Polytopes and simplexes in p-adic fields

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Abstract

We introduce topological notions of polytopes and simplexes, the latter being expected to play in p-adically closed fields the role played by real simplexes in the classical results of triangulation of semi-algebraic sets over real closed fields. We prove that the faces of every p-adic polytope are polytopes and that they form a rooted tree with respect to specialisation. Simplexes are then defined as polytopes whose faces tree is a chain. Our main result is a construction allowing to divide every p-adic polytope in a complex of p-adic simplexes with prescribed faces and shapes.

1 Introduction

Throughout all this paper we fix a p-adically closed field (K, v). The reader unfamiliar with this notion may restrict to the special case when $K = \mathbf{Q}_p$ or a finite extension of it, and v its p-adic valuation. We let R denote the valuation ring of v, and $\Gamma = v(K)$ its valuation group (augmented with one element $+\infty = v(0)$). In this introductory section we present informally what we are aiming at. Precise definitions will be given in section 2 and at the beginning of section 6.

Our long-term objective is to set a triangulation theorem which would be an acceptable analogous over K of the classical triangulation of semi-algebraic sets over the reals. Polytopes and simplexes in $\mathbf{R}^m$ are well known to have the following properties, among others (see for example [BCR98] or [vdD98]).

(Sim) They are bounded subsets of $\mathbf{R}^m$ which can be described by a finite set of linear inequalities of a specially simple sort.

(Fac) There is a notion of "faces" attached to them with good properties: every face of a polytope S is itself a polytope; if S'' is a face of S' and S' a face of S then S'' is a face of S; the union of the proper faces of S is a partition of its frontier.

(Div) Last but not least, every polytope can be divided in simplexes by a certain uniform process of "Barycentric Division" which offers a good control both on their shapes and their faces.

The goal of the present paper is to build a p-adic counterpart of real polytopes and simplexes having similar properties. Obviously we cannot transfer directly linear *inequalities* and *Barycentric* Division to *non-ordered* fields, such as the p-adic ones. Nevertheless we want our p-adic polytopes and simplexes to be defined by conditions as simple as possible, to have a notion of faces satisfying all the above properties, and most of all to come with a flexible and powerful division tool.

This is achieved here by introducing and studying first certain subsets of Γ^m called "largely continuous precells mod N", for a fixed m-tuple N of strictly positive integers. These sets will be defined by a very special triangular system of linear inequalities and congruence relations mod N. In particular they are defined simply by linear inequalities in the special case when $N = (1, \dots, 1)$ (again, see Section 2 for precise definitions and basic examples).

This paper, which is essentially self-contained, is organised as follows. The general properties of subsets of Γ^m defined by conjunctions of linear inequalities and congruence conditions are studied in section 3. Property (Fac) is proved there to hold true for for largely continuous precells mod N (while property (Sim) is a by-product of their definition). Section 4 is devoted to two technical properties preparing the proof of our main result, a construction analogous to (Div) in our context. We call it the "Monohedral Division" (see below). The whole Section 5 is devoted to its proof.

We then return to the p-adic context in the final section 6. By taking inverse images of largely continuous precells by the valuation v (which maps K^m onto Γ^m) and restricting them to certain subsets of R^m, we transport into K^m all the definitions and results built in Γ^m in the previous sections, specially the Monohedral Division (which becomes in this context the "Monotopic Division", Theorem 6.3). This latter result paves the way toward a triangulation of semi-algebraic p-adic sets, to appear in a further paper.

Monohedral division. In addition to (Sim) and (fac), every largely continuous precell A mod N has one more remarkable property which real polytopes are lacking: its proper faces, ordered by specialisation[1], form a rooted tree (Proposition 3.3(4). When this tree is a chain, we say that A is "monohedral".

Among real polytopes of a given dimension, the simplexes are those whose number of facets is minimal: a polytope $A \subseteq \mathbf{R}^m$ of dimension d has at least $d + 1$ facets, and it is a simplex if and only if it has exactly $d + 1$ facets (see Corollary 9.4 and Corollary 12.8 in [Brø83]). We expect largely continuous precells mod N to play in Γ^m the role played by polytopes in $\mathbf{R}^m$, and the monohedral ones (whose ordered set of faces is in a sense the simplest possible tree) to play the role of simplexes[2]

Indeed our main result, the "Monohedral Division" (Theorem 5.5), provides in our context a powerful tool very similar to (Div), the Barycentric Division of real polytopes. It provides in particular a "Monohedral Decomposition" (Theorem 5.6) which says that every largely continuous precell mod N in Γ^m is the disjoint union of a complex of *monohedral* largely continuous precells mod N. The latter result is in analogy with the situation in the real case, where every polytope can be divided in simplexes forming a simplicial complex.

But the Barycentric Division in $\mathbf{R}^m$ says much more than this. Roughly speaking, given a polytope A and a simplicial complex $\mathcal{D}$ partitioning the frontier

[1] The **specialisation pre-order** on the subsets of a topological space is defined as usually by $B \leq A$ if and only if B is contained in the closure or A.

[2] This is specially true when $N = (1, \dots, 1)$. However, since all the results in this paper hold true for arbitrary N it would be pointless to restrict to this case in the present paper.

of A, it allows to build a simplicial complex $\mathcal{C}$ partitioning A and "lifting" $\mathcal{D}$, in the sense that for every C in $\mathcal{C}$, the faces D of C which are outside A belong to $\mathcal{D}$. Moreover, given a strictly positive function $\varepsilon : B \to \mathbf{R}$ (where B is any proper face of A), the shapes of the elements of $\mathcal{C}$ can be required to satisfy the following condition: for every D in $\mathcal{D}$ there is a unique $C \in \mathcal{C}$ such that D is the largest proper face of C in $\mathcal{C}$, and in that case the distance of any point $x \in C$ to its projection y onto B is smaller than $\varepsilon(y)$ (see Figure 1, where the dotted curve shows how the ε function controls the shapes of the elements of $\mathcal{C}$ whose largest proper face outside A is contained in the given facet B).

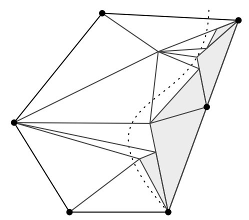

Figure 1: Division with constraints along a facet.

Although all these properties come from the Barycentric Division in $\mathbf{R}^m$, none of them involves the notion of barycenter. The strength of our Monohedral Division in Γ^m (Theorem 5.5) is to keep[3] all of them, which are finally inherited by the Monotopic Division in K^m (Theorem 6.3).

2 Notation, definitions

We let $\mathbf{N}$, $\mathbf{Z}$, $\mathbf{Q}$ denote respectively the set of positive integers, of integers and of rational numbers, and $\mathbf{N}^* = \mathbf{N} \setminus \{0\}$.

Recall that a **Z-group** is a linearly ordered group G with a smallest > 0 element such that G/nG has n elements for every integer $n \geq 1$. The reader unfamiliar with **Z**-groups may restrict to the special but prominent case of **Z** itself. Indeed a linearly ordered group is a **Z**-group if and only if it is elementarily equivalent to **Z** (in the Presburger language $\mathcal{L}_{Pres}$ defined below).

(K, v) is a p-adically closed field in the sense of [PR84], that is a henselian valued field of characteristic zero whose residue field is finite and whose value group $\mathcal{Z} = \Gamma \setminus \{+\infty\} = v(K^*)$ is a **Z**-group. A field is p-adically closed if and only if it is elementarily equivalent (in the language of rings) to a finite extension of $\mathbf{Q}_p$, so the reader unfamiliar with the formalism of model-theory may restrict to this fundamental case.

Let $\mathcal{Q}$ be the divisible hull of $\mathcal{Z}$. By identifying $\mathbf{Z}$ with the smallest non-trivial convex subgroup of $\mathcal{Z}$, we consider $\mathbf{Z}$ embedded into $\mathcal{Z}$ (and $\mathbf{Q}$ into $\mathcal{Q}$). For every $a \in \mathcal{Q}$ we let $|a| = \max(-a, a)$.

$\Omega = \mathcal{Q} \cup \{+\infty\}$ is endowed the topology generated by the open intervals and the intervals $]a, +\infty]$ for $a \in \mathcal{Q}$. Ω^m is equipped with the product topology, and

[3]In addition, the Monohedral and Monotopic Divisions even ensure that every C in $\mathcal{C}$ has no proper face inside A: either C has no proper face at all, or its unique facet is outside A (hence so are all its proper faces) and belongs to $\mathcal{D}$.

Γ^m with the induced topology. The topological closure of any set A in Ω^m is denoted $\overline{A}$. Thus for example $\Omega = \overline{\mathcal{Q}}$ and $\Gamma = \overline{\mathcal{Z}}$. The **frontier** of a subset A of Ω^m is the closure of $\overline{A} \setminus A$. We denote it ∂A.

Whenever we take an element $a \in \Omega^m$ it is understood that $a_1, \dots, a_m \in \Omega$ are its coordinates. We say that a is **positive** if all its coordinates are so. A subset A of Ω^m is positive if all its elements are so. If $m \geq 1$ we let $\widehat{a}$ (resp. $\widehat{A}$) denote the image of a (resp; A) by the coordinate projection of Ω^m onto Ω^{m-1}. We call it the **socle** of a (resp. A). If $\mathcal{A}$ is a family of subsets of Γ^m we also call $\widehat{\mathcal{A}} = \{\widehat{A} : A \in \mathcal{A}\}$ the socle of $\mathcal{A}$.

The **support of** a, denoted $\operatorname{Supp} a$, is the set of indexes i such that $a_i \neq +\infty$. When all the elements of A have the same support, we call it the **support of** A and denote it $\operatorname{Supp} A$. For every subset $I = \{i_1, \dots, i_r\}$ of $[\![1, m]\!]$ we let:

$$F_I(A) = F_{i_1,\dots,i_r}(A) = \{a \in \overline{A} : \operatorname{Supp} a = I\}.$$

When $F_I(A) \neq \emptyset$ we call it the **face of A of support** I. It is an **upward face** if moreover $m \in I$ and $F_{I\setminus\{m\}}(A)$ is non-empty. Note that if A is contained in Γ^m then so are its faces, because Γ^m is closed Ω^m. By construction, $F_I(A) = \overline{A} \cap F_I(\Omega^m)$ hence $\overline{A}$ is the disjoint union of all its faces. A **complex** in Γ^m is a finite family $\mathcal{A}$ of two-by-two disjoint subsets of Γ^m such that for every $A, B \in \mathcal{A}$, $\overline{A} \cap \overline{B}$ is the union of the *common* faces of A and B. It is a **closed complex** if moreover it contains all the faces of its members, or equivalently if $\bigcup \mathcal{A}$ is closed. Note that a finite partition $\mathcal{S}$ of a subset of Γ^m is a closed complex if and only if $\mathcal{S}$ contains the faces of all its members.

The specialisation pre-order is an order on the faces of A. The largest proper faces of A with respect to this order are called its **facets**. We say that A is **monohedral** if its faces are linearly ordered by specialisation. Note that every subset of $F_I(\Gamma^m)$ is clopen in $F_I(\Gamma^m)$. In particular if $A \subseteq F_I(\Gamma^m)$ then ∂A is the disjoint union of its proper faces. Note also that $F_J(A) = \emptyset$ whenever $J \not\subseteq I$.

Example 2.1 Let $A \subseteq \mathbf{Z}^3$ be defined by $a_1 \geq 0$, $a_2 \geq a_1$ and $a_3 = 2a_2 - 2a_1$. It has four non-empty faces: A itself, two facets $F_1(X) = \mathbf{N} \times \{+\infty\} \times \{+\infty\}$ and $F_3(A) = \{+\infty\} \times \{+\infty\} \times 2\mathbf{N}$, plus $F_\emptyset(A) = \{(+\infty, +\infty, +\infty)\}$.

We let π_I^m be the natural projection of Γ^m onto $F_I(\Gamma^m)$. When m is clear from the context, π_I^m is simply denoted π_I. If a (resp. A) is any element (resp. subset) of Γ^m we write $\widehat{a}$ and $\widehat{A}$ instead of $\pi_I^m(a)$ and $\pi_I^m(A)$ with $I = [\![1, m-1]\!]$.

Remark 2.2 Given any $A \subseteq \Gamma^m$ and $b \in F_I(A)$ it is easy to see that $\pi_I(a) = b$ for every $a \in A$ in a sufficiently small[4] neighbourhood of b. Thus $F_J(A) \subseteq \pi_J(A)$ and $\widehat{F_J(A)} \subseteq F_{\widehat{J}}(\widehat{A})$ (where $\widehat{J} = J \setminus \{m\}$).

For every $J \subseteq [\![1, m]\!]$ and $a \in \Omega^m$ we let $\Delta_J^m(a) = \min\{a_i : i \notin J\}$ (if $J = [\![1, m]\!]$ we use the convention that $\min \emptyset = +\infty$ in this definition of $\Delta_J^m(a)$). Again the superscript m is omitted whenever it is clear from the context. Note that for every $a, b \in \Omega^m$

$$\Delta_J(a + b) \geq \Delta_J(a) + \Delta_J(b).$$

[4] A sufficient condition is that $\max_{i \in I} |a_i - b_i| < 1$.

Remark 2.3 When $\mathcal{Z} = \mathbf{Z}$ the topology on Ω^m comes from the distance $d(a,b) = \max_{1\leq i\leq m} |2^{-a_i} - 2^{-b_i}|$, with the convention that $2^{-\infty} = 0$. Thus $2^{-\Delta_J(a)}$ is just the distance from a to its projection $\pi_J(a)$. In the general case the topology on Ω^m no longer comes from a distance. Nevertheless we will keep in mind this geometric intuition, that $\Delta_J(a)$ measures in a sense the distance from a to $F_J(\Omega^m)$: the bigger is $\Delta_J(a)$, the closer is a to $F_J(\Omega^m)$.

This intuition makes the following facts rather obvious.

Fact 2.4 *For every function $f : A \subseteq \Gamma^m \to \Omega$ every $b \in \Gamma^m$ with support J we have:*

1. *$b \in F_J(A)$ iff $\forall\delta \in \mathcal{Z}$, $\exists a \in A$, $\pi_J(a) = b$ and $\Delta_J(a) \geq \delta$.*
2. *If $b \in F_J(A)$ then f has limit $+\infty$ at b iff $\forall\varepsilon \in \mathcal{Z}$, $\exists\delta \in \mathcal{Z}$, $\forall a \in A$, $[\pi_J(a) = b$ and $\Delta_J(a) \geq \delta] \Rightarrow f(a) \geq \varepsilon$.*

Given a vector $u \in \mathcal{Z}^m$ we let $A + u = \{x + u : x \in A\}$. We say that u is **pointing** to some $J \subseteq [\![1,m]\!]$ if $u_i = 0$ for $i \in J$ and $u_i > 0$ for $i \notin J$.

