On Bellissima's construction of the finitely generated free Heyting algebras, and beyond

Luck Darnière *
Département de mathématiques
Faculté des sciences
Université d'Angers, France

Markus Junker **
Mathematisches Institut
Abteilung für mathematische Logik
Universität Freiburg, Deutschland

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1 Introduction

Heyting algebras are a generalisation of Boolean algebras; the most typical example is the lattice of open sets of a topological space. Heyting algebras play the same rôle for intuitionistic logic as Boolean algebras for classical logic. They are special distributive lattices, and they form a variety. They are mainly studied by universal algebraists and by logicians, hardly by model theorists. In contrast to Boolean algebras, finitely generated free Heyting algebras are infinite, as was shown in the first article on Heyting algebras by McKinsey and Tarski in the 1940s. For one generator, the free Heyting algebra is well understood, but from two generators on, the structure remains mysterious, though many properties are known. With the help of recursively described Kripke models, Bellissima has given a representation of the finitely generated free Heyting algebras $\mathcal{F}_n$ as sub-algebras of completions $\widehat{\mathcal{F}_n}$ of them. Essentially the same construction is due independently to Grigolia. Our paper offers a concise and readable account of Bellissima's construction and analyses the situation closer.

Our interest in Heyting algebras comes from model theory and geometry. Our initial questions concerned axiomatisability and decidability of structures like the lattice of Zariski closed subsets of K^n for fields K, which led us rapidly to questions about Heyting algebras. One of the problems with Heyting algebras is that they touch many subjects: logic, topology, lattice theory, universal algebra, category theory, computer science. Therefore there are many different approaches and special languages, which often produce papers that are hard to read for non-insiders. An advantage of our article should be clear proofs and the use mainly of standard mathematical terminology. There is a bit of logic that one might skip if one believes in Bellissima's theorem; and there are basic model theoretic notions involved in the section about model theoretic results.

Our paper is organised as follows: Section 2 introduces the definitions, basic properties, and reference examples. Section 3 contains an account of Bellissima's construction, a short proof, and results from his article that we are not going to prove.

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Section 4 analyses the Heyting algebra constructed by Bellissima: we show that it is the profinite completion of the finitely generated free Heyting algebra as well as the metric completion for a naturally defined metric. In Section 5, we reconstruct the Kripke model as the principal ideal spectrum and prove that the Zariski topology on this spectrum is induced by the partial ordering. Section 6 shows the Kripke model to be first order interpretable, from which several model theoretic and algebraic properties for dense sub-algebras of $\widehat{\mathcal{F}_n}$ follow. For example, we show the set of generators to be $\emptyset$-definable, and we determine automorphism groups. We solve questions of elementary equivalence, e.g. we prove that no proper sub-algebra of $\mathcal{F}_n$ is elementarily equivalent to $\mathcal{F}_n$, and that $\mathcal{F}_n$ is an elementary substructure of $\widehat{\mathcal{F}_n}$ iff both algebras are elementarily equivalent. And we settle some questions about irreducible elements: $\mathcal{F}_n$ contains the same meet-irreducible elements as $\widehat{\mathcal{F}_n}$; we characterise the join-irreducible elements, we show that there are continuum many of them in $\widehat{\mathcal{F}_n}$ and that the join-irreducibles of $\mathcal{F}_n$ remain join-irreducible in $\widehat{\mathcal{F}_n}$. Finally, Section 7 collects open problems and miscellaneous considerations.

Some of the properties we isolated were known before, some were published after we started this work in 2004. We added references where we were able to do so, and apologise for everything we have overlooked. Though not all the results are new, the proofs might be, and the way of looking at the problem is hopefully interesting.

This paper is closely related to [DJ2]; both complement each other. When we started to study finitely generated Heyting algebras, we did it in two ways: on the one hand by analysing Bellissima's construction, on the other hand by analysing the notion of dimension and codimension in dual Heyting algebras. Many insights were obtained by both approaches, but some features are proper to the free Heyting algebras, others hold for a much wider class than just the finitely generated Heyting algebras. Therefore, we decided to write two papers: this one, which collects results that follow more or less directly from Bellissima's construction, and the paper [DJ2], which analyses the structure of Heyting algebras from a more geometric point of view.

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2 Basic facts about Heyting algebras

2.1 Definitions and notations

Under a *lattice* we will always understand a distributive lattice with maximum 1 and minimum 0, join (union) and meet (intersection) being denoted by $\sqcup$ and $\sqcap$ respectively. A *Heyting algebra* is a lattice where for every a, b there exists an element

$$a \to b \ := \ \max \{ x \mid x \sqcap a = b \sqcap a \}.$$

This is expressible as a universal theory, the theory T_{HA} of Heyting algebras, in the language $\mathcal{L}_{HA} = \{0, 1, \sqcap, \sqcup, \to\}$ with constant symbols $0, 1$ and binary function symbols $\sqcap, \sqcup, \to$.

Note that everything is definable from the partial ordering

$$a \sqsubseteq b \ :\iff \ a = a \sqcap b \iff b = a \sqcup b \iff a \to b = 1$$

In a poset $(X, \leqslant)$, we call y *a successor* of x if $x < y$ and there is no z with $x < z < y$. Analogously for *predecessor*.

If $\Lambda = (\Lambda, 0, 1, \sqcap, \sqcup, \sqsubseteq)$ is a lattice, then the dual lattice $\Lambda^* := (\Lambda, 1, 0, \sqcup, \sqcap, \sqsupseteq)$ is the lattice of the reversed ordering. Thus, in the dual of a Heyting algebra, for all a, b there exists a smallest element $b - a$ with the property $a \sqcup (b - a) = a \sqcup b$. Dual Heyting algebras are the older siblings of Heyting algebras; they were born *Brouwerian algebras* in [McT], they appear under the name *topologically complemented lattices* in [Da], and are often called *co-Heyting algebras* nowadays. A lattice is a *bi-Heyting algebra* (or *double Brouwerian algebra* in [McT]) if itself and its dual are Heyting algebras.

We define $a \leftrightarrow b$ as $(a \to b) \sqcap (b \to a)$, which is dual to the "symmetric difference" $a \vartriangle b := (a - b) \sqcup (b - a)$.

2.2 Examples

- The open sets of a topological space X form a complete[1] Heyting algebra $\mathcal{O}(X)$ with the operations suggested by the notations, i.e. $a \sqcap b = a \cap b$, $a \sqcup b = a \cup b$, $a \sqsubseteq b \iff a \subseteq b$, and

  $$a \to b \;=\; \overline{a \setminus b}^{\complement} \;=\; (a^{\complement} \cup b)^{\circ}.$$

  (Here, $^{\complement}$ denotes the complement, $\overline{}$ topological closure and $^{\circ}$ the interior.) We call such an algebra a *topological Heyting algebra.*

- If $(X, \leqslant)$ is a partial ordering, then the increasing sets form a topology and hence a Heyting algebra $\mathcal{O}^{\uparrow}(X, \leqslant)$, and the decreasing sets, which are the closed sets of $\mathcal{O}^{\uparrow}(X, \leqslant)$, form a topology and Heyting algebra $\mathcal{O}_{\downarrow}(X, \leqslant)$. Hence such an algebra is bi-Heyting, and both infinite distributive laws hold.

- The propositional formulae in κ propositional variables[2], up to equivalence in the intuitionistic propositional calculus, form a Heyting algebra IPL_κ. It is freely generated by the (equivalence classes of the) propositional variables, hence isomorphic to the *free Heyting algebra* $\mathcal{F}_\kappa$ *over* κ *generators.*

- If $\pi : H \to H'$ is a non-trivial epimorphism of Heyting algebras, then the *kernel* $\pi^{-1}(1)$ is a filter. Conversely, if Φ is an arbitrary filter in H, then $\equiv_\Phi$ defined by $x \equiv_\Phi y :\iff x \leftrightarrow y \in \Phi$ is a congruence relation such that $\Phi = \pi_\Phi^{-1}(1)$ for the canonical epimorphism $\pi_\Phi : H \to H/{\equiv_\Phi}$. In particular, $\equiv_{\{1\}}$ is equality.

3 Bellissima's construction

Bellissima in [Be] has constructed an embedding of the free Heyting algebra $\mathcal{F}_n$ into the Heyting algebra $\mathcal{O}_\downarrow(\mathfrak{K}_n)$ for a "generic" Kripke model $\mathfrak{K}_n$ of intuitionistic propositional logic in n propositional variables $P_1, \ldots, P_n$. We fix $n > 0$ and this fragment IPL_n of intuitionistic logic, and we will give a short and concise account of Bellissima's construction and proof.

