{Z}})$. Take $p \in \mathcal{C}(X)$ with $\mathcal{C}(X) \models \chi_k(p)$. Then p has a single zero p_0 and $p_0 \in \Delta_k(X)$. For any $v \in \mathcal{C}(X)$ with $\mathcal{C}(X) \models p \not\sqsubseteq v$, by continuity, the set $V = \{v \neq 0\}$ is a neighborhood of p_0 in X. Hence by Lemma 6.1 there is a point $q_0 \in V \cap W_k(X)$. In particular $q_0 \in \Delta_k(X)$ by Property 2.5 and therefore $h(q_0) \in \tau^{\mathbb{Z}}$. By Remark 6.6 the assumptions on $\mathcal{K}$ ensure that X has arbitrarily many germs at p_0. Thus $\mathcal{C}(X) \models \mathrm{Int}^{\times}(h, q, \tau^*)$ with $q = \delta_{q_0}$ defined in Lemma 4.1 (2), which shows that $\mathcal{C}(X) \models \mathrm{Z}_k(h, \tau^*)$.

The proof of the second statement is similar. Replacing τ by $-\tau$ if necessary we may assume that $\tau \succ 0$. Then let $\tau^* \in \mathcal{C}(X)$ be the function with constant value τ and replace $\mathrm{Int}^{\times}(h, q, \tau^*)$ by $\mathrm{Int}^{+}_{\preccurlyeq}(h, q, \tau^*)$ in $\mathrm{Z}_k(h, \tau^*)$. $\square$

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