Remark 2.5 Let $J \subseteq I \subseteq [\![1,m]\!]$ and S be any subset of $F_I(\Gamma^m)$. Using Remark 2.2 and the above facts, one easily sees that if for every $\delta \in \mathcal{Z}$ there is $u \in \mathcal{Z}^m$ pointing to J such that $\Delta_J(u) \geq \delta$ and $S + u \subseteq S$ then $F_J(S) = \pi_J(S)$, and in particular $F_J(S) \neq \emptyset$.

A function $f : A \subseteq \Gamma^m \to \Omega$ is **largely continuous** on A if it can be extended to a continuous function on $\overline{A}$, which we will usually denote $\bar{f}$. If A has support I, we say that f is a **affine map** (resp. **linear map**) if either f is constantly equal to $+\infty$, or for some $\alpha_0 \in \mathcal{Q}$ (resp. $\alpha_0 = 0$) and some $(\alpha_i)_{i\in I} \in \mathbf{Q}^I$, we have

$$\forall a \in A, \quad f(a) = \alpha_0 + \sum_{i\in I} \alpha_i a_i. \tag{1}$$

We call α_0 the "constant coefficient" in the above expression of f. If such an expression exists for which $\alpha_0 \in \mathcal{Z}$ and $\alpha_i \in \mathbf{Z}$ for $i \in I$, we say that f is **integrally affine**. A affine map which takes values in Γ will be called **Γ-affine**. For example $f(x) = x/2$ is Γ-affine on $2\mathbf{Z}$ but is not integrally affine.

Remark 2.6 Of course there is no uniqueness in a description of f as in (1). Nevertheless, affinity and linearity are intrinsic because a function $\varphi : A \subseteq F_I(\Gamma^m) \mapsto \mathcal{Q}$ is a linear map if and only if for every $a_1, \dots, a_k \in A$ and every $\lambda_1, \dots, \lambda_k \in \mathbf{Q}^m$:

$$\sum_{1\leq i\leq k} \lambda_i a_i \in A \implies \varphi\Big(\sum_{1\leq i\leq k} \lambda_i a_i\Big) = \sum_{1\leq i\leq k} \lambda_i \varphi(a_i)$$

The symbols of the Presburger language $\mathcal{L}_{Pres} = \{0, 1, +, \leq, (\equiv_n)_{n\in\mathbf{N}^*}\}$ are interpreted as usually in $\mathcal{Z}$: the binary relation $a \equiv_n b$ says that $a - b \in n\mathcal{Z}$, and the other symbols have their obvious meanings. A subset X of $\mathcal{Z}^d$ is **$\mathcal{L}_{Pres}$-definable** if there is a first order formula $\varphi(\xi)$ in $\mathcal{L}_{Pres}$, with parameters in $\mathcal{Z}$ and a d-tuple ξ of free variables, such that $X = \{x \in \mathcal{Z}^d : \mathcal{Z} \models \varphi(x)\}$. A function $f : X \subseteq \mathcal{Z}^d \to \mathcal{Z}$ is **$\mathcal{L}_{Pres}$-definable** its graph is so.

Each $F_I(\Gamma^m)$ identifies to $\mathcal{Z}^d$ with $d = \mathrm{Card}(I)$. We say that a subset A of Γ^m is **definable** if for every $I \subseteq [\![1,m]\!]$ the set $A \cap F_I(\Gamma^m)$ is $\mathcal{L}_{Pres}$-definable by means of this identification. We say that a function $f : A \subseteq \Gamma^m \to \Omega$ is **definable** if there is an integer $N \geq 1$ such that $Nf(X) \subseteq \Gamma$ and if the restrictions of Nf to each $F_I(\Gamma^m)$ become, after this identification, either an $\mathcal{L}_{Pres}$-definable map from $\mathcal{Z}^{\mathrm{Card}(I)}$ to $\mathcal{Z}$ or the constant map $+\infty$. Note that every affine map is definable in this broader sense.

The next characterisation of definable maps and sets comes directly from Theorem 1 in [Clu03].

Theorem 2.7 (Cluckers) *For every definable function $f : A \subseteq \Gamma^m \to \Gamma$ on a positive set A, there exists for some N a partition of A in finitely many definable sets mod N, on each of which the restriction of f is an affine map.*

It is well known that the theory of $\mathbf{Z}$-groups has quantifier elimination and definable Skolem functions. At many places, without mentioning, we will use the latter property under the following form.

Theorem 2.8 (Skolem Functions) *Let $A \subseteq \mathcal{Z}^m$ and $B \subseteq \mathcal{Z}^n$ be two $\mathcal{L}_{Pres}$-definable sets. Let $\varphi(x,y)$ be a first order formula in $\mathcal{L}_{Pres}$. If for every $a \in A$ there is $b \in B$ such that $\mathcal{Z} \models \varphi(a,b)$ then there is a definable map $\lambda : A \to B$ such that $\mathcal{Z} \models \varphi(a, \lambda(a))$ for every $a \in A$.*

Since $\mathcal{Z}$ is elementarily equivalent to $\mathbf{Z}$ in the language $\mathcal{L}_{Pres}$, every non-empty $\mathcal{L}_{Pres}$-definable subset of $\mathcal{Z}$ which is bounded above (resp. below) has a maximum (resp. minimum) element. As a consequence for every $a \in \Omega$ there is in $\mathcal{Z}$ a largest element $\lfloor a \rfloor$ (resp. $\lceil a \rceil$) which is $\leq a$ (resp. $\geq a$). Note that if $f : X \subseteq \mathcal{Z}^d \to \mathcal{Q}$ is definable and $N \geq 1$ is an integer such that Nf is $\mathcal{L}_{Pres}$-definable, then for every integer $0 \leq k < N$ the set $S_k = \{x \in X : Nf(x) \equiv_n k\}$ is $\mathcal{L}_{Pres}$-definable, and so is the map $\lfloor f \rfloor(x) = (Nf(x) - k)/N$ on S_k. Thus the map $\lfloor f \rfloor : S \to \mathcal{Z}$ is $\mathcal{L}_{Pres}$-definable, and so is $\lceil f \rceil$ by a symmetric argument. Obviously the same holds true for every definable map from $A \subseteq \Gamma^m$ to Ω.

Lemma 2.9 *If $f : A \subseteq \Gamma^m \to \Omega$ is a largely continuous definable map on a positive set A, then it has a minimum in A.*

Proof: It suffices to prove the result separately for each $A \cap F_I(\Gamma^m)$ with $I \subseteq [\![1,m]\!]$. Every such piece identifies with a definable subset of $\mathcal{Z}^{\mathrm{Card}\, I}$ hence we can assume that $A \subseteq \mathcal{Z}^m$. Multiplying f by some integer $n \geq 1$ if necessary we can assume that f takes values in Γ, and even in $\mathcal{Z}$ (otherwise f is constantly $+\infty$ and the result is trivial). Since $\mathcal{Z} \equiv \mathbf{Z}$, by instantiating the parameters of a definition of $f : A \subseteq \mathcal{Z}^m \to \mathcal{Z}$ we are reduced to prove the result for every largely continuous definable function on a positive subset A of $\mathbf{Z}^m$. But in that case the topology on Γ^m comes from a metric and every positive subset of Γ^m is precompact (that is $\overline{A}$ is compact). So there is $\bar{a} \in \overline{A}$ such that $\bar{f}(\bar{a}) = \min\{\bar{f}(x) : x \in \overline{A}\}$. For any $a \in A$ close enough to $\bar{a}$ we have $f(a) = \bar{f}(\bar{a})$ (because $f(A) \subseteq \mathbf{Z}$) hence $f(a) = \min\{f(x) : x \in A\}$.
■

Lemma 2.10 *Let $f : A \subseteq \Gamma^m \to \Omega^n$ a continuous definable map. If A is positive then $f(\overline{A})$ is closed.*

Proof: As for Lemma 2.9 we can reduce to the case when $\mathcal{Z} = \mathbf{Z}$. But then $\overline{A}$ is compact, hence so is $f(\overline{A})$ since f is continuous.
■

We extend the binary congruence relations of $\mathcal{Z}$ to Γ with the convention that $a \equiv +\infty\ [N]$ for every $a \in \Gamma$ and every $N \in \mathbf{N}$. A subset A of $F_I(\Gamma^m)$ is a **basic Presburger set** if it is the set of solutions of finitely many linear inequalities and congruence relations. Although we will not use it, it is worth mentioning that, by the quantifier elimination of the theory of $\mathbf{Z}$ in the language $\mathcal{L}_{Pres}$, the definable subsets of $\mathcal{Z}^d$, and more generally of Γ^m, are exactly the finite unions of basic Presburger sets.

Cluckers has shown in [Clu03] that every definable subset of $\mathcal{Z}^d$ is actually the disjoint union of finitely many subsets of a much more restrictive sort, called cells. The following definition of precells in Γ^m, more precisely of precells mod N for a given $N \in (\mathbf{N}^*)^m$, is adapted from his (see Remark 2.11 below). Since we only need to consider positive precells it is convenient to restrict the definition to this case. If $m = 0$, Γ^0 itself is the only precell mod N in Γ^0. If $m \geq 1$, for every $I \subseteq [\![1, m]\!]$, a subset A of $F_I(\Gamma^m)$ is a **precell mod** N if: $\widehat{A}$ is a precell mod $\widehat{N}$, and there are positive affine maps $\mu, \nu : \widehat{A} \to \Omega$ and an integer ρ such that $0 \leq \rho < N_m$ and A is exactly the set of points $a \in F_I(\Gamma^m)$ such that $\widehat{a} \in \widehat{A}$ and

$$\mu(\widehat{a}) \leq a_m \leq \nu(\widehat{a}) \quad \text{and} \quad a_m \equiv \rho\ [N_m]. \tag{2}$$

We call μ, ν the **boundaries** of A, ρ a **modulo** for A, and such a triple (μ, ν, ρ) a **presentation** of A. We call it a **largely continuous presentation** of A if moreover μ, ν are largely continuous. A is a **largely continuous precell mod** N if $m = 0$, or if $\widehat{A}$ is largely continuous precell mod $\widehat{N}$ and A is a precell mod N having a largely continuous presentation.

Remark 2.11 Cells in [Clu03] are not supposed to have positive boundaries, but to be of one of these two types : either $\mu - \nu$ is not finitely bounded or $\mu = \nu$. Unfortunately this condition seems to be too restrictive for our constructions, we had to relax it. Thus our precells are *not* cells in the sense of [Clu03] but a restriction (we require positive boundaries) of a slight generalisation ($\mu - \nu$ can be finitely bounded and non-zero) of them.

3 Faces and projections

In this section we consider a non-empty basic Presburger set $A \subseteq F_I(\Gamma^m)$ defined by

$$\bigwedge_{1 \leq l \leq l_0} \varphi_l(x) \geq \gamma_l \quad \text{and} \quad \bigwedge_{1 \leq l \leq l_1} \psi_l(x) \equiv \rho_l [n_l] \tag{3}$$

where $\varphi_l, \psi_l : F_I(\Gamma^m) \to \mathcal{Z}$ are integrally linear maps, $\gamma_l \in \mathcal{Z}$, ρ_l and n_l are integers such that $0 \leq \rho_l < n_l$. We prove some basic properties on the faces of A and the affine maps on A. Finally we derive from these facts that every face of a largely continuous precell A mod N is a largely continuous precell mod N and has a presentation inherited from A in a uniform way (Proposition 3.11).

Example 2.1 shows that precell mod N (here $N = (1, 1, 1)$) can have a facet which is no longer a precell mod N. But even worse is possible: the next example shows that a precell mod N can have a facet which is not even a basic Presburger set.

Example 3.1 $A \subseteq \mathbf{Z}^3$ is defined by $0 \leq x_1 \leq x_2$ and $(x_1 + 3x_2)/3 \leq z \leq (x_1 + 3x_2 + 1)/3$. Its unique facet $F_1(A)$ is defined by $0 \leq x_1$ and either $x_1 \equiv 0$ [3] or $x_1 \equiv 2$ [3].

Lemma 3.2 *Let $A \subseteq F_I(\Gamma^m)$ be defined by (3). Let J be any subset of $[\![1, m]\!]$. Then $F_J(A) \neq 0$ if and only if for every $\delta \in \mathcal{Z}$ there is $u \in \mathcal{Z}^m$ pointing to J such that $\Delta_J(u) \geq \delta$ and $A + u \subseteq A$.*

Proof: It suffices to prove the result when $I = [\![1, m]\!]$. One direction is general by Remark 2.5, so let us prove the converse. Assume that $F_J(A) \neq \emptyset$ and fix any $\delta \in \mathcal{Z}$. W.l.o.g. we can assume that $\delta > 0$. Pick $y_0 \in F_J(A)$ and let $A_0 = \{x \in A : \pi_J(x) = y_0\}$. By Remark 2.2, $F_J(A_0) = \{y_0\}$.

Assume that for some $k \in [\![0, l_0 - 1]\!]$ we have found a definable subset A_k of A_0 such that $F_J(A_k) = \{y_0\}$ and for every $l \in [\![1, k]\!]$, either φ_l is constant on A_k or $\varphi_l(x)$ tends to $+\infty$ as x tends to y in A_k. If the same holds true for φ_k, let $A_{k+1} = A_k$. Otherwise, there is some $\alpha \in \mathcal{Z}$ such that for every $\omega \in \mathcal{Z}$ there is $x \in A_k$ such that $\Delta_J(x) \geq \omega$ and $\varphi_k(x) \leq \alpha$. The set A of these α's is definable, non-empty, and bounded below since $\varphi_k \geq \gamma_k$ on A_k. Hence it has a minimum, say β. By minimality of β there is $\omega_0 \in \mathcal{Z}$ such that for every $x \in A_k$ such that $\Delta_J(x) \geq \omega_0$, $\varphi_k(x) > \beta - 1$. Thus, for every $\omega \in \mathcal{Z}$ there is $x \in A_0$ such that $\Delta_J(x) \geq \omega$ and $\varphi_k(x) = \beta$ (because $\beta \in A$). With other words, the set A_{k+1} defined by

$$A_{k+1} = \big\{x \in A_k : \varphi_k(x) = \beta\big\}$$

is such that $F_J(A_{k+1}) \neq 0$ (see fact 2.4). It obviously has all the other required properties since it is contained in A_k.