We start with some terminology: we identify the set Val_n of all valuations (assignments) of the propositional variables with the power set of $\{P_1, \ldots, P_n\}$, namely

[1] *I.e.* complete as a lattice. Note that in general only one of the infinite distributive laws holds in topological Heyting algebras, namely $a \sqcap \bigsqcup_{i \in I} b_i = \bigsqcup_{i \in I} (a \sqcap b_i)$.

[2] For these "intuitionistic formulae", the system of connectives $\{\bot, \wedge, \vee, \to\}$ is used, and the following abbreviations: $\top := \bot \to \bot$, $\neg A := A \to \bot$, $A \leftrightarrow B := (A \to B) \wedge (B \to A)$.

a valuation with the set of variables to which it assigns "true". A *Kripke model* $\mathfrak{K} = (K, \leqslant, \mathrm{val})$ for IPL_n consists of a reflexive partial order[3] $\leqslant$ on a set K and a function $\mathrm{val} : K \to \mathrm{Val}_n$ satisfying the following monotonicity condition: If $P_i \in \mathrm{val}(w)$ for $w \in K$, written $w \vDash P_i$, and if $w' \leqslant w$, then $w' \vDash P_i$. Validity of formulae at a point w is then defined by induction in the classical way for the connectives $\bot$, $\vee$ and $\wedge$, and for $\varphi \to \chi$ by the condition: $w' \vDash \varphi \Rightarrow w' \vDash \chi$ for all points $w' \in K$ with $w' \leqslant w$. By induction, validity of all intuitionistic formulae obeys the monotonicity condition. It follows from this definition that the map

$$\varphi \ \mapsto \ [\![\varphi]\!] := \{w \in K \mid w \vDash \varphi\}$$

induces a homomorphism of Heyting algebras from IPL_n to $\mathcal{O}_\downarrow(K, \leqslant)$. The kernel of this morphism is the *theory of the model*: all formulae valid at every point of the model. The *theory of a point* w consists of all formulae valid at w. (See e.g. [Fi] for more details.)

A Kripke model is *reduced* if any two points differ either by their valuations or by some (third) point below. Precisely: there are no two distinct points w_1, w_2 with the same valuation and (1) such that $w \leqslant w_1 \iff w \leqslant w_2$ for all w or (2) such that w_1 is the unique predecessor of w_2. (One can show that a finite model is reduced if and only if two distinct points have distinct theories.) One can reduce a finite model by applying the following two operations:

- identify points with same valuation and same points below;
- delete a point with only one predecessor, if both carry the same valuation;

and one can check by induction that these reductions do not change the theory of the model. (This is well known in modal logic: it is a special case of a bisimulation, see [BMV].) Therefore it follows:

Fact 3.1 (Lemma 2.3 of [Be]) *For every finite model of* IPL_n*, there is a reduced finite model with the same theory.*

3.1 The construction of the generic Kripke model $\mathfrak{K}_n$

The idea of the construction of $\mathfrak{K}_n$ is to ensure that all finite reduced models embed as an initial segment. $(K_n, \leqslant)$ will be a well-founded partial ordering of rank ω and $\mathfrak{K}_n$ will be an increasing union of Kripke models $\mathfrak{K}_n^d = (K_n^d, \leqslant, \mathrm{val})$. We define $\mathfrak{K}_n^d$ by induction on d as follows (cf. Figure 1):

- We let $K_n^{-1} = \emptyset$. Then $K_n^d \setminus K_n^{d-1}$ consists of all possible elements $w_{\beta,Y}$ such that:
  - Y is a decreasing set in K_n^{d-1} and $Y \not\subseteq K_n^{d-2}$ (for $d = 0$ the last condition is empty, therefore $Y = \emptyset$);
  - β is a valuation in Val_n such that $\beta \subseteq \mathrm{val}(w')$ for all points $w' \in Y$;
  - if Y is the decreasing set generated by an element $w_{\beta',Y'}$, then $\beta \neq \beta'$.
- The valuation of $w_{\beta,Y}$ is defined to be β.
- The partial ordering on K_n^{d-1} is extended to K_n^d by

$$w \leqslant w_{\beta,Y} :\iff (w \in Y \text{ or } w = w_{\beta,Y}).$$

In particular, one sees that by construction every K_n^d is finite, K_n^d is an initial part of K_n^{d+1}, and K_n^d is the set of points of K_n of foundation rank $\leqslant d$. One can check that $\mathfrak{K}_n^d$ is the maximal reduced Kripke model of foundation rank d for IPL_n.

[3] Note that the order is reversed with respect to the usual approach to Kripke models. This is for the sake of an easy description and to be in coherence with the order of the Heyting algebra, see Remark 4.4, and the order on the spectrum, compare with Fact 5.3.

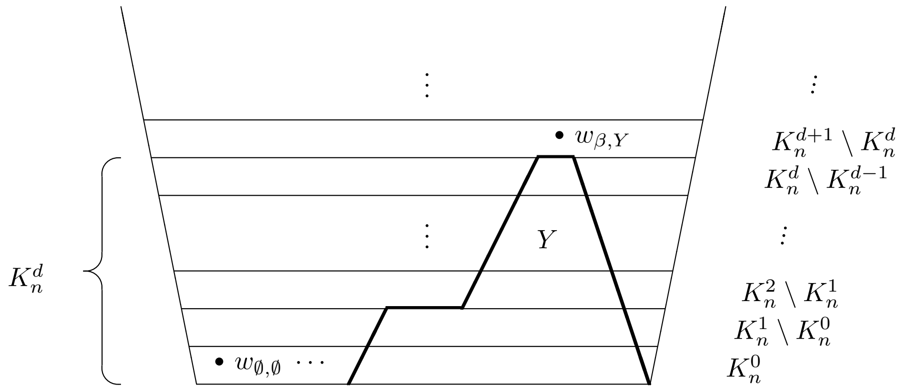

Figure 1: The construction of $\mathfrak{K}_n$

Theorem 3.2 (Bellissima in [Be]) *The map $\varphi \mapsto [\![\varphi]\!] = \{w \in K_n \mid w \vDash \varphi\}$ induces an embedding of* IPL_n*, and hence of the free Heyting algebra $\mathcal{F}_n$ with a fixed enumeration of n free generators, into the Heyting algebra*

$$\widehat{\mathcal{F}_n} := \mathcal{O}_{\downarrow}(K_n, \leqslant).$$

This map identifies a set of n free generators of $\mathcal{F}_n$ with the propositional variables $P_1, \ldots, P_n$ that were used in the construction of $\mathfrak{K}_n$. We will see in Corollary 6.3 that there is only one set of free generators of $\mathcal{F}_n$, therefore the embedding is unique up to the action of $\mathrm{Sym}(n)$ on the free generators.

PROOF[4]: We have to show that the homomorphism is injective, which amounts to show that the theory of $\mathfrak{K}_n$ consists exactly of all intuitionistic tautologies. Each finite reduced Kripke model embeds by a straightforward induction onto an initial segment of $(\mathfrak{K}_n, \leqslant)$. Because validity of formulae is preserved under "going down" along $\leqslant$, the theory of $\mathfrak{K}_n$ is contained in that of all finite reduced models. On the other hand, as intuitionistic logic has the finite model property, a non-tautology is already false in some finite model, hence also in some finite reduced model. Thus the theory of $\mathfrak{K}_n$ consists exactly of the intuitionistic tautologies. □

For Grigolia's version of this construction see e.g. [Gr2] or the account in [Bz] which offers a wider context.

From now on, we will identify $\mathcal{F}_n$ with its image in $\widehat{\mathcal{F}_n}$. Thus the free generators become $[\![P_1]\!], \ldots, [\![P_n]\!]$, and the operations can be computed in the topological Heyting algebra $\widehat{\mathcal{F}_n}$ as indicated in Example 2.2. Moreover, we speak of *finite elements* of $\mathcal{F}_n$ or $\widehat{\mathcal{F}_n}$ meaning elements which are finite subsets of K_n.

3.2 Results from [Be]

In this section we collect all results from [Be] that we are going to use.

We call an element a in a lattice $\sqcup$-*irreducible* if it is different from 0 and can not be written as a union $b_1 \sqcup b_2$ with $b_i \neq a$. It is *completely* $\sqcup$-*irreducible*, or $\bigsqcup$-*irreducible* for short, if it can not be written as any proper union of other elements

[4]The proof is essentially Bellissima's: we have simplified notations, separated the general facts from the special situation and left out some detailed elaborations, e.g. a proof of Fact 3.1.

(possibly infinite, possibly empty), thus if and only if it has a unique predecessor $a^- := \bigsqcup\{x \mid x < a\}$. The dual notions apply for $\sqcap$. In particular, a $\sqcap$-irreducible element is by convention different from 1, and a $\bigsqcap$-irreducible has a unique successor $a^+ := \bigsqcap\{x \mid a < x\}$.