By repeating the process until $k = l_0$ we get a definable set A_{l_0} as above. Pick any $a \in A_{l_0}$, by construction there is $\omega \in \mathcal{Z}$ such that for every $x \in A_{l_0}$ if $\Delta_J(x) \geq \omega$ then $\varphi_l(x) \geq \varphi_l(a)$ for every $l \in [\![1, l_0]\!]$. Pick any $b \in A_{l_0}$ such that $\Delta_J(b) \geq \omega$ and $\Delta_J(b) \geq \delta + a_i$ for every $i \notin J$. It remains to check that $u = b - a$ gives the conclusion. For every $j \in J$, $a_j = b_j = y_{0,j}$ because $a, b \in A_{l_0} \subseteq A_0$ and $\pi_J(A_0) = \{y_0\}$, hence $u_j = 0$. For $i \notin J$ we have $b_i \geq \Delta_J(b) \geq \delta + a_i$, hence $u_i \geq \delta > 0$. In particular u points to J and $\Delta_J(u) \geq \delta$. Finally let x be any element of A_{l_0}. For every $l \leq l_0$ we have $\varphi_l(x) \geq \gamma_l$ since $x \in A$, and by linearity of φ_l

$$\varphi_l(x + u) = \varphi_l(x) + \varphi_l(u) \geq \gamma_l + \varphi_l(u). \tag{4}$$

We also have $\varphi_l(b) = \varphi_l(a) + \varphi_l(u)$ by linearity, and $\varphi_l(b) \geq \varphi_l(a)$ because $\Delta_J(b) \geq \omega$, hence $\varphi_l(u) \geq 0$. It follows that $\varphi_l(x + u) \geq 0$ by (4). On the other hand, for every $l \in [\![1, l_1]\!]$ we have $\psi(x) \equiv \rho_l$ $[n_l]$ because $x \in A$, $\psi(a) \equiv \rho_l$ $[n_l]$ and $\psi(b) \equiv \rho_l$ $[n_l]$ for the same reason, hence $\psi_l(x+u) = \psi_l(x) + \psi_l(a) - \psi_l(b) \equiv \rho_l$ $[n_l]$. Thus $x + u \in A$ for every $x \in A$, which proves the result.
■

Proposition 3.3 *Let $A \subseteq F_I(\Gamma^m)$ be a basic Presburger set, J and H be any subsets of $[\![1, m]\!]$ such that $F_J(A)$ and $F_H(A)$ are non-empty.*

1. *$F_J(A) = \pi_J(A)$.*
2. *If $H \subseteq J$ then $F_H(A) = F_H(F_J(A))$.*

3. $F_H(A) \subseteq \overline{F_J(A)}$ *if and only if* $H \subseteq J$*. In particular the faces of* A *are linearly ordered by specialisation if and only if their supports are linearly ordered by inclusion.*

4. $F_{H\cap J}(A)$ *is non-empty.*

We will refer to the n-th point of Proposition 3.3 as to Proposition 3.3(n).

Remark 3.4 Proposition 3.3(4) shows that the set of faces of A ordered by specialisation is a distributive lower semi-lattice with one smallest element. If S is any monohedral subset of Γ^m, Proposition 3.3(3) implies that every basic Presburger subset A of S is monohedral, and Proposition 3.3(2) that every face of A is again monohedral.

Proof: The first point $F_J(A) = \pi_J(A)$ follows from Lemma 3.2, by Remark 2.5 applied to $S = A$.

For the second point, $H \subseteq J$ implies that $\pi_H(A) = \pi_H(\pi_J(A))$. Since $F_H(A) = \pi_H(A)$ and $F_J(A) = \pi_J(A)$ by the first point, it suffices to prove that $F_H(\pi_J(A)) = \pi_H(\pi_J(A))$. For every $\delta \in \mathcal{Z}$ there is by Lemma 3.2 a vector $u \in \mathcal{Z}^m$ pointing to H such that $\Delta_H(u) \geq \delta$ and $A + u \subseteq A$. Then obviously $\pi_J(A) + u = \pi_J(A + u) \subseteq \pi_H(A)$, and the conclusion follows by Remark 2.5 applied to $S = \pi_J(A)$.

For the third point, one direction follows from the second point and the other direction is general since $F_H(A) \subseteq F_H(\Gamma^m)$, $\overline{F_J(A)} \subseteq \overline{F_J(\Gamma^m)}$, and $F_H(\Gamma^m)$ is disjoint from $\overline{F_J(A)}$ if H is not contained in J.

It remains to prove the last point. For every $\delta \in \mathcal{Z}$, Lemma 3.2 gives u_J and u_H in $\mathcal{Z}^m$ pointing to J and H respectively such that $\Delta_J(u_J) \geq \delta$, $A + u_J \subseteq A$ and similarly for u_H. W.l.o.g. we can assume that $\delta > 0$ hence for every $i \notin J \cap H$, $u_{J,i} + u_{H,i} \geq \delta > 0$. In particular $u_J + u_H$ points to $J \cap H$ and $\Delta_{J\cap H}(u_J + u_H) \geq \delta$. Obviously $A + u_J + u_H$ is contained in A. So $F_{J\cap H}(A)$ is non-empty by Remark 2.5.
■

Proposition 3.5 *Let* $A \subseteq F_I(\Gamma^m)$ *be a basic Presburger set defined by (3),* $f : A \to \Omega$ *be an affine map,* $J \subseteq I$ *and* $B = F_J(A)$*. Assume that* B *is not empty and that* f *extends to a continuous map* $f^* : A \cup B \to \Omega$*. Then* f^* *is affine, and if* $f^* \neq +\infty$ *then* $f = f^*_{|B} \circ \pi_{J|A}$*. In particular if* $f^* \neq +\infty$ *then* $f(A) = f^*(B)$*.*

If f is Γ-affine then so is f^* of course. However, if f is integrally affine we cannot conclude that f^* will be integrally affine as well, even if f is largely continuous, as the following example shows.

Example 3.6 Keep $A \subseteq \mathbf{Z}^3$ as in Example 2.1. The map $f(x) = x_2 - x_1$ is integrally affine and largely continuous on A, with $\overline{f}(x) = x_3/2$ on ∂A. This is no longer an integrally affine map on $B = F_3(A) = \{+\infty\} \times \{+\infty\} \times 2\mathbf{N}$.

Proof: It suffices to prove the result when $I = [\![1, m]\!]$, $f < +\infty$ is an integrally linear map and f^* is not constantly equal to $+\infty$. Let φ be an integrally linear map on $\mathcal{Z}^m$ extending f, and $b \in B$ such that $f^*(b) < +\infty$. Since $f(A) \subseteq \mathcal{Z}$ and $f(x)$ tends to $f^*(b)$ at b, there exists $\delta \in \mathcal{Z}$ such that for every $x \in A$, if $\pi_J(x) = b$ and $\Delta_J(x) \geq \delta$ then $f(x) = f^*(b)$. Pick any $a \in A$ such that $\pi_J(a) = b$ and $\Delta_J(a) \geq \omega$, hence $f(a) = f^*(b)$.

Now assume for a contradiction that $f(x_0) \neq f^*(\pi_J(x_0))$ for some $x_0 \in A$. Let $y_0 = \pi_J(x_0)$, since $f(x)$ tends to $f^*(y_0)$ at y_0 and $f^*(y_0) \neq f(x_0)$ there exists $\omega \in \mathcal{Z}$ such that for every $x \in A$, if $\pi_J(x) = y_0$ and $\Delta_J(x) \geq \omega$ then $f(x) \neq f(x_0)$. Lemma 3.2 gives $u \in \mathcal{Z}^m$ pointing to J such that $\Delta_J(u) \geq \omega - \Delta_J(x_0)$ and $A + u \subseteq A$. Then $\pi_J(x_0 + u) = \pi_J(x_0) = y_0$ and $\Delta_J(x_0 + u) \geq \Delta_J(x_0) + \Delta_J(u) \geq \omega$, hence $f(x_0 + u) \neq f(x_0)$. By linearity it follows that $\varphi(u) = f(x_0 + u) - f(x_0) \neq 0$. On the other hand we have $\Delta_J(a + u) \geq \Delta_J(a) + \Delta_J(u) \geq \delta$ and $\pi_J(a + u) = \pi_J(a) = b$ hence $f(a + u) = f^*(b) = f(a)$, and thus by linearity $\varphi(u) = f(a + u) - f(a) = 0$, a contradiction.

■

Proposition 3.7 *Let $A \subseteq F_I(\Gamma^m)$ be a basic Presburger set with $m \geq 1$, and $X = \widehat{A}$. Then for every face $B = F_J(A)$, $\widehat{B} = F_{\widehat{J}}(\widehat{A})$ is a face of $\widehat{A}$. If moreover A is positive, then conversely for every face Y of X there is a face B of A such that $\widehat{B} = Y$. In that case $B = Y \times \{+\infty\}$ if $m \notin \operatorname{Supp} B$, and $B = (Y \times \mathcal{Z}) \cap \overline{A}$ if $m \in \operatorname{Supp} B$.*

Remark 3.8 The last assertion on B is general: for every subset S of $F_I(\Gamma^m)$ and every face $T = F_J(\Gamma^m)$ with socle Y, we have $T = Y \times \{+\infty\}$ if $m \notin J$, and $T = (Y \times \mathcal{Z}) \cap \overline{S}$ if $m \in J$. Indeed $T = F_J(\Gamma^m) \cap \overline{S}$ and $F_J(\Gamma^m)$ is equal to $F_{\widehat{J}}(\Gamma^{m-1}) \times \{+\infty\}$ if $m \notin J$ and to $F_{\widehat{J}}(\Gamma^{m-1}) \times \mathcal{Z}$ otherwise.

Example 3.9 Let $A = \{x \in \mathbf{Z}^3 : x_1 - x_2 - x_3 = 0\}$, $B = F_3(A)$ its unique proper face, $J = \{3\}$. Then $\widehat{A} = \mathbf{Z}^2$ has two facets $\mathbf{Z} \times \{+\infty\}$ and $\{+\infty\} \times \mathbf{Z}$ which are not the socle of any face of A.

This example shows that the assumption that A is positive is necessary for the second part of Proposition 3.7 to hold. Note that $\widehat{B} = \{(+\infty, +\infty)\}$ is not a facet of $\widehat{A} = \mathbf{Z}^2$, which shows that the positivity of A is mandatory also in Corollary 3.10.

Proof: Given that $B = F_J(A)$ is a face of A, hence non-empty, let us prove that $F_{\widehat{J}}(\widehat{A}) = \pi_{\widehat{J}}(\widehat{A})$. For every $\delta \in \mathcal{Z}$ we can find a vector $u \in \mathcal{Z}^m$ pointing to J such that $\Delta_J(u) \geq \delta$ and $A + u \subseteq A$, in particular $\widehat{u}$ points to $\widehat{J}$, $\Delta_{\widehat{J}}(\widehat{u}) \geq \delta$ and $\widehat{A} + \widehat{u} \subseteq \widehat{A}$. Thus $F_{\widehat{J}}(\widehat{A}) = \pi_{\widehat{J}}(\widehat{A})$ by Remark 2.5 applied to $S = \widehat{A}$. Since $B = \pi_J(A)$ by Proposition 3.3(1), and obviously $\widehat{\pi_J(A)} = \pi_{\widehat{J}}(\widehat{A})$, it follows that $\widehat{B} = F_{\widehat{J}}(\widehat{A})$.

Now assume that A is positive. Then the socle of $\overline{A}$ is closed by Lemma 2.10. It contains X, hence $\overline{X}$. In particular it contains Y, which is non-empty. So there is $b \in \overline{A}$ whose socle $\widehat{b}$ belongs to Y. Let $J = \operatorname{Supp} b$ and $B = F_J(A)$. Since B contains b it is non-empty, hence a face of A. Then $\widehat{B}$ is a face of X by the first point. Since $\widehat{b}$ belongs both to Y and $\widehat{B}$, it follows that $Y = \widehat{B}$.

■

Corollary 3.10 *Let $A \subseteq F_I(\Gamma^m)$ be a non-closed positive basic Presburger set with socle X. Let B be a facet of A with socle Y. Then $Y = X$ or Y is a facet of X.*

Proof: By Proposition 3.7, Y is a face of X. If $Y \neq X$ then there is a facet X' of X whose closure contains Y. It remains to show that $Y = Y'$. Proposition 3.7 gives a face B' of A with socle X'. Let J, J', H, H' be the supports of A, A', B, B' respectively. Obviously $H = J \setminus \{m\}$ and $H' = J' \setminus \{m\}$. If $B = B'$ then $J = J'$ hence $H = H'$ and thus $Y = Y'$. Now assume that $B \neq B'$. Since B is a facet of A it is not smaller than B' (with respect to the specialisation order) hence $J \not\subseteq J'$ by Proposition 3.3(3). On the other hand $Y \leq Y'$ hence $H \subseteq H'$ (otherwise $F_J(\Gamma^{m-1})$ is disjoint from the closure of $F_{J'}(\Gamma^{m-1})$). Altogether this implies that $J = J' \cup \{m\}$. In particular $H = J \setminus \{m\} = J' \setminus \{m\} = H'$ hence $Y = Y'$ is a facet of X.

■

Proposition 3.11 *Let $A \subseteq F_I(\Gamma^m)$ be a largely continuous precell mod N with $m \geq 1$. Let (μ, ν, ρ) be a largely continuous presentation of A, J a subset of I, $\widehat{J} = J \setminus \{m\}$ and $Y = F_{\widehat{J}}(\widehat{A})$. Then $F_J(A) \neq \emptyset$ if and only if either $m \in J$ and $\bar{\mu} < +\infty$ on Y, or $m \notin J$ and $\bar{\nu} = +\infty$ on Y. In any case*

$$F_J(A) = \big\{b \in F_J(\Gamma^m) : \widehat{b} \in Y,\ \bar{\mu}(\widehat{b}) \leq b_m \leq \bar{\nu}(\widehat{b}) \text{ and } b_m \equiv \rho\ [N_m]\big\}.$$

In particular, if $F_J(A)$ is non-empty then it is a largely continuous precell A mod N and $(\bar{\mu}_{|Y}, \bar{\nu}_{|Y}, \rho)$ is a presentation of $F_J(A)$.

Remark 3.12 Combining the last point of the above result with Remark 3.4, we get that if A is a *monohedral* largely continuous precell mod N in Γ^m then every face of A is so.

Proof: Let X be the socle of A. Recall that $Y = F_{\widehat{J}}(X)$ is a face of X and the socle of $F_J(A)$ by Proposition 3.3(1). Let B be the set of $a \in F_J(\Gamma^m)$ such that $\widehat{a} \in F_{\widehat{J}}(\widehat{A}) = Y$, $\bar{\mu}(\widehat{a}) \leq a_m \leq \bar{\nu}(\widehat{a})$ and $a_m \equiv \rho\ [N_m]$. Large inequalities and congruence relations valid on A pass to the limits, hence remain valid on $F_J(A)$. So $B \subseteq F_J(A)$, and if $F_J(A) \neq \emptyset$ then necessarily one of the two alternatives of the first point hold true.

Conversely, take any point $a \in A$ and let $b = \pi_J(a)$. Assume first that $m \in J$ and $\bar{\mu} < +\infty$. By Proposition 3.5, $\bar{\mu} \circ \pi_J = \mu$ on X hence $\bar{\mu}(\widehat{b}) = \mu(\widehat{a}) \leq a_m = b_m$. If $\bar{\nu} < +\infty$ then similarly $b_m \leq \bar{\nu}(\widehat{b})$. Otherwise $\bar{\nu} = +\infty$ and $b_m \leq \bar{\nu}(\widehat{b}$ is obvious. Since $b_m = a_m \equiv \rho\ [N]$ it follows in both cases that $b \in B$. Now assume that $m \notin J$ and $\bar{\nu} = +\infty$. Then $b_m = +\infty$, hence obviously $\bar{\mu}(\widehat{b}) \leq +\infty = b_m = \bar{\nu}(\widehat{b})$ and $b_m = +\infty \equiv \rho\ [N]$. Thus $b \in B$, which proves that $\pi_J(A) \subseteq B$. In particular $B \neq \emptyset$, hence $F_J(A) \neq \emptyset$ since it contains B. This proves the first point. Moreover by Proposition 3.3(1) it follows that $F_J(A) = \pi_J(A) \subseteq B$. Hence $F_J(A) = B$, which proves the second point. In particular $F_J(A)$ is a largely continuous precell if $F_{\widehat{J}}(Y)$ is so. The remaining of the conclusion then follows by a straightforward induction.

■

4 Bounding functions

Proposition 4.1 *Let $A \subseteq F_I(\Gamma^m)$ be a definable set, and $f_1, \dots, f_r$ be definable maps from A to $\mathcal{Q}$. Assume that the coordinates of all the points of A are* *positive. Then there exists a largely continuous, strictly positive, integrally affine map* $f : A \to \mathcal{Z}$ *such that* $f(x) \geq \max_j f_j(x)$ *on* A *and* $\bar{f} = +\infty$ *on* ∂A*. More precisely* f *can be taken of the form* $f(x) = \beta + \alpha \sum_{i \in I} x_i$ *on* A*, for some strictly positive* $\alpha \in \mathbf{Z}$ *and* $\beta \in \mathcal{Z}$*.*

Proof: W.l.o.g. we can assume that $I = [\![1, m]\!]$. Theorem 2.7 reduces to the case when the f_j's are affine. Each f_j then writes

$$f_j(x) = \alpha_{0,j} + \sum_{1 \leq i \leq m} \alpha_{i,j} x_i$$

for some $\alpha_{i,j} \in \mathbf{Z}$ for $i \geq 1$ and some $\alpha_{0,j} \in \mathcal{Z}$. Let $\alpha \geq 1$ be an integer greater than $\alpha_{i,j}$ for every $i, j \geq 1$, and $\beta \geq 1$ an element of $\mathcal{Z}$ greater the $\alpha_{0,j}$ for every $j \geq 1$. For every x in A and every $i, j \geq 1$, since $x_i \geq 0$ we have $\alpha x_i \geq \alpha_{i,j} x_i$. So the function $f(x) = \beta + \alpha \sum_{1 \leq i \leq m} x_i$ has all the required properties.
■

Lemma 4.2 *Let* $A \subseteq \mathcal{Z}^m$ *be a largely continuous precell mod* N*,* X *its socle,* (μ, ν, ρ) *a largely continuous presentation of* A *and* f *a largely continuous affine map on* A *such that* $\bar{f} = +\infty$ *on* ∂A*. Let* $(\alpha_i)_{1 \leq i \leq m} \in \mathbf{Q}^m$ *and* $\beta \in \mathcal{Q}$ *be such that* $f(a) = \beta + \sum_{1 \leq i \leq m} \alpha_i a_i$ *on* A*. Extend* f *to* $\mathcal{Q}^m$ *by means of this expression. For every* $x \in \widehat{A}$ *let* $\hat{f}(x) = f(x, \mu(x))$ *if* $\alpha_m \geq 0$*, and* $\hat{f}(x) = f(x, \nu(x))$ *otherwise. Then* $\hat{f}$ *is a well-defined largely continuous affine map on* X *with limit* $+\infty$ *at every point of* ∂X*, and* $\min f(A) - |\alpha_m| N_m \leq \hat{f}(\widehat{a}) \leq f(a)$ *for every* $a \in A$*.*

Proof: The only possible problem in the definition of $\hat{f}$ is when $\nu = +\infty$. But then $\alpha_m \geq 0$ because otherwise, given any $x \in X$ we have $(x, +\infty) \in \partial A$ and f has limit $-\infty$ at $(x, +\infty)$, a contradiction. Thus $\hat{f}(x, \mu(x))$ is well-defined in this case too.

Let $\lambda = \mu$ if $\alpha_m \geq 0$ and $\lambda = \nu$ otherwise. Then $\hat{f}(x) = f(x, \lambda(x))$ is an affine map and $\alpha_m(a_m - \lambda(\widehat{a}))$ is positive on A by construction, hence

$$f(a) = f\big(\widehat{a}, \lambda(\widehat{a})\big) + \alpha_m\big(a_m - \lambda(\widehat{a})\big) \geq \hat{f}(\widehat{a}).$$

For every $x \in X$ there is a point $a \in A$ such that $\widehat{a} = x$ and $|a_m - \lambda(x)| \leq N_m$. So there is a definable function $\delta : X \to \mathcal{Z}$ such that $(x, \delta(x)) \in A$ and $|\delta(x) - \lambda(x)| \leq N_m$ for every $x \in X$. We have

$$\begin{aligned} f(x, \lambda(x)) &= f\big(x, \delta(x)\big) + \alpha_m\big(\lambda(x) - \delta(x)\big) \\ &\geq f\big(x, \delta(x)\big) - |\alpha_m| N_m \end{aligned}$$

In particular $\hat{f}(x) \geq \min f(A) - \alpha_m N_m$ on X.

It only remains to check that, given any $y \in \partial X$, $f(x, \lambda(x))$ tends to $+\infty$ when $x \in A$ tends to y. By the above inequality it suffices to prove that $f(x, \delta(x))$ tends to $+\infty$ when $x \in A$ tends to y. Since $(x, \delta(x)) \in A$ for every $x \in X$ and $\bar{f} = +\infty$ on ∂A, it is sufficient to show that $\delta(x)$ tends to a limit $l \in \Gamma$ as $x \in X$ tends to y. Indeed, since $y \in \partial X$ we will then have that $(x, \delta(x))$ tends to $(y, l) \in \partial A$ so the conclusion. We prove it only when $\alpha_m \geq 0$, the case when $\alpha_m < 0$ being similar.

If $\bar{\mu}(y) = +\infty$, then obviously $\delta(x)$ tends to $+\infty$ since $\mu(x) \leq \delta(x)$. If $\bar{\mu}(y) < +\infty$ then $\mu(x) = \bar{\mu}(y)$ for every $x \in X$ close enough to y. Hence $\delta(x)$, which is the smallest element t in Γ such that $\mu(x) \leq t$ and $t \equiv \rho\ [N_m]$, remains constant too. In particular it has a limit in $\mathcal{Z}$ as $x \in X$ tends to y.

■

Proposition 4.3 *Let $A \subseteq F_I(\Gamma^m)$ be a largely continuous precell mod N, and $f_1, \dots, f_r$ be largely continuous affine maps on A such that $\overline{f_j} = +\infty$ on ∂A for every j. Then there exists a largely continuous affine map f on A such that $\overline{f} = +\infty$ on ∂A and $f(x) \leq \min_j f_j(x)$ for every $x \in A$. If, moreover, each f_j is strictly positive on A then f can be chosen strictly positive on A.*

Proof: W.l.o.g. we can assume that $A \subseteq \mathcal{Z}^m$ and $f_j < +\infty$ for every j. By Lemma 2.9 there is $\gamma \in \mathcal{Q}$ such that $\gamma = \min \bigcup_j f_j(A)$. Given an arbitrary $\gamma' < \gamma$ in $\mathcal{Q}$ we are going to show that there exists a largely continuous map $f : A \to \mathcal{Q}$ such that $\bar{f} = +\infty$ on ∂A and $\gamma' \leq f(x) \leq \min_j f_j(x)$ on A. This will prove simultaneously the last statement, since if each f_j is strictly positive then $\gamma > 0$ hence taking for example $\gamma' = \gamma/2$ will give that $0 < \gamma/2 \leq f$ on A.

The proof goes, needless to say, by induction on m. If $m = 0$, and more generally if A is closed, the constant function $f = \gamma$ has the required properties. So we can assume that A is not closed, $m \geq 1$ and the result is proved for smaller integers. Replacing each f_j by $f_j - \gamma$ we can assume that $\gamma = 0$. Replacing $\gamma' < 0$ by a bigger one if necessary we can assume that $\gamma' \in \mathbf{Q}$.

Let $\alpha_{i,j} \in \mathbf{Q}$ and $\beta_j \in \mathcal{Q}$ such that $f_j(x) = \beta_j + \sum_{1 \leq i \leq m} \alpha_{i,j} x_i$. Let $\hat{f}_j : X \to \mathcal{Q}$ be defined as in Lemma 4.2, and $\eta = \min \bigcup_j \hat{f}_j(X)$. By Lemma 4.2 the induction hypothesis applies to these functions. Given any $\eta' < \eta$, it gives a largely continuous affine map $g : X \to \mathcal{Q}$ such that $\bar{g} = +\infty$ on ∂X and $\eta' \leq g(x) \leq \hat{f}_j(x)$ on X for $1 \leq j \leq r$. We do this for $\eta' = -(\max_j |\alpha_{m,j}| N_m + 1)$. Indeed by Lemma 4.2, $-|\alpha_{m,j}| N_m \leq \hat{f}_j$ on X for $1 \leq j \leq r$ hence $\eta' \leq \eta - 1 < \eta$. Since $\eta' < 0$, replacing γ' by a bigger one if necessary we can assume that $\eta' \leq \gamma'$.

Case 1: $\nu_A = +\infty$. Then for $1 \leq j \leq r$ the coefficient $\alpha_{m,j}$ of x_m in the above expression of f_j is strictly positive (see the proof of lemma 4.2), hence $\hat{f}_j(x) = f_j(x, \mu(x))$ and $\alpha = \min_{j \leq r} \alpha_{m,j}$ is strictly positive. Let $G(a) = g(\widehat{a}) + \alpha(a_m - \mu(\widehat{a}))$ on A. For $1 \leq j \leq r$ we have

$$G(a) \leq \hat{f}_j(x) + \alpha_{m,j}\big(a_m - \mu(\widehat{a})\big) = f_j\big(\widehat{a}, \mu(\widehat{a})\big) + \alpha_{m,j}\big(a_m - \mu(\widehat{a})\big) = f_j(a).$$

Every $b \in \partial A$ either belongs to $X \times \{+\infty\}$ or to $\partial X \times \Gamma$. If $b \in X \times \{+\infty\}$ then $G(a) = g(\widehat{a}) + \alpha(a_m - \mu(\widehat{a}))$ tends to $+\infty$ as $a \in A$ tends to b, because $\widehat{a}$ then tends to $\widehat{b}$, a_m tends to $+\infty$ and $\alpha > 0$. If $b \in \partial X \times \Gamma$ then $G(a) \geq g(\widehat{a})$ tends to $+\infty$ as $a \in A$ tends to b, because $\widehat{a}$ then tends to $\widehat{b}$. Hence G is largely continuous and $\bar{G} = +\infty$ on ∂A.

Case 2: $\nu_A < +\infty$. Then every $b \in \partial A$ belongs to $\partial X \times \Gamma$ hence $g(\widehat{a})$ tends to $+\infty$ as $a \in A$ tends to b. Moreover $g(\widehat{a}) \leq \hat{f}_j(\widehat{a}) \leq f_j(a)$ for $1 \leq j \leq r$.

Cases 1 and 2: In both cases, it remains to modify G so that its minimum becomes greater than γ'. By construction $G(a) \geq g(\widehat{a}) \geq \eta'$ on A. Recall that $\eta' = -(\max_j |\alpha_{m,j}|N_m + 1)$ and $\gamma' \geq \eta'$ are strictly negative *rational* number. Thus we can define $f(a) = (\gamma'/\eta')G(a)$ on A. Clearly f is a largely continuous affine function on A with $\bar{f} = +\infty$ on ∂A, and $f \geq (\gamma'/\eta')\eta' = \gamma'$ since $\gamma'/\eta' \geq 0$ and $G \geq \eta'$ on A. Moreover $0 \leq \gamma'/\eta' \leq 1$ hence for every $a \in A$:

$$f(a) = \frac{\gamma'}{\eta'}G(a) \leq \max(0, G(a)) \leq \min_{1 \leq j \leq r} f_j(a)$$

■

5 Monohedral division

Lemma 5.1 *Let $A \subseteq F_I(\Gamma^m)$ be a non-closed largely continuous precell mod N. Let B be a facet of A, J its support, $f : B \to \mathcal{Z}$ a definable map. Let $\mathcal{D}$ be a family of largely continuous monohedral precells mod N such that $\bigcup \mathcal{D} = B$. Then there exists a pair $(\mathcal{C},\mathcal{U})$ of families of largely continuous precells mod N contained in A and an integrally affine map $\delta : B \to \mathcal{Z}$ such that $\mathcal{U}$ is a finite partition of $A \setminus \bigcup \mathcal{C}$, the proper faces of every precell in $\mathcal{U}$ are proper faces of A, and $\mathcal{C}$ is a family $(C_D)_{D \in \mathcal{D}}$ of precells with the following properties:*

(Fac) *C_D has a unique facet which is D.*

(Sub) *$C_D \subseteq \{a \in A : \pi_J(a) \in D$ and $\Delta_J(a) \geq f \circ \pi_J(a)\}$.*

(Sup) *$C_D \supseteq \{a \in A : \pi_J(a) \in D$ and $\Delta_J(a) \geq \delta \circ \pi_J(a)\}$.*

(Diff) *For every $E \in \mathcal{D}$, $\pi_J(C_D \setminus C_E) \subseteq D \setminus E$.*

Remark 5.2 In every application of Lemma 5.1, $\mathcal{D}$ will be a partition of B. So the condition (Diff) simply says that the precells in $\mathcal{C}$ are two-by-two disjoint, hence that $\mathcal{C} \cup \mathcal{U}$ is a partition of A. However we can not restrict to this case because it may happen that $\mathcal{D}$ is a partition of B and $\widehat{\mathcal{D}}$ is not a partition of $\widehat{B}$, which will be crippling when proving the result by induction on m.