For $X \subseteq K_n$, we let $X_\downarrow$ be the $\leqslant$-decreasing set and $X^\uparrow$ the $\leqslant$-increasing set generated by X.[5] Thus $X_\downarrow$ and $X^{\uparrow\complement}$ are both open in $\mathcal{O}_\downarrow(K_n, \leqslant)$ and hence elements of $\widehat{\mathcal{F}_n}$. (Note that $X_\downarrow$ and $X^\uparrow$ are the closures of X in the topologies $\mathcal{O}^\uparrow(K_n, \leqslant)$ and $\mathcal{O}_\downarrow(K_n, \leqslant)$ respectively.) We call a set $\{w\}_\downarrow$ for $w \in K_n$ a *principal set* and a set $\{w\}^{\uparrow\complement}$ a *co-principal set* (see Figure 2).

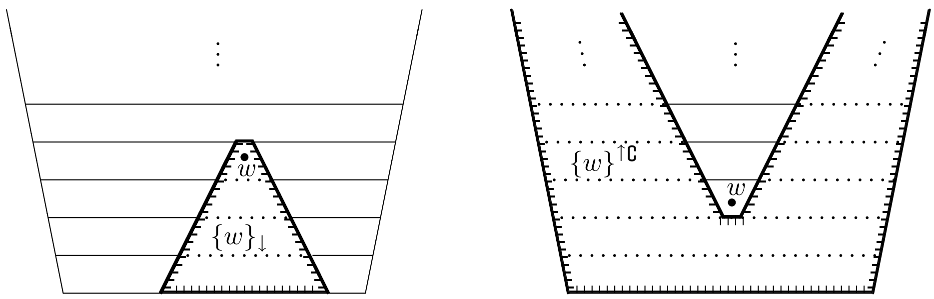

Figure 2: A principle set $\{w\}_\downarrow$ and a co-principle set $\{w\}^{\uparrow\complement}$

The following theorem and its corollary are main results of Bellissima[6]. We follow his proof except that we simplify notations and arguments and that we get shorter formulae.[7] Of course, an empty disjunction stands for the formula $\bot$ and an empty conjunction for $\top$.

Theorem 3.3 *For every $w \in K_n$, there are formulae ψ_w and ψ'_w such that $[\![\psi_w]\!] = \{w\}_\downarrow$ and $[\![\psi'_w]\!] = \{w\}^{\uparrow\complement}$. They can be defined by induction on the foundation rank of w as follows. If $Y_{\max}$ denotes the maximal elements of Y, then*

$$\psi_{w_{\beta,Y}} := \Big(\Big(\bigvee_{w \in Y_{\max}} \psi'_w \vee \bigvee_{P_i \notin \beta} P_i\Big) \to \bigvee_{w \in Y_{\max}} \psi_w\Big) \wedge \bigwedge_{P_i \in \beta} P_i$$

$$\psi'_{w_{\beta,Y}} := \psi_{w_{\beta,Y}} \to \bigvee_{w \in Y_{\max}} \psi_w$$

It follows that if w has foundation rank d, then the implication depth of ψ_w is at most $2d+1$ and of ψ'_w at most $2d+2$.

Proof: Let $w_{\beta,Y}$ be of foundation rank $d+1$. We assume the proposition to be shown for all points of smaller foundation rank in K_n, and conclude by induction.

By induction, $[\![\bigvee_{w \in Y_{\max}} \psi_w]\!] = Y$ (if $d = -1$, then $Y = \emptyset$, and everything works as well). Now $\psi_{w_{\beta,Y}}$ is intuitionistically equivalent to a conjunction of three formulae that define the following subsets of K_n:

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[5] Because we are working here with the reversed order, the arrows are the other way round compared to Bellissima.

[6] Lemma 2.6, Theorem 2.7 and Corollary 2.8 in [Be]; our Corollary 3.4 (b) is implicit in Bellissima's Lemma 2.6.

[7] This is mainly because Bellissima's φ_1 is implied by the second conjunct of his φ_2.

$$A_1 := [\![\bigvee_{w\in Y_{\max}} \psi'_w \to \bigvee_{w\in Y_{\max}} \psi_w]\!] = \Big\{ w \;\Big|\; \forall v \leqslant w \;\Big(v \in \bigcap_{z\in Y_{\max}} [\![\psi'_z]\!]^{\complement} \cup \bigcup_{z\in Y_{\max}} [\![\psi_z]\!] \Big) \Big\}$$

$$= \big\{ w \mid \forall v \leqslant w \;\big(Y \subseteq \{v\}_{\downarrow} \;\text{ or }\; v \in Y \big) \big\}$$

$$\subseteq \quad B_1 := \big\{ w \mid \{w\}_{\downarrow} \cap K_n^d = Y \big\} \cup Y$$

$$A_2 := [\![\bigvee_{P_i\notin\beta} P_i \to \bigvee_{w\in Y_{\max}} \psi_w]\!] = \Big\{ w \;\Big|\; \forall v \leqslant w \;\Big(v \in \bigcap_{P_i\notin\beta} [\![P_i]\!]^{\complement} \cup \bigcup_{z\in Y_{\max}} [\![\psi_z]\!] \Big) \Big\}$$

$$= \big\{ w \mid \forall v \leqslant w \;\big(\mathrm{val}(v) \subseteq \beta \;\text{ or }\; v \in Y \big) \big\}$$

$$\text{and} \qquad A_3 := [\![\bigwedge_{P_i\in\beta} P_i]\!] = \big\{ w \in K_n \mid \beta \subseteq \mathrm{val}(w) \big\}.$$

One sees that $Y \subseteq A_i \cap K_n^d$ for all i, and $A_1 \cap K_n^d = Y$. Thus $[\![\psi_{w_{\beta,Y}}]\!] \cap K_n^d = Y$. Moreover, if $w \in [\![\psi_{w_{\beta,Y}}]\!] \setminus K_n^d$, then w has the following property: for all $v \leqslant w$, either $v \in Y$, or $\mathrm{val}(v) = \beta$ (from A_2 and A_3) and $\{v\}_{\downarrow} \cap K_n^d = Y$ (from B_1). By construction of $\mathfrak{K}_n$, there is only one such point, namely $w_{\beta,Y}$.

Then $[\![\psi'_{w_{\beta,Y}}]\!]$ is by definition the largest decreasing set contained in $[\![\psi_{w_{\beta,Y}}]\!]^{\complement} \cup [\![\bigvee_{w\in Y_{\max}} \psi_w]\!] = K_n \setminus \{w_{\beta,Y}\}$, which is exactly $\{w_{\beta,Y}\}^{\uparrow\complement}$. □

Corollary 3.4

(a) *The principal and the co-principle sets are in $\mathcal{F}_n$, hence also all finite sets.*

(b) *For $w \in K_n$, if $a = \{w\}_{\downarrow}$ and $b = \{w\}^{\uparrow\complement}$, then $b = a \to a^-$.*

PROOF: (b) follows because $\{w_{\beta,Y}\}_{\downarrow}{}^{-} = Y = [\![\bigvee_{w\in Y_{\max}} \psi_w]\!]$. □

From the construction in Theorem 3.2 we will mainly use two properties: The "filtration" of the Kripke model into levels of finite foundation rank. And the property that any finite set of at least two incomparable elements in K_n has a common successor without other predecessors. Moreover, we need the following result from [Be]:

Fact 3.5 (Theorem 3.0 in [Be])

(a) *The principal sets are exactly the $\bigsqcup$-irreducible elements of both algebras, $\mathcal{F}_n$ and $\widehat{\mathcal{F}_n}$.*

(b) *The co-principal sets are exactly the $\sqcap$-irreducible sets of both algebras.*

The theorem in [Be] is formulated for $\mathcal{F}_n$ only, but the proof works as well for $\widehat{\mathcal{F}_n}$. For the sake of completeness, we add a sketch of the proof:

PROOF: (a) For $X \in \widehat{\mathcal{F}_n}$ we have $X = \bigcup_{w\in X}\{w\}_{\downarrow}$. It follows that X is $\bigsqcup$-irreducible, if and only if there is a greatest element w_0 in X, if and only if $X = \{w_0\}_{\downarrow}$.

(b) If there are two minimal elements $w_0, w_1 \in K_n \setminus X$, then $X = (X \cup \{w_0\}) \cap (X \cup \{w_1\})$ is not $\sqcap$-irreducible. Conversely, $\{w\}^{\uparrow\complement}$ has a unique successor, namely $\{w\}^{\uparrow\complement} \cup \{w\}$, hence $\{w\}^{\uparrow\complement}$ is $\bigsqcap$-irreducible. □

Let $\breve{\mathcal{F}}_n$ be the sub-Heyting algebra of $\mathcal{F}_n$ generated by all $\bigsqcup$-irreducible elements of $\mathcal{F}_n$. Thus $\breve{\mathcal{F}}_n = \mathbf{B}_n$ in Bellissima's notation in [Be].