Before entering in the somewhat intricate proof of this lemma, let us make a few preliminary observations.

Claim 5.3 *With the notation of Lemma 5.1, B is not a face of any $U \in \mathcal{U}$.*

Proof: For every $b \in B$ there is $D \in \mathcal{D}$ such that $b \in C_D$. By (Sup) every point in A such that $\pi_J(a) = b$ and $\Delta_J(a) \geq \delta \circ \pi_J(a)$ belongs to C_D, hence not to U. Thus $b \in \overline{U}$, that is $B \cap \overline{U} = \emptyset$.

■

Claim 5.4 *Let $A \subseteq F_I(\Gamma^m)$ be a non-closed largely continuous precell mod N, B a facet of A, $J = \operatorname{Supp} A$. Let C_D, D be precells mod N contained in A, B respectively. Let $f, \delta : B \to \mathcal{Z}$ be two definable maps such that properties (Sub) and (Sup) of Lemma 5.1 hold true. If f is largely continuous and $\bar{f} = +\infty$ on ∂B then (Fac) holds true.*

Proof: For every $b \in D$ and every $\varepsilon \in \mathcal{Z}$, $b \in \overline{A}$ hence there exists $a \in A$ such that $\pi_J(a) = b$ and $\Delta_J(a) \geq \max(\delta(b), \varepsilon)$. By (Sup) this point a belongs to C_D, hence b is in the closure of C_D. So $D \subseteq F_J(C_D)$, and conversely (Sub) implies that $\pi_J(C_D) \subseteq D$, hence $F_J(C_D) = \pi_J(C_D) = D$ by Proposition 3.3(1).

Assume for a contradiction that C_D has a proper face $F_H(C_D)$ not contained in $\overline{D}$. Pick any c in $F_H(C_D)$. By Proposition 3.3(3), H is not contained in J so pick any $k \in H \setminus J$. By Proposition 3.3(4), $F_{J\cap H}(C_D) \neq 0$ hence by the remaining of Proposition 3.3, $\pi_{J\cap H}(C_D) = F_{J\cap H}(C_D) = F_{J\cap H}(F_H(C_D)) \subseteq F_{J\cap H}(B) \subseteq \partial B$. So $\pi_{J\cap H}(c) \in \partial B$, hence f has limit $+\infty$ at $\pi_{J\cap H}(c)$. In particular there is $\delta \in \mathcal{Z}$ such that for every $b \in B$

$$\big[\pi_{J\cap H}(b) = \pi_{J\cap H}(c) \text{ and } \Delta_{J\cap H}(b) \geq \delta\big] \Rightarrow f(b) > c_k \tag{5}$$

On the other hand $c \in F_H(C_D)$ hence there is $a \in C_D$ such that $\pi_H(a) = c$ and $\Delta_H(a) \geq \delta$. Let $b = \pi_J(a)$, then $\pi_{J\cap H}(b) = \pi_{J\cap H}(a) = c$ and

$$\Delta_{J\cap H}(b) = \min_{j\notin H} b_j = \min_{j\in J\setminus H} a_j \geq \min_{i\notin H} a_i = \Delta_H(a) \geq \delta.$$

By (5) this implies that $f(b) > c_k$, that is $f \circ \pi_J(a) > c_k$. By (Sub) it follows that $\Delta_J(a) > c_k$, a contradiction since $\Delta_J(a) = \min_{j\notin J} a_j \leq a_k$ (because $k \notin J$) and $a_k = c_k$ (because $k \in H$ and $\pi_H(a) = c$.

■

Proof: Let (μ, ν, ρ) be a largely continuous presentation of A. Let X, Y be the socles of A, B respectively, $\widehat{I} = \operatorname{Supp} X$ and $\widehat{J} = \operatorname{Supp} Y$. Since B is a facet of A, by Proposition 3.7 either $Y = X$ and $B = X \times \{+\infty\}$, or Y is a facet of A and either $B = Y \times \{+\infty\}$ or $B = (Y \times \mathcal{Z}) \cap \overline{A}$. For each $D \in \mathcal{D}$ let (μ_D, ν_D, ρ_D) be a largely continuous presentation of D.

If $m = 0$ the result is trivially true because there is no non-closed precell contained in Γ^0. So we can assume that $m \geq 1$ and the result is proved for smaller integers. If $\mu = +\infty$ then $A = X \times \{+\infty\}$ identifies to X, to which the induction hypothesis applies. So we can assume that $\mu < +\infty$. Proposition 4.1 gives strictly positive $\alpha \in \mathbf{Z}$ and $\beta \in \mathcal{Z}$ such that $f(x) \leq \beta + \alpha \sum_{j\in J} x_j$ on B. W.l.o.g. we can assume that equality holds on B, and we still denote by f the corresponding extension of f to $F_J(\Omega^m)$.

Thanks to Claim 5.4, (Fac) will automatically follow from (Sub) and (Sup). We will only check that $\pi_J(C_D) = D$, so that (Sub) and (Diff) boil down respectively to the properties that $\Delta_J \geq f \circ \pi_J$ on C_D, and that $\pi_J(D \setminus E)$ is disjoint from E. Note also that it suffices to prove (Sup) with $\delta : B \to \mathcal{Q}$ any definable map since it will then hold true for every larger map, and Proposition 4.1 provides an integrally affine one.

Case 1: $Y = X$.
Then $B = X \times \{+\infty\}$ hence $\nu = +\infty$ and $J = I \setminus \{m\}$, thus $\Delta_J(a) = a_m$ and $\pi_J(a) = (\widehat{a}, +\infty)$ for every $a \in A$. Proposition 4.1 gives a largely continuous affine function $\lambda : X \to \mathcal{Z}$ such that $\lambda(x) \geq \max(f(x, +\infty), \mu(x) + N_m)$ on X. Let U be the set of $a \in F_I(\Gamma^m)$ such that $\widehat{a} \in X$, $\mu(\widehat{a}) \leq a_m \leq \lambda(\widehat{a})$ and $a_m \equiv \rho\ [N_m]$. It is clearly a largely continuous precell mod N (with socle X since $\lambda \geq \mu + N_m$). For each $D \in \mathcal{D}$ let C_D be the set of $a \in F_I(\Gamma^m)$ such that $\widehat{a} \in \widehat{C}$, $\lambda(\widehat{a}) + 1 \leq a_m$ and $a_m \equiv \rho\ [N_m]$. This is a largely continuous precell mod

N with socle $\widehat{D}$. Let $\mathcal{C} = \{C_D : D \in \mathcal{D}\}$ and $\mathcal{U} = \{U\}$. Obviously $\bigcup \mathcal{C} = A \setminus U$, $\partial U = \partial B$ and for every $D, E \in \mathcal{D}$, $\pi_J(C_D \setminus C_E) = D \setminus E$.

By construction, for every $D \in \mathcal{D}$ and every $a \in C_D$ we have

$$\Delta_J(a) = a_m > \lambda(\widehat{a}) \geq f(\widehat{a}, +\infty) = f\big(\pi_J(a)\big).$$

Conversely, for every $a \in A$ such that $\pi_J(a) \in D$ and $\Delta_J(a) \geq \lambda(\widehat{a})$, we have $\widehat{a} \in \widehat{D}$, $a_m \geq \lambda(\widehat{a})$ and $a_m \equiv \rho\ [N_m]$ hence $a \in C_D$. This proves (Sub) and (Sup), with $\delta(b) = \lambda(\widehat{b})$ on B.

Case 2: Y is a facet of A and $B = Y \times \{+\infty\}$.
Then $J = \widehat{J}$, $\bar{\mu} = +\infty$ on Y (otherwise by Proposition 3.11, $F_{J \cup \{m\}}(A) \neq \emptyset$ is a proper face of A larger than B) and $\nu < +\infty$ (otherwise $X \times \{+\infty\}$ is a proper face of A larger than B). In particular $\mu(x) \geq f(y, +\infty)$ for every $y \in Y$ and every $x \in X$ close enough to y, so there is a definable map $\eta : Y \to \mathcal{Z}$ such that for every $x \in X$

$$\Delta_{\widehat{J}}(x) \geq \eta\big(\pi_{\widehat{J}}(x)\big) \Rightarrow \mu(x) \geq f\big(\pi_{\widehat{J}}(x), +\infty\big). \tag{6}$$

The induction hypothesis applies to X, Y, $\widehat{\mathcal{D}}$ and $g(y) = \max(f(y, +\infty), \eta(y))$ on Y. It gives a definable map $\varepsilon : Y \to \mathcal{Z}$ and a pair $(\mathcal{S}, \mathcal{W})$ of families of precells. For each $W \in \mathcal{W}$ (resp. $D \in \mathcal{D}$) let U_W (resp. C_D) be the set of $a \in F_J(\Gamma^m)$ such that $\widehat{a} \in W$ (resp. $\widehat{a}$ belongs to the unique precell $S_{\widehat{D}} \in \mathcal{S}$ whose facet is $\widehat{D}$), $\mu(\widehat{a}) \leq a_m \leq \nu(\widehat{a})$ and $a_m \equiv \rho\ [N_m]$. This is obviously a largely continuous precell mod N with socle W (resp. $S_{\widehat{D}}$), and exactly the set of $a \in A$ such that $\widehat{a} \in W$) (resp. $S_{\widehat{D}}$). In particular it is contained in A, and if we let $\mathcal{U} = \{U_W : W \in \mathcal{W}\}$ and $\mathcal{C} = \{C_D : D \in \mathcal{D}\}$ then $\mathcal{U}$ is a partition $A \setminus \bigcup \mathcal{C}$ by induction hypothesis on $(\mathcal{S}, \mathcal{W})$.

For every $W \in \mathcal{W}$, every proper face of W is a proper face Z of X. Let H be its support. Then by Proposition 3.11, $(\bar{\mu}_{|Z}, \bar{\nu}_{|Z}, \rho)$ is a presentation of $F_H(U_W)$, but also of $F_H(A)$ hence $F_H(U_W) = F_H(A)$ is a proper face of A. For every $D \in \mathcal{D}$, since $\bar{\mu} = +\infty$ on Y we have $F_J(C_D) = \widehat{D} \times \{+\infty\} = D$ by Proposition 3.11, hence $\pi_F(C_D) = D$ by Proposition 3.3. Moreover for every $E \in \mathcal{E}$, $\pi_{\widehat{J}}(S_{\widehat{D}} \setminus S_{\widehat{E}}) \subseteq \widehat{D} \setminus \widehat{E}$ by induction hypothesis hence

$$\pi_J(C_D \setminus C_E) = \big[\pi_{\widehat{J}}(S_{\widehat{D}}) \setminus \pi_{\widehat{J}}(S_{\widehat{E}})\big] \times \{+\infty\} \subseteq (\widehat{D} \setminus \widehat{E}) \times \{+\infty\} = D \setminus E.$$

For every $a \in C_D$, since $J = \widehat{J}$ we have $\Delta_J(a) = \min(a_m, \Delta_{\widehat{J}}(\widehat{a}))$ and $\pi_J(a) = (\pi_{\widehat{J}}(\widehat{a}), +\infty)$. By induction hypothesis $\Delta_{\widehat{J}}(\widehat{a}) \geq \eta \circ \pi_{\widehat{J}}(\widehat{a})$ and $\Delta_{\widehat{J}}(\widehat{a}) \geq f(\pi_{\widehat{J}}(\widehat{a}), +\infty)$ because $\widehat{a} \in S_{\widehat{D}}$. The first inequality implies that $a_m \geq \mu(x) \geq f(\pi_{\widehat{J}}(\widehat{a}), +\infty)$ by (6). Together with the second inequality this gives that $\min(a_m, \Delta_{\widehat{J}}(\widehat{a})) \geq f(\pi_{\widehat{J}}(\widehat{a}), +\infty)$. That is $\Delta_J(a) \geq f(\pi_J(a))$, and (Sub) follows.

Conversely, since C_D is clearly the set of $a \in A$ such that $\widehat{a} \in S_{\widehat{D}}$, for every $a \in A$ such that $\pi_J(a) \in D$ (hence $\pi_{\widehat{J}}(\widehat{a}) \in \widehat{D}$) and $\Delta_J(a) \geq \varepsilon \circ \pi_{\widehat{J}}(\widehat{a})$ we have $\widehat{a} \in S_{\widehat{D}}$ by induction hypothesis on ε and $\widehat{D}$ hence $a \in C_D$. This proves (Sup) with $\delta(b) = \varepsilon(\widehat{b})$ on B.

Case 3: Y is a facet of X and $B = (Y \times \mathcal{Z}) \cap \overline{A}$.
Then $m \in \operatorname{Supp} B = J$, hence $\bar{\mu} < +\infty$ on Y, $\mu_D < +\infty$ for every $D \in \mathcal{D}$, and for every $a \in A$:

$$\Delta_J(a) = \Delta_{\widehat{J}}(\widehat{a}) \quad \text{and} \quad \pi_J(a) = (\pi_{\widehat{J}}(\widehat{a}), a_m) \tag{7}$$

Note that $\rho = \rho_D$ for every $D \in \mathcal{D}$ because, given any $b \in D \subseteq B$, we have $b_m \neq +\infty$ and on one hand $b_m \equiv \rho_D\ [N_m]$, on the other hand $b_m \equiv \rho\ [N_m]$ (using the presentation of $B = F_J(A)$ given by Proposition 3.11).

Sub-case 3.1: $\nu < \infty$.
Let $g : Y \to \mathcal{Z}$ be a strictly positive affine map given by Proposition 4.1 such that $g(y) \geq f(y, 0) + \alpha(\bar{\mu}(y) + N_m)$ on Y and $\bar{g} = +\infty$ on ∂Y. Given any $y \in Y$, since $g(y) < +\infty$ and $\nu - \mu$ has limit $+\infty$ at y, we have $\nu(x) - \mu(x) > 2N_m + 1 + g(y)$ for every $x \in X$ close enough to $\widehat{y}$. So there is a definable function $\eta_1 : Y \to \mathcal{Z}$ such that for every $x \in X$

$$\Delta_{\widehat{J}}(x) \geq \eta_1\big(\pi_{\widehat{J}}(x)\big) \Rightarrow \nu(x) - \mu(x) > 2N_m + 1 + g\big(\pi_{\widehat{J}}(x)\big). \tag{8}$$

The induction hypothesis applies to X, Y, $\{Y\}$ and $\max(\eta_1, 2g)$. It gives a definable map $\varepsilon_1 : Y \to \mathcal{Z}$ and a pair $(\mathcal{S}_1, \mathcal{W}_1)$ of families of precells. In the present case $\mathcal{S}_1$ consists of a single largely continuous precell X° mod N contained in X, such that $\Delta_{\widehat{J}} \geq \max(\eta_1 \circ \pi_{\widehat{J}}, 2g \circ \pi_{\widehat{J}})$ on X°, and every $x \in X$ such that $\pi_{\widehat{J}}(x) \in Y$ and $\Delta_{\widehat{J}}(x) \geq \varepsilon_1(\pi_{\widehat{J}}(x))$ belongs to X°. The family $\mathcal{W}_1$ is a finite partition of $X \setminus X^\circ$ in largely continuous precells mod N. Let $\mathcal{U}_1 = \{U_W : W \in \mathcal{W}_1\}$ where $U_W = (W \times \mathcal{Z}) \cap A$ for every $W \in \mathcal{W}$. Since $\nu < +\infty$, the proper faces of U_W are proper faces of A by Claim 5.3.

For every $k \notin \widehat{J}$ and every $x \in X^\circ$, we have $x_k \geq \Delta_{\widehat{J}}(x)$ because $k \notin \widehat{J}$, and $\Delta_{\widehat{J}} \geq 2g \circ \pi_{\widehat{J}}(x)$ on X° by induction hypothesis. Thus on one hand $x_k - g \circ \pi_{\widehat{J}}(x) \geq g \circ \pi_{\widehat{J}}(x) \geq 1$, and on the other hand $x_k - g \circ \pi_{\widehat{J}}(x) \geq x_k/2$. In particular $x \mapsto x_k - g \circ \pi_{\widehat{J}}(x)$ is a largely continuous strictly positive affine function on X° with limit $+\infty$ at every point of ∂X°. We also have $\Delta_{\widehat{J}}(x) \geq \eta_1(\pi_{\widehat{J}}(x))$ by induction hypothesis, hence $\nu(x) - \mu(x) > 2N_m + 1 + g(\pi_{\widehat{J}}(x))$ by (8). In particular the restriction of $\nu - \mu - 2N_m - 1$ to X° is a strictly positive affine function with limit $+\infty$ at every point of ∂X°. Proposition 4.3 then gives a largely continuous strictly positive affine function $\lambda : X^\circ \to \mathcal{Q}$ such that $\bar{\lambda} = +\infty$ on ∂X°, $\lambda \leq \nu - \mu - 2N_m - 1$ on X° and $\lambda(x) \leq (x_k - g \circ \pi_{\widehat{J}}(x))/\alpha$ for every $k \notin \widehat{J}$. Let us quote for further use that in particular

$$\alpha\lambda(x) \leq \min_{k \notin \widehat{J}} \big(x_k - g(\pi_{\widehat{J}}(x))\big) = \Delta_{\widehat{J}}(x) - g(\pi_{\widehat{J}}(x)). \tag{9}$$

Note that $\partial X^\circ = \overline{Y}$ because X° has a unique facet which is Y by Claim 5.4, hence $\bar{\lambda} = +\infty$ on $\bar{Y}$. Let $n \geq 1$ an integer such that $n\lambda$ is integrally affine, so that $\lambda(x) > t$ if and only if $\lambda(x) \geq t + 1/n$ for every $(x, t) \in X^\circ \times \mathcal{Z}$. Let $\zeta = \mu + \lambda + N_m$ on X°, and V (resp. A°) be the set of $a \in F_I(\Gamma^m)$ such that $\widehat{a} \in X^\circ$, $\zeta(\widehat{a}) + 1/n \leq a_m \leq \nu(\widehat{a})$ (resp. $\mu(\widehat{a}) \leq a_m \leq \zeta(\widehat{a})$) and $a_m \equiv \rho\ [N_m]$. By construction ζ is a largely continuous affine map on X° with $\bar{\zeta} = +\infty$ on ∂X°. Moreover on X° we have

$$\zeta + 1/n + N_m = \mu + \lambda + 2N_m + 1/n \leq \nu$$

(because that $\lambda \leq \nu - \mu - 2N_m - 1$ by construction) hence the socle of V is X°. Obviously $\mu + N_m \leq \mu + \lambda + N_m = \zeta$ (because $\lambda > 0$ by construction) hence the socle of A° is X°. Thus both V and A° are largely continuous precells mod N contained in $(X^\circ \times \mathcal{Z}) \cap A$. Moreover $a_m > \zeta(\widehat{a}) = \mu(\widehat{a}) + \lambda(x\widehat{a}) + N_m$ if and only if $a_m \geq \mu(\widehat{a}) + \lambda(x\widehat{a}) + 1/n + N_m = \zeta(\widehat{a}) + 1/n$. Thus V and A° form a partition of $(X^\circ \times \mathcal{Z}) \cap A$, or equivalently $\mathcal{U}_1 \cup \{V\}$ is a partition of $A \setminus A^\circ$. Since $\bar{\zeta} = +\infty$ on $\partial X^\circ = \overline{Y}$, by Proposition 3.11 every proper face V' of V is of type $Z \times \{+\infty\}$ for Z a face of Y. In particular V' is a proper face of A.

For every $D \in \mathcal{D}$ let $\zeta_D = \nu_D$ if $\nu_D < +\infty$ and $\zeta_D = \mu_D + N_m$ otherwise. Since $\bar{\zeta} = +\infty$ on Y there is a definable function $\eta_2 : Y \to \mathcal{Z}$ such that for every $x \in X^\circ$ and every $D \in \mathcal{D}$ such that $\pi_{\widehat{J}}(x) \in \widehat{D}$ we have

$$\Delta_{\widehat{J}}(x) \geq \eta_2\big(\pi_{\widehat{J}}(x)\big) \Rightarrow \zeta(x) \geq \zeta_D\big(\pi_{\widehat{J}}(x)\big). \tag{10}$$

The induction hypothesis applies to X°, Y, $\widehat{\mathcal{D}}$ and η_2. It gives a definable map $\varepsilon_2 : Y \to \mathcal{Z}$ and a pair $(\mathcal{S}_2, \mathcal{W}_2)$ of families of precells. For each $W \in \mathcal{W}_2$ let $U_W = (W \times \mathcal{Z}) \cap A^\circ$. Clearly the family $\mathcal{U}_2 = \{U_W : W \in \mathcal{W}_2\}$ is a finite partition in largely continuous precells mod N of the complement in A° of the set $A^{\circ\circ} = (\bigcup \mathcal{S}_2 \times \mathcal{Z}) \cap A^\circ$. Equivalently, $\mathcal{U}_1 \cup \{V\} \cup \mathcal{U}_2$ is a finite partition of $A \setminus A^{\circ\circ}$. Since $\nu < +\infty$, by Claim 5.3 the proper faces of U_W are proper faces of A for every $W \in \mathcal{W}_2$.

For each $D \in \mathcal{D}$ let $S_{\widehat{D}}$ be the precell in $\mathcal{S}$ given by induction hypothesis, so that conditions (Fac), (Sub), (Sup), (diff) apply to $S_{\widehat{D}}$, η_2 and ε_2. If $\nu_D = +\infty$ (resp. $\nu_D < +\infty$) let C_D be the set of $a \in F_I(\Gamma^m)$ such that $\widehat{a} \in S_{\widehat{D}}$, $\mu_D(\pi_J(\widehat{a})) \leq a_m \leq \zeta(\widehat{a})$ (resp. $\mu_D(\pi_J(\widehat{a})) \leq a_m \leq \nu_D(\pi_J(\widehat{a}))$) and $a_m \equiv \rho\ [N_m]$. For every $x \in S_{\widehat{D}}$ we have $\pi_{\widehat{J}}(x) \in \widehat{D}$, because $\widehat{D} = F_{\widehat{J}}(S_{\widehat{D}})$ by (Fac), and $F_{\widehat{J}}(S_{\widehat{D}}) = \pi_{\widehat{J}}(S_{\widehat{D}})$ by Proposition 3.3(1). So there is $b \in D$ such that $\widehat{b} = x$, $\mu_D(\pi_{\widehat{J}}(x)) \leq b_m \leq \nu_D(\pi_{\widehat{J}}(x))$ and $b_m \equiv \rho\ [N_m]$. We can (and do) require in addition that $b_m \leq \mu_D(\pi_{\widehat{J}}(x)) + N_m$, hence $b_m \leq \zeta_D(\pi_{\widehat{J}}(x))$. Because $x \in S_{\widehat{D}}$ we also have $\Delta_{\widehat{J}}(x) \geq \eta_2 \circ \pi_{\widehat{J}}(x)$ by (Sub), hence $\zeta_D(x) \leq \zeta(x)$ by (10). Altogether this proves that $(x, b_m) \in C_D$, hence x belongs to the socle of C_D. So the socle of C_D is exactly $S_{\widehat{D}}$ and C_D is then a largely continuous precell mod N. The presentation of $F_J(C_D)$ given by Proposition 3.11 is exactly (μ_D, ν_D, ρ), hence $F_J(C_D) = D$ since $\rho_D = \rho$. In particular $\pi_J(C_D) = D$ by Proposition 3.3(1). More precisely, the above computations show that we have

$$C_D = \big\{a \in A^{\circ\circ} : \widehat{a} \in S_{\widehat{D}} \text{ and } \pi_J(a) \in D\big\}. \tag{11}$$

Let $\mathcal{C} = \{C_D : D \in \mathcal{D}\}$ and $\mathcal{U} = \mathcal{U}_1 \cup \{V\} \cup \mathcal{U}_2$. We already know that $\mathcal{U}$ is a finite partition of $A \setminus A^{\circ\circ}$ in largely continuous precells mod N whose proper faces are proper faces of A, and that each $C_D \in \mathcal{C}$ is a largely continuous precell mod N contained in $A^{\circ\circ}$ with socle $S_{\widehat{D}}$ and $F_J(C_D) = \pi_J(C_D) = D$. Let us check that $\bigcup \mathcal{C} = A^{\circ\circ}$. In order to do so, we are claiming that

$$\forall a \in A^{\circ\circ}, \forall E \in \mathcal{D},\ \pi_J(a) \in E \Rightarrow a \in C_E. \tag{12}$$

Assume the contrary and let $a \in A^{\circ\circ}$, $E \in \mathcal{E}$ be such that $\pi_J(a) \in E$ and $a \notin C_E$. By (11) this implies that $\widehat{a} \notin S_{\widehat{E}}$. But the socle of $A^{\circ\circ}$ is $\bigcup \mathcal{S}$, hence $\widehat{a} \in S_{\widehat{D}}$ for some $D \in \mathcal{D}$. Thus $\pi_{\widehat{J}}(\widehat{a})$ belongs to $\pi_{\widehat{J}}(S_{\widehat{D}} \setminus S_{\widehat{E}})$. By induction hypothesis the latter is contained in $\widehat{D} \setminus \widehat{E}$, hence $\pi_{\widehat{J}}(\widehat{a}) \notin \widehat{E}$. But $\pi_{\widehat{J}}(\widehat{a})$ is also the socle of $\pi_J(a)$. Since $\pi_J(a) \in E$ it follows that $\pi_{\widehat{J}}(\widehat{a}) \in \widehat{E}$, a contradiction.

That $A^{\circ\circ} \subseteq \bigcup \mathcal{C}$ then follows immediately from (12) and the fact that $\pi_J(A^{\circ\circ}) \subseteq \pi_J(A) = B \subseteq \bigcup \mathcal{D}$. So $A^{\circ\circ} = \bigcup \mathcal{C}$ and it only remains to check (Sub), (Sup) and (Diff) for any fixed $D \in \mathcal{D}$.

We start with (Diff). Pick any $E \in \mathcal{D}$, assume that there is a point b in $\pi_J(C_D \setminus C_E)$ which belongs to E. Then $b = \pi_J(a)$ for some $a \in C_D \setminus C_E$. We have $a \in A^{\circ\circ}$ and $a \notin C_E$, hence $\pi_J(a) \in E$ by (12). In particular $a \notin E$, hence$\pi_J(C_D \setminus C_E)$ is disjoint from E. Property (Diff) follows since $\pi_J(C_D) = D$.

Let us turn now to (Sup). For every $b \in B$, since $\bar{\zeta} = +\infty$ on $\partial X^\circ = \overline{Y}$ and $\widehat{b} = Y$ we have $\zeta(x) \geq b_m$ whenever $x \in X^\circ$ is close enough to $\widehat{b}$ (that is whenever $\pi_{\widehat{J}}(x) = \widehat{b}$ and $\Delta_{\widehat{J}}$ is large enough). So there is a definable function $\eta_3 : B \to \mathcal{Z}$ such that for every $a \in (X^\circ \times \mathcal{Z}) \cap A$

$$\Delta_{\widehat{J}}(\widehat{a}) \geq \eta_3\big(\pi_J(a)\big) \Rightarrow \zeta(\widehat{a}) \geq a_m. \tag{13}$$

Let $\delta : b \in B \mapsto \max(\varepsilon_1(\widehat{b}), \eta_3(b), \varepsilon_2(\widehat{b}))$. For every $a \in A$ such that $\pi_J(a) \in D$ and $\Delta_J(a) \geq \delta \circ \pi_J(a)$, since $\Delta_J(a) = \Delta_{\widehat{J}}(\widehat{a})$ by (7) we have in particular $\pi_{\widehat{J}}(\widehat{a}) \in Y$ and $\Delta_{\widehat{J}}(\widehat{a}) \geq \varepsilon_1(\pi_{\widehat{J}}(\widehat{a}))$, hence $\widehat{a} \in X^\circ$ by construction. So $a \in (X^\circ \times \mathcal{Z}) \cap A$ and $\Delta_{\widehat{J}}(\widehat{a}) \geq \eta_3\big(\pi_J(a)\big)$, which implies that $a_m \leq \zeta(\widehat{a})$ by (13), hence $a \in A^\circ$ by construction. On the other hand, since $\widehat{a} \in X^\circ$, $\pi_{\widehat{J}}(\widehat{a}) \in \widehat{D}$ and $\Delta_{\widehat{J}}(\widehat{a}) \geq \varepsilon_2(\pi_{\widehat{J}}(\widehat{a}))$, we get that $\widehat{a} \in S_{\widehat{D}}$ by construction. In particular $\widehat{a} \in \bigcup \mathcal{S}$, hence $a \in A^{\circ\circ}$ since $A^{\circ\circ} = (\bigcup \mathcal{S}_2 \times \mathcal{Z}) \cap A^\circ$. Altogether we have $a \in A^{\circ\circ}$, $\widehat{a} \in S_{\widehat{D}}$ and $\pi_J(a) \in D$ hence that $a \in C_D$ by (11), which proves (Sup).