Fact 3.6 (Theorem 4.4 in [Be]) *For $n > 1$, the algebra $\breve{\mathcal{F}}_n$ is not finitely generated, because the sub-algebra generated by all $\{w\}_\downarrow$ for w of foundation rank $\leqslant d$ can't separate points of K_n of higher foundation rank that differ only by their valuations.*

In particular, it follows that $\breve{\mathcal{F}}_n$ is not isomorphic to $\mathcal{F}_n$ for $n > 1$. To our knowledge, Grigolia has shown that no proper sub-algebra of $\mathcal{F}_n$ is isomorphic to $\mathcal{F}_n$. Without being explicitly mentioned, it is clear in [Be] that $\breve{\mathcal{F}}_1 = \mathcal{F}_1 = \widehat{\mathcal{F}_1}$ and that $\mathcal{F}_n \neq \widehat{\mathcal{F}_n}$ for $n > 1$.

Fact 3.7 (Lemma 4.1 in [Be]) *For $n > 1$, there is an infinite antichain in K_n.*

4 Some consequences

This section collects some rather immediate consequences from Bellissima's construction that are not explicitly mentioned in Bellissima's paper, and which we will use in our analysis of Bellissima's setting. Other consequences are collected in section 7. Many of the results hold in a much wider context, see [DJ2].

4.1 More on irreducible elements

For a in a Heyting algebra, let us define the *supports*

$$\begin{aligned} \mathrm{supp}_{\bigsqcup}(a) &:= \big\{x \ \textstyle\bigsqcup\text{-irreducible} \ \big|\ x \sqsubseteq a\big\}, \\ \mathrm{supp}_{\bigsqcap}(a) &:= \big\{x \ \textstyle\bigsqcap\text{-irreducible} \ \big|\ a \sqsubseteq x\big\}. \\ \mathrm{supp}^{\min}_{\bigsqcap}(a) &:= \text{ the minimal elements in } \mathrm{supp}_{\bigsqcap}(a) \end{aligned}$$

Lemma 4.1

(a) *The $\sqcap$-irreducible elements are $\bigsqcap$-irreducible in both, $\mathcal{F}_n$ and $\widehat{\mathcal{F}_n}$.*

(b) *For each $a \in \widehat{\mathcal{F}_n}$, we have*

$$a = \bigsqcup \mathrm{supp}_{\bigsqcup}(a) = \bigsqcap \mathrm{supp}^{\min}_{\bigsqcap}(a),$$

and $a \neq \bigsqcap S$ for every proper subset $S \subset \mathrm{supp}^{\min}_{\bigsqcap}(a)$.

It follows from part (a) of this Lemma and from Fact 3.5 (a) that the (well-founded) partial ordering of K_n equals the partial ordering of the $\bigsqcap$-irreducibles, and that of the $\bigsqcup$-irreducibles. In particular, the (co-)foundation rank of w in K_n equals the (co-)foundation rank of $\{w\}_\downarrow$ in the partial ordering of the $\bigsqcup$-irreducibles and also the (co-)foundation rank of $\{w\}^{\uparrow\complement}$ in the partial ordering of the $\bigsqcap$-irreducibles. Hence all these foundation ranks are finite, and all these co-foundation ranks equal to ∞, due to the existence of an infinite chain $w < w' < w'' < \cdots$ in K_n. Foundation and co-foundation ranks are used to compute dimensions and co-dimensions, cf. [DJ2] remark 6.9.

Proof: (a) has already be shown in the proof of Fact 3.5 (b).

(b) For any decreasing set $a \subseteq K_n$, we have $a = \bigcup_{w \in a}\{w\}_\downarrow = \bigcap_{w \notin a}\{w\}^{\uparrow\complement}$, which proves the first equality and the second for the full $\mathrm{supp}_{\bigsqcap}$. But because the order is well-founded on the $\bigsqcap$-irreducible elements, it is enough to keep the minimal elements in the support. The last statement is clear by definition. □

Corollary 4.2

(a) *The $\bigsqcup$-irreducibles and $\bigsqcap$-irreducibles of $\widehat{\mathcal{F}_n}$ and of $\mathcal{F}_n$ are the same.*

(b) *Any element of $\widehat{\mathcal{F}_n}$ is a (possibly infinite) union as well as a (possibly infinite) intersection of elements of $\mathcal{F}_n$.*

A characterisation of the $\sqcup$-irreducible elements can be found in Section 7.2.

Example 4.3 Lemma 4.1 doesn't work with the maximal elements in $\mathrm{supp}_{\bigsqcup}$, because the order on the $\bigsqcup$-irreducible elements is not anti-well-founded. For example, if $n \geqslant 2$, then there are no maximal elements in $\mathrm{supp}_{\bigsqcup}(\llbracket P_i \rrbracket)$. However, for $\widehat{\mathcal{F}_n}$ Corollary 6.10 in [DJ2] provides a decomposition into $\sqcup$-irreducible components.

Proof: Say $i = 2$. One can show that for each k, there are at least two elements in K_n of foundation rank k and with P_2 in the valuation. For example because there is a copy of the Kripke model $\mathfrak{K}_1$ inside the points w of K_n with $P_2 \in \mathrm{val}(w)$ —just add P_2 to the valuations of the points of $\mathfrak{K}_1$, i.e. start with the points $w_{\{P_2\},\emptyset}$ and $w_{\{P_1,P_2\},\emptyset}$— and $\mathfrak{K}_1$ is well known to have two points of each foundation rank. Now if $w \in \llbracket P_2 \rrbracket$, choose $v \in \llbracket P_2 \rrbracket$, $v \neq w$, of same foundation rank. Then the point $w_{\{P_2\},\{v,w\}_\downarrow}$ shows that $\{w\}_\downarrow$ is not maximal in $\mathrm{supp}_{\bigsqcup}(\llbracket P_2 \rrbracket)$. □

Remark 4.4 We can identify the underlying partially ordered set of the Kripke model $\mathfrak{K}_n$ with either the $\bigsqcup$-irreducibles via $w \mapsto \{w\}_\downarrow$, or with the $\bigsqcap$-irreducibles via $w \mapsto \{w\}^{\uparrow\complement}$, both with the order induced by the partial order $\sqsubseteq$ of the Heyting algebra $\mathcal{F}_n$ (which is one of the reasons to work with the reversed order on the Kripke model). We can further identify the $\bigsqcap$-irreducibles with the principal prime ideals of $\mathcal{F}_n$ that they generate, ordered by inclusion. By Lemma 4.1 (a), all principal prime ideals are of that form. The valuations of the Kripke model can also be recovered from $\mathcal{F}_n$, as will be explained in Section 5.

A survey about the relationship between Kripke models and Heyting algebras can be found in the doctoral thesis of Nick Bezhanishvili [Bz].

Remark 4.5 The decomposition $a = \bigsqcap \mathrm{supp}^{\min}_{\bigsqcap}(a)$ of Lemma 4.1 provides a sort of infinite (conjunctive) normal form for elements of $\widehat{\mathcal{F}_n}$. Provided the partial order on the $\bigsqcap$-irreducibles is known, the Heyting algebra operations can be computed from the supports in a first order way: First note that $\mathrm{supp}^{\min}_{\bigsqcap}$ and $\mathrm{supp}_{\bigsqcap}$ can be computed from each other, and then

$$\begin{aligned}
\mathrm{supp}^{\min}_{\bigsqcap}(a \sqcup b) &= \text{the minimal elements in } \mathrm{supp}_{\bigsqcap}(a) \cap \mathrm{supp}_{\bigsqcap}(b) \\
\mathrm{supp}^{\min}_{\bigsqcap}(a \sqcap b) &= \text{the minimal elements in } \mathrm{supp}_{\bigsqcap}(a) \cup \mathrm{supp}_{\bigsqcap}(b) \\
\mathrm{supp}^{\min}_{\bigsqcap}(a \to b) &= \text{the minimal elements in } \mathrm{supp}_{\bigsqcap}(b) \setminus \mathrm{supp}_{\bigsqcap}(a)
\end{aligned}$$

Note that every set of pairwise incomparable $\bigsqcap$-irreducible elements forms the $\mathrm{supp}^{\min}_{\bigsqcap}$ of an element of $\widehat{\mathcal{F}_n}$, namely of its intersection.

Question 4.6 Can we characterise those sets that correspond to elements of $\mathcal{F}_n$?

Because $\{w\}_\downarrow \in \mathrm{supp}_{\bigsqcup}(a) \iff \{w\}^{\uparrow\complement} \notin \mathrm{supp}_{\bigsqcap}(a)$, one can translate the above rules into computations of the Heyting algebra operations from the $\bigsqcup$-supports, but they are less nice.