In order to get (Sub), it only remains to check that $\Delta_J \geq f \circ \pi_J$ on C_D. This is the moment to recall (9), which says that $\alpha\lambda \leq \Delta_{\widehat{J}} - g \circ \pi_{\widehat{J}}$ on X°. Recall also that $g(y) \geq f(y, 0) + \alpha(\bar{\mu}(y) + N_m)$ on Y by definition of g. Thus on X° we have

$$\alpha\lambda(x) \leq \Delta_{\widehat{J}}(x) - f\big(\pi_{\widehat{J}}(x), 0\big) - \alpha\bar{\mu}\big(\pi_{\widehat{J}}(x)\big) - \alpha N_m. \tag{14}$$

For every $a \in C_D$, $\widehat{a} \in X^\circ$ and $a \in A^\circ$ hence $a_m \leq \zeta(\widehat{a}) = \mu(\widehat{a}) + \lambda(\widehat{a}) + N_m$. We also have $\mu(\widehat{a}) = \bar{\mu}\big(\pi_{\widehat{J}}(\widehat{a})\big)$ by Proposition 3.5. Combining all this with (14) we get that

$$\alpha a_m \leq \alpha\lambda(\widehat{a}) \leq \Delta_{\widehat{J}}(\widehat{a}) - f\big(\pi_{\widehat{J}}(\widehat{a}), 0\big). \tag{15}$$

Since $f(\pi_J(a)) = f\big(\pi_{\widehat{J}}(\widehat{a}), 0\big) + \alpha a_m$ by definition of f, and $\Delta_J(a) = \Delta_{\widehat{J}}(\widehat{a})$ by (7), we finally get from (15) that $\Delta_J(a) = \Delta_{\widehat{J}}(\widehat{a}) \geq f(\pi_J(a))$.

Sub-case 3.2: $\nu = +\infty$.

Proposition 4.1 gives a largely continuous integrally affine map λ on X such that $\bar{\lambda} = +\infty$ on ∂X and $\lambda \geq \mu + N_m$. Let A^- (resp. A^+) be the set of $a \in F_I(\Gamma^m)$ such that $\widehat{a} \in X$, $\mu(\widehat{a}) \leq a_m \leq \lambda(\widehat{a})$ (resp. $\lambda(\widehat{a}) + 1 \leq a_m$) and $a_m \equiv \rho\ [N_m]$. Its socle is X (for A^- we use that $\lambda \geq \mu + N_m$) hence it is a largely continuous precell mod N. Since λ takes values in $\mathcal{Z}$, A^- and A^+ form a partition of A. The presentation of the faces of A, A^- A^+ given by Proposition 3.11 gives that every proper face of A^- and A^+ is a proper faces of A, and B is a face of A^-. The previous sub-case 2.1 applies to A^-, B, $\mathcal{D}$ and f. It gives a pair $(\mathcal{C}^-, \mathcal{W}^-)$ of families of largely continuous precells mod N and an integrally affine map $\delta^- : B \to; \mathcal{Z}$. Then $(\mathcal{C}^-, \mathcal{W}^- \cup \{A^+\})$ and δ^- have all the required properties for A, $\mathcal{D}$ and f, except possibly (Sup). We remedy this by replacing δ^- by a larger function δ defined as follows.

For every $b \in B$, we have $\lambda(x) \geq b_m$ for every $x \in X$ close enough to $\widehat{b}$ since $\bar{\lambda} = +\infty$ on Y. So there is a definable function $\eta : B \to \mathcal{Z}$ such that for every

$a \in A$

$$\Delta_J(a) \geq \eta\big(\pi_J(a)\big) \Rightarrow \lambda(\widehat{a}) \geq a_m. \tag{16}$$

Let $\delta = \max(\eta, \delta^-$, then for every $D \in \mathcal{D}$ and every $a \in A$ such that $\pi_J(a) \in D$ and $\Delta_J(a) \geq \delta(\pi_J(a))$ we have in particular $\Delta_J(a) \geq \eta(\pi_J(a))$ hence $a_m \leq \lambda(\widehat{a})$ by (16), that is $a \in A^-$. On the other hand we have $\pi_J(a) \in D$ and $\Delta_J(a) \geq \delta^-(\pi_J(a))$. Altogether this implies that a belongs to $C_D \in \mathcal{C}^-$, which in turn proves (Sup).

■

Theorem 5.5 (Monohedral Division) *Let $A \subseteq F_I(\Gamma^m)$ be a largely continuous precell mod N, $f : \partial A \to \mathcal{Z}$ a definable function, and $\mathcal{D}$ a complex of monohedral largely continuous precells mod N such that $\bigcup \mathcal{D} = \partial A$. Then there exists a finite partition $\mathcal{C}$ of A in monohedral largely continuous precells mod N such that $\mathcal{C} \cup \mathcal{D}$ is a closed complex, $\mathcal{C}$ contains for every $D \in \mathcal{D}$ a unique precell C with facet D, and moreover $\Delta_J \geq f \circ \pi_J$ on C where $J = \operatorname{Supp} D$.*

Proof: The proof goes by induction on the number n of proper faces of A. If $n = 0$ then $\mathcal{D} = \emptyset$ and A is monohedral, hence $\mathcal{C} = \{A\}$ gives the conclusion. So let us assume that $n \geq 1$ and the result is proved for smaller integers. Let B be a facet of A. Lemma 5.1 applied to A, B, $\mathcal{D}$ and the restriction of f to B gives a pair $(\mathcal{U}, \mathcal{C}_B)$ of families of precells. For every $U \in \mathcal{U}$, the proper faces of U are proper faces of A. So the family $\mathcal{D}_U = \{D \in \mathcal{D} : D \subseteq \partial U\}$ is a complex and $\bigcup \mathcal{D}_U = \partial U$. Since B is not a proper face of U by Claim 5.3, the induction hypothesis applies to U, $\mathcal{D}_U$ and the restriction of f to ∂U. It gives a family $\mathcal{C}_U$ of precells. Let $\mathcal{C}$ be the union of $\mathcal{C}_B$ and $\mathcal{C}_U$ for $U \in \mathcal{U}$. This is a family of largely continuous precells mod N partitioning A. By construction $\mathcal{C}$ contains for every $D \in \mathcal{D}$ a unique precell C with facet D, and $\Delta_J \geq f \circ \pi_J$ on C with $J = \operatorname{Supp} D$. In particular $\mathcal{C} \cup \mathcal{D}$ is a partition of $\overline{A}$ which contains the faces of all its members, since $\mathcal{D}$ is a closed complex (because $\mathcal{D}$ is a complex and $\bigcup \mathcal{D} = \partial B$ is closed). So $\mathcal{C} \cup \mathcal{D}$ is a closed complex.

■

Theorem 5.6 (Monohedral Decomposition) *Let $A \subseteq F_I(\Gamma^m)$ be a largely continuous precell mod N. Then there exists a complex $\mathcal{U}$ of monohedral largely continuous precells mod N such that such that $A = \bigcup \mathcal{U}$.*

Proof: We are going to show that given any closed complex $\mathcal{A}$ of largely continuous precells mod N in Γ^m, there is a closed complex $\mathcal{C}$ of largely continuous *monohedral* precells mod N such that $\bigcup \mathcal{C} = \bigcup \mathcal{A}$ and $\mathcal{C}$ refines $\mathcal{A}$ (that is every $C \in \mathcal{C}$ is contained in some $A \in \mathcal{A}$). The conclusion for A will follow, by applying this to the closed complex consisting of all the faces of A. The proof goes by induction on the cardinality n of $\mathcal{A}$. If $n = 0$ then $\mathcal{C} = \mathcal{A} = \emptyset$ proves the result. Assume that $n \geq 1$ and the result is proved for smaller integers. Let A be a maximal element of $\mathcal{A}$ with respect to specialisation, and $\mathcal{B} = \mathcal{A} \setminus \{A\}$. By maximality of A, $\mathcal{B}$ is again a closed complex. The induction hypothesis gives a closed complex $\mathcal{D}$ of largely continuous monohedral precells mod N such that $\bigcup \mathcal{D} = \bigcup \mathcal{B}$ and $\mathcal{D}$ refines $\mathcal{B}$. If A is closed then obviously $\mathcal{C} = \mathcal{D} \cup \{A\}$ proves the result for $\mathcal{A}$. Otherwise let $\mathcal{D}_A = \{D \in \mathcal{D} : D \subseteq \partial A\}$. The Monohedral Division Theorem 5.5 applied to A, $\mathcal{D}_A$ and the constant function $f = 0$ gives a finite partition $\mathcal{C}_A$ of A in monohedral largely continuous precells mod N such

that the family $\mathcal{C}_A \cup \mathcal{D}_A$ is a closed complex. The family $\mathcal{C} = \mathcal{C}_A \cup \mathcal{D}$ is a partition of $A \cup \bigcup \mathcal{B} = \bigcup \mathcal{A}$. Since $\mathcal{D}$ is a closed complex and every precell in $\mathcal{C}_A$ has a unique facet which belongs to $\mathcal{D}$, it follows that $\mathcal{C}$ is a complex.
■

We finish this section with another, much more elementary, division result. Contrary to the above ones, it is drastically different from what occurs in the real situation, where the polytopes are connected sets.

Proposition 5.7 *Let $A \subseteq F_I(\Gamma^m)$ be a non-closed monohedral largely continuous precell mod N. For every integer $n \geq 1$ there exists for some $N' \in (\mathbf{N}^*)^m$ a partition $(A_i)_{1\leq i\leq n}$ of A in largely continuous precells mod N' such that $\partial A_i = \partial A$ for $1 \leq i \leq n$.*

Proof: The proof goes by induction on m. The result is trivially true for $m = 0$ since there is no non-closed precell in Γ^0. Assume that $m \geq 1$ and the result is proved for smaller integers. Let (μ, ν, ρ) be a presentation of A. By induction hypothesis we can assume that $m \in \operatorname{Supp} A$ hence $\mu < +\infty$. If $\nu = +\infty$, for $1 \leq i \leq n$ let A_i be the set of $a \in F_I(\Gamma^m)$ such that $\widehat{a} \in \widehat{A}$, $\mu(\widehat{a}) \leq a_m \leq \nu(\widehat{a})$ and $a_m \equiv \rho + iN_m\ [nN_m]$. This is obviously a partition of A in largely continuous precells mod $N' = (\widehat{N}, nN_m)$ having the same boundaries as A. On the other hand, if $\nu < +\infty$ then $\widehat{A}$ is not closed (otherwise A would be closed) hence the induction hypothesis gives for some $P' \in (\mathbf{N}^*)^m$ a partition $(X_i)_{1\leq i\leq n}$ of $\widehat{A}$ in largely continuous precells mod P' such that $\partial X_i = \partial X$ for every i. Let $A_i = (X_i \times \mathcal{Z}) \cap A$ for every i. Then $(A_i)_{1\leq i\leq n}$ is easily seen to give the conclusion, thanks to the description of the faces of A and A_i given by Proposition 3.11.
■

6 Polytopes in p-adic fields

Recall that K is a p-adically closed field, v its p-valuation, R its valuation ring and $\Gamma = v(K)$. We still denote by v the map $(v, \dots, v)$ from K^m to Γ^m.

We are going to define polytopes[5] mod N in K^m by means of the inverse image by v of largely continuous precells mod N in Γ^m. However, the p-adic triangulation theorem that we are aiming at requires a more versatile definition. It involves semi-algebraic subgroups $Q_{1,M}$ of multiplicative group $K^\times = K\setminus\{0\}$, where M is a positive integer. In the special case when K is a finite extension of $\mathbf{Q}_p$, we have

$$Q_{1,M} = \bigcup_{k\in\mathbf{Z}} \pi^k(1 + \pi^M R).$$

where π is any generator of the maximal ideal of R. Since in this paper we will only use that $v(Q_{1,M}) = \mathcal{Z}$, we refer the reader to [CL12] for a general definition of $Q_{N,M}$ for every integers $N, M \geq 1$ in arbitrary p-adically closed fields.

We let $D^M R = (\{0\} \cup Q_{1,M}) \cap R$. Given an m-tuple $N \in (\mathbf{N}^*)^m$ we call a set $S \subseteq K^m$ a **polytope mod N in $D^M R^m$** if $v(S)$ is a largely continuous precell mod N in Γ^m and $S = v^{-1}(v(S)) \cap D^M R^m$. The **faces** and **facets** $F_J(S)$ of a

[5] We don't call them largely continuous precells because they are much more special than the usual p-adic cells as defined in [Den86].

subset S of $D^M R^m$ are defined as the inverse images, by the restriction of v to $D^M R^m$, of the faces and facets of $v(S)$. The **support** of S (resp. of $x \in K^m$) is the support of $v(S)$ (resp. of $v(x)$, so that:

$$\mathrm{Supp}(x) = \big\{i \in [\![1, m]\!] : x_i \neq 0\big\}$$

$$F_J(S) = \big\{x \in \overline{S} : \mathrm{Supp}\, x = J\big\}$$

We say that S is **monohedral** if $v(S)$ is so, that is if the faces of S are linearly ordered by specialisation, in which case we call S a **monotope mod** N **in** $D^M R^m$.

A family $\mathcal{C}$ of polytopes mod N in $D^M R^m$ is a **complex** if it is finite and for every $S, T \in \mathcal{C}$, $\overline{S} \cap \overline{T}$ is the union of the common faces of S and T. It is a **closed complex** if moreover it contains all the faces of its members. Every complex $\mathcal{S}$ of polytopes mod N is contained in a smallest closed complex, namely the family of all the faces of the members of $\mathcal{S}$. We call it the **closure** of $\mathcal{S}$ and denote it $\overline{\mathcal{S}}$.

In order to ease the notation, we write vS for $v(S)$, and $v\mathcal{C}$ for $\{vS : S \in \mathcal{C}\}$. Clearly $\mathcal{C}$ is a (closed) complex if and only if vC is so.

Proposition 6.1 *Let S be a polytope mod N in $D^M R^m$, and $T = F_J(S)$ be any of its faces. Then T is a polytope mod N equal to $\pi_J(S)$.*

Proof: Due to the correspondence between the faces of S and vS, this follows directly from Proposition 3.11 and Proposition 3.3(1).

■

More generally, all the points of Proposition 3.3, as well as Proposition 3.7, Corollary 3.10, the Monohedral Decomposition (Theorem 5.6) and Proposition 5.7 immediately transfer to polytopes mod N in $D^M R^m$. Only the Monohedral Division (Theorem 5.5) requires a bit more of preparation.

For seek of generality we want the p-adic analogous of the Monohedral Division Theorem in Γ^m to hold true not only with a map $\varepsilon : \partial S \subseteq K^m \to K^*$ definable in the language of rings (*i.e.* semi-algebraic) but also with a map definable in various expansions $(K, \mathcal{L})$ of the ring structure of K. The proof of Theorem 6.3 below shows that it is sufficient for this to make the following assumptions on $(K, \mathcal{L})$:

1. For every definable function $f : X \subseteq K^m \to K^*$, if f is continuous and X is closed and bounded, then $v(f(x))$ takes a maximum value at some point $x \in X$.
2. The image by the valuation of every subset of K^m definable in $(K, \mathcal{L})$, is $\mathcal{L}_{Pres}$-definable.

Remark 6.2 If K is a finite extension of $\mathbf{Q}_p$ then condition (1) holds true for every continuous function by the Extreme Value Theorem. But this condition, when restricted to definable continuous functions, is preserved by elementary equivalence. Hence it will be satisfied whenever the complete theory of $(K, \mathcal{L})$ has a p-adic model (that is a model whose underlying field is a finite extension of $\mathbf{Q}_p$). On the other hand, if $(K, \mathcal{L})$ is strongly p-minimal (a.k.a P-minimal in [HM97]), Theorem 6 in [Clu03] proves that condition (2) is satisfied. In particular Theorem 6.3 applies for example to every subanalytic map ε, and more generally to every map ε which is definable in a strongly p-minimal structure $(K, \mathcal{L})$ which has a p-adic model.

For every $x \in K^m$ we let $w(x) = \min_{1 \le i \le m} v(x_i)$. If $v(K) = \mathbf{Z}$ this is the valuative counterpart of the usual norm on K^m, which measures the distance of x to the origin (see also Remark 2.3).

Theorem 6.3 (Monotopic Division) *Let S be a polytope mod N in $D^M R^m$, $\varepsilon : \partial S \to K^*$ a definable function, and $\mathcal{T}$ a complex of monotopes mod N in $D^M R^m$ such that $\bigcup \mathcal{T} = \partial S$. Assume that the restriction of $v \circ \varepsilon$ to every proper face of S is continuous. Then there exists a finite partition $\mathcal{U}$ of S in monotopes mod N in $D^M R^m$ such that $\mathcal{U} \cup \mathcal{T}$ is a closed complex, $\mathcal{U}$ contains for every $T \in \mathcal{T}$ a unique cell U with facet T, and moreover for every $u \in U$*

$$w\big(u - \pi_J(u)\big) \ge v\big(\varepsilon(\pi_J(u))\big)$$

where $J = \mathrm{Supp}(T)$.

Proof: For every proper face $F_J(S)$ of S, and every $s \in F_J(S)$, the function $t \mapsto v(\varepsilon(t))$ is continuous on $v^{-1}(\{v(s)\}) \cap F_J(S)$, which is a closed and bounded domain. Thus it attains a maximum value $e(s)$ (see Remark 6.2). So let

$$G_J = \big\{(s,t) \in F_J(S) \times K : v(t) = e(s)\big\}.$$

This is a definable set hence $v(G_J)$ is $\mathcal{L}_{Pres}$-definable (see Remark 6.2). Moreover by construction $v(G_J)$ is the graph of a function $g_J : vF_J(S) = F_J(vS) \to \mathcal{Z}$, such that $v(\varepsilon(s)) \le g_J(v(s))$ for every $s \in S$. Let $g : \partial(vS) \to \mathcal{Z}$ be the function whose restriction to each $F_J(vS)$ is g_J.

The Monotopic Division (Theorem 5.5) applies to vS, g and vT. It gives a finite partition $\mathcal{C}$ of vS in monotopes mod N such that $\mathcal{C} \cup v\mathcal{T}$ is a complex, every non-closed $C \in \mathcal{C}$ has a unique facet D which belongs to $v\mathcal{T}$ and $\Delta_J \ge f \circ \pi_J$ on C where $J = \mathrm{Supp}\, D$. Let $\mathcal{U}$ be the family of $v^{-1}(C) \cap D^M R^m$ for $C \in \mathcal{C}$. This is clearly a finite partition of S in monotopes mod N in $D^M R^m$. Every $U \in \mathcal{U}$ has a unique facet $T \in \mathcal{T}$, and $\Delta_J \ge g \circ \pi_J$ on vT where $J = \mathrm{Supp}\, vT = \mathrm{Supp}\, T$. That is, for every $u \in U$ we have

$$w\big(u - \pi_J(u)\big) = \min_{i \notin J} v(u_i) = \Delta_J(v(u)) \ge g \circ \pi_J(v(u)) \tag{17}$$

By construction $\pi_J(v(u)) = v(\pi_J(u))$ and $g(v(t)) \ge v(\varepsilon(t))$ for every $t \in T$, hence

$$g \circ \pi_J\big(v(u)\big) = g\big(v(\pi_J(u))\big) \ge v\big(\varepsilon(\pi_J(u))\big).$$

Together with (17), this proves the last point.
∎

Finally, let us mention for further works the following generalisation of Proposition 5.7.

Proposition 6.4 *Let $A \subseteq D^M R^m$ be a relatively open[6] set. Assume that A is the union of a complex $\mathcal{A}$ of monotopes mod N in $D^M R^m$. Then for every integer $n \ge 1$ there exists a finite partition of A in semi-algebraic sets $A_1, \dots, A_n$ such that $\partial A_k = \partial A$ for every k.*

[6] A subset A of a toplogical set is called **relatively open** if it is open in its closure, that is $\overline{A} \setminus A$ is closed.

Proof: Thanks to the correspondence between the faces of the monotopes mod N in $D^M R^m$ and their faces, it suffices to prove the result for a relatively open set $A \subseteq \Gamma^m$ which is the union of a complex of monotopes mod N in Γ^m.

Let $\mathcal{C} = \overline{\mathcal{A}} \setminus \mathcal{A}$ and $C = \bigcup \mathcal{C} = \overline{A} \setminus A$. By assumption A is relatively open hence C is closed, so $\mathcal{C}$ is a closed complex. Let $U_1, \dots, U_r$ be the list of minimal elements of $\mathcal{A}$. Every $S \in \overline{\mathcal{A}}$ such that $U_i \leq S$ for some i belongs to $\mathcal{A}$ (otherwise $S \in \overline{\mathcal{A}} \setminus \mathcal{A} = \mathcal{C}$ which is closed, hence $U_i \in \mathcal{C}$, a contradiction since $U_i \in \mathcal{A}$). At the contrary every $T \in \overline{\mathcal{A}} \setminus \mathcal{A}$ is a proper face of some U_i (because T is a face of some $S \in \mathcal{A}$ and $U_i \leq S$ for some i, hence $T < U_i$ or $U_i \leq T$ because S is a monotope, and the second case is excluded because $T \notin \mathcal{A}$). In particular $\partial A = \overline{A} \setminus A$ is the union of the sets $T \in \overline{\mathcal{A}}$ such $T < U_i$ for some i, that is $\partial A = \bigcup_{i \leq r} \partial U_i$.

For each $i \leq r$ let $\mathcal{B}_i$ be the family of $S \in \mathcal{S}$ such that $S \geq U_i$, and $B_i = \bigcup \mathcal{B}_i$. The families $\mathcal{B}_i$ are two-by-two disjoint, and so are the sets B_i since $\mathcal{A}$ is a complex. By the same argument as above (replacing $\mathcal{A}$ by $\mathcal{B}_i$) $\overline{B}_i \setminus B_i = \bigcup (\overline{\mathcal{B}}_i \setminus \mathcal{B}_i) = \partial U_i$, hence B_i is relatively open and $\partial B_i = \partial U_i$. It suffices to prove the result separately for each B_i. Indeed, assume that for each $i \leq r$ we have found a partition $(B_{i,j})_{1 \leq j \leq n}$ of B_i in definable sets such that $\partial B_{i,j} = \partial B_i$. Then let $A_j = \bigcup_{i \leq r} B_{i,j}$ for each j. By construction these sets form a partition of A and

$$\overline{A}_j \setminus A_j = \overline{A}_j \setminus A = \bigcup_{i \leq r} \overline{B}_{i,j} \setminus A = \bigcup_{i \leq r} \overline{B}_i \setminus A = \bigcup_{i \leq r} \partial B_i = \partial A.$$

Thus replacing A and $\mathcal{A}$ by B_i and $\mathcal{B}_i$ if necessary, we can assume that $\mathcal{A}$ has a unique smallest element U_0. If U_0 is closed, then $\partial A = \partial U_0 = \emptyset$ (by minimality of U_0), and it suffices to take $A_1 = A$ and $A_k = \emptyset$ for $k \geq 2$. So from now we assume that U_0 is not closed. Proposition 6.4 then applies to U_0 and gives for some N' a partition $A_1(U_0), \dots, A_n(U_0)$ of U_0 in largely continuous monotopic cells mod N' such that $\partial A_i(U_0) = \partial U_0$ for every i. In particular each $A_i(U_0)$ is a basic Presburger set. Let $H = \operatorname{Supp} U_0$, and for every $S \in \mathcal{S}$ and $i \in [\![1, n]\!]$ let $A_i(S) = \pi_H^{-1}(A_i(U_0)) \cap S$. Note that this is a basic Presburger set. Indeed, S itself is a basic Presburger set, and $\pi_H^{-1}(A_i(U_0)) \cap F_I(\Gamma^m)$ with $I = \operatorname{Supp} S$ is a basic Presburger set because $A_i(U_0)$ is so (replace every condition $f(x) \geq 0$ defining $A_i(U_0)$ by $f \circ \pi_H(x) \geq 0$). Hence their intersection $A_i(S)$ is a basic Presburger set too. For every $i \leq n$ let $A_i = \bigcup \{A_i(S) : S \in \mathcal{A}\}$. This defines a partition of A. In order to conclude it only remains to show that $\overline{A_i} = A_i \cup \partial U_0$ for each i, so that $\partial A_i = \partial U_0 = \partial A$. Since $\overline{A_i} = \bigcup \{\overline{A_i(S)} : S \in \mathcal{A}\}$, it suffices to check that for every $S \in \mathcal{A}$

$$\overline{A_i(S)} = \bigcup \{A_i(T) : T \in \mathcal{A},\ U_0 \leq T \leq S\} \cup \partial U_0. \tag{18}$$

Let $I = \operatorname{Supp} A_i(S) = \operatorname{Supp} S$, and $J \subseteq I$ be the support of any face of $A_i(S)$. Note that $F_J(A_i(S)) \neq \emptyset$ implies that $F_J(S) \neq \emptyset$, thus J is the support of a face $T = F_J(S)$ of S. This face T belongs to $\overline{\mathcal{A}}$, hence to $\mathcal{S}$ if $U_0 \leq T$. We are claiming that $F_J(A_i(S)) = A_i(T)$ in that case, and that $F_J(T) = T = F_J(U_0)$ if $T < U_0$. This will finish the proof since $\overline{A_i(S)}$ is the union of its faces, and (18) then follows immediately.

Assume first that $U_0 \leq T$. Since $A_i(S)$ and S are basic Presburger set, we know by Proposition 3.3(1) that $F_J(A_i(S)) = \pi_J(A_i(S))$ and $F_J(S) = \pi_J(S)$, that is $T = \pi_J(S)$. Since $U_0 \le T$ we have $H \subseteq J$ hence $\pi_J(\pi_H^{-1}(A_i(U_0))) = \pi_J^{-1}(A_i(U_0))$. It follows that

$$\pi_J\big(\pi_H^{-1}(A_i(U_0)) \cap S\big) \subseteq \pi_J\big(\pi_H^{-1}(A_i(U_0))\big) \cap \pi_J(S) = \pi_J^{-1}(A_i(U_0)) \cap T$$

that is $\pi_J\big(A_i(S)\big) \subseteq A_i(T)$.

Conversely, for every $y \in A_i(T)$ we have on one hand $y \in T = \pi_J(S)$ so there is $x \in S$ such that $\pi_J(x) = y$, and on the other hand $y \in \pi_H^{-1}(A_i(U_0))$ so $\pi_H(x) = \pi_H(\pi_J(x)) = \pi_H(y) \in A_i(U_0)$. Thus $x \in \pi_H^{-1}(A_i(U_0)) \cap S = A_i(S)$, and since $y = \pi_J(x)$ this proves that $A_i(T) \subseteq \pi_J(A_i(S))$. The first case our the claim follows.

Now assume that $T < U_0$. Then $J \subset H$ hence $\pi_J(A_i(S)) = \pi_J(\pi_H(A_i(S)))$. We already know that $\pi_H(A_i(S)) = A_i(U_0)$ be the previous case, and that $\partial A_i(U_0) = \partial U_0$ by construction. In particular $F_J(A_i(U_0)) = F_J(U_0)$. But $F_J(U_0) = T$ since $\mathcal{A}$ is a complex and $T < U_0$. Altogether, using Proposition 3.3(1) for $A_i(S)$ and $A_i(U_0)$ we get

$$F_J(A_i(S)) = \pi_J(A_i(S)) = \pi_J(A_i(U_0)) = F_J(A_i(U_0)) = T.$$

■

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