4.2 The completion as a profinite limit

Consider in $\mathcal{F}_n$ for each i the filter $(K_n^i)^\uparrow = \{a \in \mathcal{F}_n \mid K_n^i \subseteq a\}$ generated by K_n^i, and denote the corresponding congruence relation by $\equiv_i$. Thus

$$a \equiv_i b \iff a \cap K_n^i = b \cap K_n^i.$$

We denote the quotient $\mathcal{F}_n/\equiv_i$ by $\mathcal{F}_n^i$ and the canonical epimorphism by π_i. It extends to an epimorphism $\widehat{\pi}_i : \widehat{\mathcal{F}_n} \to \mathcal{F}_n^i$, where $\widehat{\pi}_i^{-1}(1) = \{a \in \widehat{\mathcal{F}_n} \mid K_n^i \subseteq a\}$ is the filter generated by K_n^i in $\widehat{\mathcal{F}_n}$. For simplicity, we denote it and the corresponding congruence relation again by $(K_n^i)^\uparrow$ and $\equiv_i$ and we will identify $\widehat{\mathcal{F}_n}/\equiv_i$ with $\mathcal{F}_n^i$.

$\mathcal{F}_n^i$ is naturally isomorphic to the finite Heyting algebra $\mathcal{O}_\downarrow(K_n^i, \leqslant)$ via "truncation" $\pi_i(a) \mapsto a \cap K_n^i$ (the surjectivity needs part (a) of Corollary 4.2). Again, we will identify both without further mentioning.[8]

Remark 4.7 There is a natural notion of dimension (more precisely: dual codimension, see [DJ2] and [Da]) in lattices such that $\mathcal{F}_n^i$ is the free Heyting algebra over n generators of dimension i. Among universal algebraists, it is known as the free algebra generated by n elements in the variety of Heyting algebras satisfying $P_{i+1} = 1$ (where P_{i+1} is defined inductively as $x_{i+1} \sqcup (x_{i+1} \to P_i)$ with $P_0 = 0$).

Proposition 4.8 *$\mathcal{F}_n$ and $\widehat{\mathcal{F}_n}$ are residually finite, i.e. for each element $a \neq 1$ there is a homomorphism ψ onto a finite Heyting algebra such that $\psi(a) \neq 1$. Moreover, ψ can be chosen to be some π_i or $\widehat{\pi}_i$ respectively.*

Proof: This is clear from the above considerations and the construction of K_n as union of the K_n^i. □

Because $K_n^i \subseteq K_n^j$ for $i < j$, the morphism $\pi_i : \mathcal{F}_n \to \mathcal{F}_n^i$ factors through $\mathcal{F}_n^j$ for $i < j$ and yields an epimorphism $\pi_{ji} : \mathcal{F}_n^j \to \mathcal{F}_n^i$. The system of maps π_{ji} is compatible, hence the projective limit $\varprojlim \mathcal{F}_n^i$ exists. Because of the universal property of the projective limit and because the system of morphisms $\widehat{\pi}_i : \widehat{\mathcal{F}_n} \to \mathcal{F}_n^i$ is compatible as well, there is a natural morphism $\widehat{\mathcal{F}_n} \to \varprojlim \mathcal{F}_n^i$.

$$\begin{array}{ccccccccc}
\widehat{\mathcal{F}_n} & & \hookleftarrow & & & \mathcal{F}_n & & & \\
\downarrow & & & \cdots & {\scriptstyle \pi_{i+1}}\swarrow & & {\scriptstyle \pi_i}\searrow & & \cdots \\
\varprojlim \mathcal{F}_n^i & \twoheadrightarrow & \cdots & \twoheadrightarrow & \mathcal{F}_n^{i+1} & \underset{\pi_{i+1,i}}{\twoheadrightarrow} & \mathcal{F}_n^i & \twoheadrightarrow & \cdots
\end{array}$$

Proposition 4.9 *$\widehat{\mathcal{F}_n}$ is the projective limit of the finite Heyting algebras $\mathcal{F}_n^i$, in symbols*

$$\widehat{\mathcal{F}_n} = \varprojlim_{i \in \mathbb{N}} \mathcal{F}_n^i.$$

Any finite quotient of $\mathcal{F}_n$ factors through some $\mathcal{F}_n^i$, hence $\widehat{\mathcal{F}_n}$ is the profinite completion of $\mathcal{F}_n$, i.e. the projective limit of all finite epimorphic images of $\mathcal{F}_n$.

The content of this proposition or parts of it is, in various forms, contained in several articles. To our knowledge, Grigolia was the first to embed $\mathcal{F}_n$ into $\varprojlim \mathcal{F}_n^i$,

[8] There is a certain ambiguity here that should not harm: As K_n^i is also an element of $\mathcal{F}_n$, there is a map $\mathcal{F}_n \to \mathcal{F}_n$, $a \mapsto a \cap K_n^i$. The image of this map is a sub-poset (but not a sub-algebra) of $\mathcal{F}_n$ isomorphic to $\mathcal{F}_n^i$.

see [Gr1] or [Gr3]. More about profinite Heyting algebras can be found in [BGMM] and [Bz[2]]. In the first of these two articles, the profinite completion of a Heyting algebra is embedded in the topological Heyting algebra of its dual space, i.e. its prime spectrum.

Proof: The natural morphism $\widehat{\mathcal{F}_n} \to \varprojlim \mathcal{F}_n^i$ is in fact the map $a \mapsto (\widehat{\pi}_i(a))_{i\in\mathbb{N}} = (a \cap K_n^i)_{i\in\mathbb{N}}$. It is injective by Proposition 4.8 and it is surjective because for any compatible system $(a_i)_{i\in\mathbb{N}}$ of decreasing sets $a_i \subseteq K_n^i$, the union $\bigcup_{i\in\mathbb{N}} a_i$ is a pre-image in $\widehat{\mathcal{F}_n}$.

Now let Φ be a filter such that $\mathcal{F}_n/{\equiv_\Phi}$ is finite. Choose an i such that for all elements in $\mathcal{F}_n/{\equiv_\Phi}$ there is a pre-image in $\mathcal{F}_n$ with pairwise distinct images in $\mathcal{F}_n^i$. Then $\mathcal{F}_n \to \mathcal{F}_n/{\equiv_\Phi}$ factors through $\mathcal{F}_n^i$. □

The epimorphism $\pi_{i+1,i} : \mathcal{O}_\downarrow(K_n^{i+1}, \leqslant) \to \mathcal{O}_\downarrow(K_n^i, \leqslant)$ is induced from the inclusion $(K_n^i, \leqslant) \hookrightarrow (K_n^{i+1}, \leqslant)$. The theorem shows that $\mathcal{O}_\downarrow$ behaves like a contravariant functor, i.e.

$$\widehat{\mathcal{F}_n} \;=\; \mathcal{O}_\downarrow(K_n, \leqslant) \;=\; \mathcal{O}_\downarrow\big(\varinjlim_{i\in\mathbb{N}} (K_n^i, \leqslant)\big) \;=\; \varprojlim_{i\in\mathbb{N}} \mathcal{O}_\downarrow(K_n^i, \leqslant) \;=\; \varprojlim_{i\in\mathbb{N}} \mathcal{F}_n^i$$

Note that any profinite lattice is a complete lattice as projective limit of complete lattices, and that any profinite structure is well known to be a compact Hausdorff topological space (the topology being the initial topology for the projections onto the finite quotients in the defining system, equipped with the discrete topology). We are going to examine this topology a little further:

Definition For $a, b \in \widehat{\mathcal{F}_n}$, define their distance to be

$$\mathrm{d}(a, b) \;:=\; 2^{-\min\{i \,|\, \widehat{\pi}_i(a) \neq \widehat{\pi}_i(b)\}} \;=\; 2^{-\min\{i \,|\, a\cap K_n^i \neq b \cap K_n^i\}}$$

with the convention $2^{-\min\emptyset} = 0$.

Theorem 4.10 *$(\widehat{\mathcal{F}_n}, \mathrm{d})$ is a compact Hausdorff metric space. The metric topology is the profinite topology, and the Heyting algebra operations are continuous. $\widehat{\mathcal{F}_n}$ is the metric completion of its dense subset $\mathcal{F}_n$.*

This is analogous to properties of the p-adic integers $\mathbb{Z}_p$ as a projective limit of the rings $\mathbb{Z}/p^n\mathbb{Z}$. See also Theorem 6.1 in [DJ2] for a generalisation to a wider class of Heyting algebras comprising the finitely presented ones.

Proof: It is straightforward to check that d is a metric: symmetry and the ultrametric triangular inequality $\mathrm{d}(a, c) \leqslant \max\big\{\mathrm{d}(a, b), \mathrm{d}(b, c)\big\}$ are immediate from the definition, and $\mathrm{d}(a, b) = 0 \iff a = b$ holds by Proposition 4.8.

The profinite topology and the metric topology coincide because they have the same basis of (cl-)open sets $\widehat{\pi}_i^{-1}(a) = \{x \mid \mathrm{d}(x, x_0) < 2^{-i+1}\}$ for $a \in \mathcal{F}_n^i$ and x_0 with $\widehat{\pi}_i(x_0) = a$.

It is a standard result that a projective limit of compact Hausdorff spaces is again compact and Hausdorff. (Here, this is easily seen directly: Metric topologies are always Hausdorff. If a family of closed sets with the finite intersection property is given, we may suppose the closed sets to be of the form $\widehat{\pi}_i^{-1}(a_i)$. The fip implies that every i appears only once and that the a_i are compatible. Hence $(a_i)_{i\in\omega}$ is an element of $\varprojlim \mathcal{F}_n^i$ in the intersection of the family and the topology is compact.)

By definition of the profinite topology, the maps $\widehat{\pi}_i$ are continuous. Then the continuity of the Heyting algebra operations on the $\mathcal{F}_n^i$ (with discrete topology!) lifts to $\widehat{\mathcal{F}_n}$, e.g. $\sqcap^{-1}\big(\widehat{\pi}_i^{\,-1}(a)\big) = (\widehat{\pi}_i \times \widehat{\pi}_i)^{-1}\big(\sqcap^{-1}(a)\big)$ is open. An element $a \in \widehat{\mathcal{F}_n}$ is a limit of the sequence $(a \cap K_n^i)_{i\in\omega}$ in $\mathcal{F}_n$. Therefore $\widehat{\mathcal{F}_n}$ is the metric completion and thus $\mathcal{F}_n$ is dense in $\widehat{\mathcal{F}_n}$. □

As points are closed, we get that any term in the language of Heyting algebras (even with parameters) defines a continuous map.

Remark 4.11 There is a fundamental difference between the cases $n = 1$ and $n > 1$. In case $n = 1$, the map $\pi_{i+1\,i} : \mathcal{F}_1^{i+1} \to \mathcal{F}_1^i$ has a kernel of 3 elements, but is otherwise injective. It follows that the map $\pi_i : \widehat{\mathcal{F}_1} \to \mathcal{F}_1^i$ is injective on $\widehat{\mathcal{F}_1} \setminus (K_n^i)^{\uparrow}$, and that $\mathcal{F}_1 = \widehat{\mathcal{F}_1}$. In case $n > 1$, the size of $\mathcal{F}_n^i$ is growing faster than exponentially with i. Moreover, the maps $\pi_{i+1\,i} : \mathcal{F}_n^{i+1} \to \mathcal{F}_n^i$ are non-injective enough to allow a tree of elements $a_{s_1,\dots,s_i} \in \mathcal{F}_n^i$ with $s_j \in \{0,1\}$ such that $\pi_{i+1\,i}(a_{s_1,\dots,s_{i+1}}) = a_{s_1,\dots,s_i}$. Hence $\widehat{\mathcal{F}_n}$ has size continuum and differs from $\mathcal{F}_n$.

4.3 Dense sub-algebras

Definition We say that H is *dense in* $\widehat{\mathcal{F}_n}$ if H is a dense Heyting sub-algebra of $\widehat{\mathcal{F}_n}$ in the metric topology. By Theorem 4.10, the metric topology equals the profinite topology, hence density of a sub-algebra H of $\widehat{\mathcal{F}_n}$ means that all the induced maps $\pi_d : H \to \mathcal{F}_n^d$ are surjective.

Lemma 4.12 *The isolated points in $\widehat{\mathcal{F}_n}$ are exactly the finite elements. They are dense in $\widehat{\mathcal{F}_n}$.*

PROOF: If a is finite $\subseteq K_n^i$, then there is no other element in the $2^{-(i+2)}$-ball around a, hence a is isolated. If a is infinite, then a is the limit of a sequence of finite elements distinct from a, namely $a = \lim_{i\in\omega}(a \cap K_n^i)$. Hence a is not isolated, but a limit of isolated points. □

Recall that $\breve{\mathcal{F}}_n$ is the sub-Heyting algebra of $\mathcal{F}_n$ generated by all $\bigsqcup$-irreducible elements of $\mathcal{F}_n$, i.e. the sub-algebra generated by all finite elements. Thus Lemma 4.12 immediately implies:

Proposition 4.13 *$\breve{\mathcal{F}}_n$ is the smallest dense sub-Heyting algebra of $\mathcal{F}_n$.*

As we have mentioned in Fact 3.6, Bellissima has shown that $\breve{\mathcal{F}}_n$ is not isomorphic to $\mathcal{F}_n$ for $n \geqslant 2$. If one knew that $\breve{\mathcal{F}}_n \not\equiv \mathcal{F}_n$, then Proposition 4.13 would yield an easier proof, because $\mathcal{F}_n$ then has a proper dense sub-algebra, namely $\breve{\mathcal{F}}_n$, whereas $\breve{\mathcal{F}}_n$ does not. We will see later that $\breve{\mathcal{F}}_n \not\equiv \mathcal{F}_n$ for $n \geqslant 2$.

Lemma 4.14 *If H is dense in $\widehat{\mathcal{F}_n}$, then H has the same $\bigsqcup$-irreducibles and the same $\bigsqcap$-irreducibles as $\widehat{\mathcal{F}_n}$.*

PROOF: Because of the density, all the finite sets are in H. Thus in particular all the principal sets are in H, and as they are $\bigsqcup$-irreducible in $\widehat{\mathcal{F}_n}$, they remain irreducible in H. If $X \in H$ is not principal, then X is the proper union of its $\mathrm{supp}_{\bigsqcup}$, hence not $\bigsqcup$-irreducible.

Because of Corollary 4.2 (b), H also contains all the $\bigsqcap$-irreducibles of $\widehat{\mathcal{F}_n}$, and they remain for trivial reasons $\bigsqcap$-irreducible in H. With the same argument as above, one sees that there are no other $\bigsqcap$-irreducible elements in H, because any other element if the proper intersection of its $\mathrm{supp}_{\bigsqcap}$. $\square$

It follows that $\breve{\mathcal{F}}_n$ is also the sub-Heyting algebra generated by the $\bigsqcap$-irreducible elements of $\widehat{\mathcal{F}_n}$, because any finite set is the intersection of finitely many $\bigsqcap$-irreducible elements, namely its $\mathrm{supp}^{\min}_{\bigsqcap}$. See also section 7.3 for more on $\breve{\mathcal{F}}_n$, and Proposition 6.12 of [DJ2] for a generalisation of $\breve{\mathcal{F}}_n$ to, among others, finitely presented Heyting algebras.

Remark 4.15 If $n > 1$ and b_1, b_2 are two incomparable $\bigsqcap$-irreducible elements in $\mathcal{F}_n$, then $\mathrm{supp}^{\min}_{\bigsqcap}(b_1 \sqcup b_2)$ is infinite. Thus not all elements of $\breve{\mathcal{F}}_n$ have finite $\mathrm{supp}^{\min}_{\bigsqcap}$. This is a reason why an explicit description of $\breve{\mathcal{F}}_n$ is not as easy as one could first think.

Proof: If $b_i = \{w_i\}^{\uparrow\complement}$, then $\{w_{\emptyset,\{w_1,w_2,v\}_\downarrow}\}^{\uparrow\complement}$ is in $\mathrm{supp}^{\min}_{\bigsqcap}(b_1 \sqcup b_2)$ for any v that is incomparable with w_1 or with w_2. If $n > 1$, there are infinitely many such v. $\square$

Remark 4.16 There are two canonical sections of the projection map $\pi_i : \widehat{\mathcal{F}_n} \to \mathcal{F}^i_n$:

$$\begin{aligned} \text{the } \textit{minimal section} \quad & \sigma^{\min}_i : x \;\mapsto\; \bigsqcap \left\{ y \in \widehat{\mathcal{F}_n} \mid \pi_i(y) = x \right\} \\ \text{and the } \textit{maximal section} \quad & \sigma^{\max}_i : x \;\mapsto\; \bigsqcup \left\{ y \in \widehat{\mathcal{F}_n} \mid \pi_i(y) = x \right\} \end{aligned}$$

In fact, both have images in $\breve{\mathcal{F}}_n$: each $\sigma^{\min}_i(x)$ is a finite set, thus a finite union of $\bigsqcup$-irreducibles, and each $\sigma^{\max}_i(x)$ is what one might call a "co-finite set", namely a finite intersection of $\bigsqcap$-irreducibles. If one sees $x \in \mathcal{F}^i_n$ as a subset of the Kripke model $\mathfrak{K}^i_n$ for $\mathcal{F}^i_n$, then the minimal section maps x to itself but now seen as a subset of the Kripke model $\mathfrak{K}_n$. One can compute $\sigma^{\min}_i(x) = K^i_n \sqcap \sigma^{\max}_i(x)$ and $\sigma^{\max}_i(x) = (K^i_n \to \sigma^{\min}_i(x))$, and one can check that $\sigma^{\min}_i$ is a $\{0, \sqcap, \sqcup, \sqsubseteq\}$-homomorphism and $\sigma^{\max}_i$ is a $\{0, 1, \sqcap, \to, \sqsubseteq\}$-homomorphism.

The quotient $\mathcal{F}^0_n$ of $\mathcal{F}_n$ is isomorphic to $\mathfrak{P}(K^0_n)$, which, via the valuation, can be identified with the free Boolean algebra $\mathfrak{P}(\mathfrak{P}(P_1, \ldots, P_n))$ over the free generators $P_1, \ldots, P_n$. The image of the maximal section of π_0 consists exactly of the regular elements of $\mathcal{F}_n$. Thus the regular elements form (as a sub-poset, but not as a sub-algebra) a free Boolean algebra over n generators. (This is easy to see with Bellissima's characterisation of regular elements in Corollary 2.8 of [Be], and much of it is already in [McT].)

5 Reconstructing the Kripke model

5.1 Another duality

The following might be well known in lattice theory. In a (sufficiently) complete lattice, define

$$a^{\sqcap} \;:=\; \bigsqcup \{x \mid a \not\sqsubseteq x\} \qquad \text{and} \qquad a^{\sqcup} \;:=\; \bigsqcap \{x \mid x \not\sqsubseteq a\}$$

Lemma 5.1 *Suppose Λ is a complete lattice that satisfies both infinite distributive laws. Then the maps $a \mapsto a^{\sqcap}$ and $b \mapsto b^{\sqcup}$ are inverse order-preserving bijections between the $\bigsqcup$-irreducibles and the $\bigsqcap$-irreducibles. Moreover, $a \not\sqsubseteq a^{\sqcap}$ and $b^{\sqcup} \not\sqsubseteq b$.*

Proof: The maps are order-preserving by definition (and the transitivity of $\sqsubseteq$). Let a be $\bigsqcup$-irreducible and suppose $a \sqsubseteq a^{\sqcap}$. Then $a = a \sqcap a^{\sqcap} = \bigsqcup\{a \sqcap x \mid a \not\sqsubseteq x\}$. Then the $\bigsqcup$-irreducibility of a implies $a = a \sqcap x$ for some $x \not\sqsupseteq a$: contradiction. Hence $a^{\sqcap}$ is the greatest element x with the property $a \not\sqsubseteq x$. It follows that $a^{\sqcap} \sqsubset a \sqcup a^{\sqcap}$, and if $a^{\sqcap} \sqsubset b$, then $a \sqsubseteq b$. Thus $a^{\sqcap} \sqcup a$ is the unique successor of $a^{\sqcap}$, which therefore is $\bigsqcap$-irreducible. Since dually $a^{\sqcap\sqcup}$ is the minimal element x with the property $x \not\sqsubseteq a^{\sqcap}$, and $a \not\sqsubseteq a^{\sqcap}$, we get $a^{\sqcap\sqcup} \sqsubseteq a$. If we had $a \not\sqsubseteq a^{\sqcap\sqcup}$, then $a^{\sqcap\sqcup} \sqsubseteq a^{\sqcap}$ by definition of the latter. But dually to the argument above, $a^{\sqcap\sqcup}$ is the smallest element x with $x \not\sqsubseteq a^{\sqcap}$: contradiction and $a = a^{\sqcap\sqcup}$. The remaining parts are by duality. $\square$

One can reformulate Lemma 5.1 partially as

$$\mathcal{O}_{\downarrow}\Big(\textstyle\bigsqcap\text{-irreducibles}, \sqsubseteq\Big) \;\cong\; \mathcal{O}_{\downarrow}\Big(\textstyle\bigsqcup\text{-irreducibles}, \sqsubseteq\Big).$$

Remark 5.2 (1) As $x \notin \operatorname{supp}_{\bigsqcap}(a) \iff a \not\sqsubseteq x \iff x^{\sqcup} \sqsubseteq a$ by definition of $x^{\sqcup}$, it follows that

$$\operatorname{supp}_{\bigsqcup}(a) \;=\; \big\{\, x^{\sqcup} \mid x \ \textstyle\bigsqcap\text{-irreducible}, x \notin \operatorname{supp}_{\bigsqcap}(a)\big\} \quad \text{if } a \neq 0,$$

and of course the dual statement also holds.

(2) $\widehat{\mathcal{F}_n}$ as a lattice of sets satisfies the hypotheses of Lemma 5.1. In that special situation, we have that any $\bigsqcup$-irreducible a is of the form $\{w\}_{\downarrow}$. The element $a^{\sqcap}$ is then the corresponding co-principal set $\{w\}^{\uparrow\complement}$, and we have seen in Corollary 3.4 (b) that $a^{\sqcap} = a \to a^{-}$. For a $\bigsqcap$-irreducible a however, we have $a^{+} \to a = a$. Considering $\widehat{\mathcal{F}_n}$ with its co-Heyting structure, we get the "dual" rule $a^{\sqcup} = a^{+} - a$. (Recall from the Example 2.2 that all partial orders are bi-Heyting.)

5.2 The spectrum

If H is a Heyting algebra, the *ideal spectrum* $\operatorname{Spec}_{\downarrow}(H)$ is the partially ordered set of all prime ideals of H endowed with the *Zariski topology*, a basis of which consists of the sets $\bar{I}(a) := \{\mathfrak{i} \in \operatorname{Spec}_{\downarrow}(H) \mid a \notin \mathfrak{i}\}$. The first part of the following fact goes back to Marshall Stone in [St].

Fact 5.3 *The map $a \mapsto \bar{I}(a)$ defines (functorially) an embedding of Heyting algebras*

$$H \;\hookrightarrow\; \mathcal{O}(\operatorname{Spec}_{\downarrow}(H)).$$

Moreover, if H is generated by $g_1, \ldots, g_n$, then $\operatorname{Spec}_{\downarrow}(H)$, partially ordered by inclusion, can be turned into a Kripke model of IPL_n *by defining* $\operatorname{val}(\mathfrak{i}) := \{P_i \mid i \in \bar{I}(g_i)\}$.

In the special case of $\mathcal{F}_n$, the Kripke model $\mathfrak{K}_n$ constructed by Bellissima is naturally isomorphic to the restriction of this construction to the principal ideal spectrum, as we will show in the remaining of this section.

Remark 5.4 There is a bijection $\mathfrak{i} \mapsto \mathfrak{i}^{\complement}$ between the set of prime ideals $\operatorname{Spec}_{\downarrow}(H)$ and the set of prime filters $\operatorname{Spec}^{\uparrow}(H) = \operatorname{Spec}_{\downarrow}(H^{*})$. Therefore there is also an embedding

$$\begin{aligned} H &\hookrightarrow \mathcal{O}\big(\overline{\operatorname{Spec}}^{\uparrow}(H)\big) \\ a &\mapsto F(a) := \{\mathfrak{p} \mid \mathfrak{p} \text{ prime filter}, a \in \mathfrak{p}\} \end{aligned}$$

where $\overline{\operatorname{Spec}}^{\uparrow}(H)$ denotes the space of prime filters endowed with the co-Zariski topology, a basis of open sets of which is given by the $F(a)$'s.

A *principal ideal* in a lattice is the decreasing set generated by an element $x \neq 1$, which we denote by $(x)_\downarrow$. A principal ideal is prime if and only if its generator is $\sqcap$-irreducible. Let the *principal ideal spectrum* $\mathrm{Spec}^0_\downarrow(H)$ be the space of all principal prime ideals endowed with the (trace of the) Zariski topology. The continuous inclusion map $\iota : \mathrm{Spec}^0_\downarrow(H) \to \mathrm{Spec}_\downarrow(H)$ induces an epimorphism of Heyting algebras $\iota^* : \mathcal{O}(\mathrm{Spec}_\downarrow(H)) \to \mathcal{O}(\mathrm{Spec}^0_\downarrow(H))$, given by $U \mapsto U \cap \mathrm{Spec}^0_\downarrow(H)$.

Definition We denote by $\bar{I}_0$ the map $\iota^* \circ \bar{I} :\ H \to \mathcal{O}\big(\mathrm{Spec}^0_\downarrow(H)\big)$.

Proposition 5.5 *For H dense in $\widehat{\mathcal{F}_n}$, we have that*

$$\mathcal{O}\big(\mathrm{Spec}^0_\downarrow(H)\big) \;=\; \mathcal{O}_\downarrow\big(\mathrm{Spec}^0_\downarrow(H), \subseteq\big),$$

that is, the Zariski topology on the principal ideal spectrum is the topology of decreasing sets.

Proof: By Lemma 4.14, H has the same $\bigsqcup$- and $\bigsqcap$-irreducibles as $\widehat{\mathcal{F}_n}$. As the latter satisfies the assumptions of Lemma 5.1, we may use it freely. In fact, the proposition holds more generally for a Heyting algebra satisfying the duality in Lemma 5.1.

By definition of the Zariski topology on the ideal spectrum, the inclusion "$\subseteq$" is clear. Conversely, let X be a decreasing set in $\mathrm{Spec}^0_\downarrow(\mathcal{F}_n)$ with respect to inclusion. We have to show that X is open in the Zariski topology. In fact, we are proving $X = \bigcup \big\{\bar{I}_0(a^\sqcup) \bigm| (a)_\downarrow \in X\big\}$:

From $a^\sqcup \not\sqsubseteq a$ (Lemma 5.1) it follows that $a^\sqcup \notin (a)_\downarrow$, *i.e.* $(a)_\downarrow \in \bar{I}_0(a^\sqcup)$ and thus "$\subseteq$". Conversely, let $(b)_\downarrow \in \bar{I}_0(a^\sqcup)$ for some $(a)_\downarrow \in X$, i.e. $a^\sqcup \notin (b)_\downarrow$. This means $a^\sqcup \not\sqsubseteq b$, whence (by definition of $^\sqcap$, see Lemma 5.1) $b \sqsubseteq a^{\sqcup\sqcap} = a$. This implies $(b)_\downarrow \subseteq (a)_\downarrow$, and because X is decreasing, $(b)_\downarrow \in X$. □

Remark 5.6 By Lemma 4.1, in $\mathcal{F}_n$ as well as in $\widehat{\mathcal{F}_n}$, the $\sqcap$-irreducibles are $\bigsqcap$-irreducible, and they are the same. Therefore, and with Proposition 5.5,

$$\begin{array}{ccccc}
\mathcal{O}\big(\mathrm{Spec}^0_\downarrow(\mathcal{F}_n)\big) & = & \mathcal{O}_\downarrow\big(\mathrm{Spec}^0_\downarrow(\mathcal{F}_n), \subseteq\big) & \cong_{\text{naturally}} & \mathcal{O}_\downarrow\big(\bigsqcap\text{-irreducibles of } \mathcal{F}_n, \sqsubseteq\big) \\
 & & & & \| \\
\mathcal{O}\big(\mathrm{Spec}^0_\downarrow(\widehat{\mathcal{F}_n})\big) & = & \mathcal{O}_\downarrow\big(\mathrm{Spec}^0_\downarrow(\widehat{\mathcal{F}_n}), \subseteq\big) & \cong_{\text{naturally}} & \mathcal{O}_\downarrow\big(\bigsqcap\text{-irreducibles of } \widehat{\mathcal{F}_n}, \sqsubseteq\big)
\end{array}$$

In particular, $\mathrm{Spec}^0_\downarrow(\widehat{\mathcal{F}_n})$ and $\mathrm{Spec}^0_\downarrow(\mathcal{F}_n)$ are naturally homeomorphic. (Note that $\mathrm{Spec}_\downarrow(\widehat{\mathcal{F}_n})$ and $\mathrm{Spec}_\downarrow(\mathcal{F}_n)$ are not homeomorphic.)

Theorem 5.7 *The generic Kripke model $\mathfrak{K}_n$ is, as a partial order, the principal ideal spectrum of $\mathcal{F}_n$ (equivalently of $\widehat{\mathcal{F}_n}$) with inclusion. The valuations are determined by the images $\bar{I}_0(g_j)$ of the free generators g_j of $\mathcal{F}_n$, namely $P_j \in \mathrm{val}(\mathfrak{i})$ for some principal ideal $\mathfrak{i}$ iff $g_j \notin \mathfrak{i}$, or equivalently, $\mathfrak{i} \in \bar{I}_0(g_j)$.*

Proof: By Fact 3.5 (a) and Lemma 4.1, an element of $\mathcal{F}_n$ or $\widehat{\mathcal{F}_n}$ is determined by its $\mathrm{supp}_{\bigsqcap}$, that is by the $\bigsqcap$-irreducibles of the algebra. It follows that the map $\bar{I}_0$ is injective, Together with Proposition 5.5, we get an embedding $\mathcal{F}_n \hookrightarrow \mathcal{O}_\downarrow\big(\mathrm{Spec}^0_\downarrow(\mathcal{F}_n), \subseteq\big)$, and the right side is, by the previous remark, naturally isomorphic to $\mathcal{O}_\downarrow\big(\bigsqcap\text{-irreducibles of } \mathcal{F}_n, \sqsubseteq\big)$. The $\bigsqcap$-irreducibles are of the form $\{w\}^{\uparrow\mathsf{C}}$ for $w \in K_n$ by Fact 3.5 (a), and $v \leqslant w \iff \{v\}^{\uparrow\mathsf{C}} \sqsubseteq \{w\}^{\uparrow\mathsf{C}}$. This proves the first statement.

A principal prime ideal of $\mathcal{F}_n$ is generated by some $\{w\}^{\uparrow\complement}$ for $w \in K_n$. As the unique successor of $\{w\}^{\uparrow\complement}$ is $\{w\}^{\uparrow\complement} \cup \{w\}$, an element x of $\mathcal{F}_n$ is not in the ideal generated by $\{w\}^{\uparrow\complement}$ iff $w \in x$. Thus the free generators g_j are mapped on

$$\bar{I}_0(g_j) \;=\; \{\mathfrak{i} \in \mathrm{Spec}^0_{\downarrow}(\mathcal{F}_n) \mid g_j \notin \mathfrak{i}\} \;\hat{=}\; \{w \in K_n \mid w \in g_j\},$$

where "$\hat{=}$" stands for the image under the natural isomorphism. On the other hand the image of g_j is $\{w \in K_n \mid P_j \in \mathrm{val}(w)\}$. This proves the second part of the theorem. □

Remark 5.8 Theorem 5.7 identifies an element a of $\mathcal{F}_n$ with $\mathrm{supp}_{\sqcap}(a)$, whereas Bellissima's construction is more easily understood as identifying it with $\mathrm{supp}_{\bigsqcup}(a)$, that is with the underlying embedding

$$\mathcal{F}_n \;\hookrightarrow\; \mathcal{O}\big(\overline{\mathrm{Spec}}^{\uparrow}_c(\mathcal{F}_n)\big) \;\cong\; \mathcal{O}_{\downarrow}\big(\textstyle\bigsqcup\text{-irreducibles}, \sqsubseteq\big),$$

where $\overline{\mathrm{Spec}}^{\uparrow}_c(\mathcal{F}_n)$ is the space of "completely prime principal filters". Algebraically, this space is less natural than the principal ideal spectrum — the lack of complete duality comes from the fact that not all $\sqcup$-irreducible elements are completely $\sqcup$-irreducible. However, up to the duality of Lemma 5.1, the embedding is the same as in Theorem 5.7. In the light of Remark 5.6, one sees that this duality is nothing else than the order preserving homeomorphism between $\mathrm{Spec}^0_{\downarrow}(\mathcal{F}_n)$ and $\overline{\mathrm{Spec}}^{\uparrow}_c(\mathcal{F}_n)$ mapping a principal prime ideal on its complement, which is exactly the map $(a)_{\downarrow} \mapsto (a^{\sqcup})^{\uparrow}$.

6 Some model theory of finitely generated free Heyting algebras

The basic model theoretic notions like elementary equivalence $\equiv$, elementary substructure $\preccurlyeq$, definability and interpretability, are explained in any newer model theory textbook, see for example [Ho]. "Definable" means definable with parameters, and "A-definable" with parameters in A.

The theory of Heyting algebras has a model completion (in [GhZ], as a consequence of a result by Pitts [Pi]), and there are some results about (un)decidability (see for example [Ry] and [Id]), but otherwise little seems to be known about the model theory of Heyting algebras.

6.1 First order definition of the Kripke model

Theorem 6.1 *Fix free generators $g_1, \dots, g_n$ of $\mathcal{F}_n$. Let H be dense in $\widehat{\mathcal{F}_n}$ and containing $g_1, \dots, g_n$. Then the set $\{g_1, \dots, g_n\}$ is $\emptyset$-definable in H.*

Proof: First we note that the partial order $(K_n, \leqslant)$ of the Kripke model $\mathfrak{K}_n$ is $\emptyset$-definable in H: the underlying set can be identified with the $\bigsqcup$-irreducibles of H by Lemma 4.14. It is $\emptyset$-definable as the set of those elements having a unique predecessor. They are ordered by the restriction of the partial order of H. According to Remark 4.4, this order can be identified with $(K_n, \leqslant)$.

In the sequel of the proof, we will simply write K_n for the definable set of $\bigsqcup$-irreducibles of H. We have then a $\emptyset$-definable injection $H \to \mathfrak{P}(K_n)$ that maps an element a on its $\{a\}$-definable *support* $\mathrm{supp}_{\bigsqcup}(a) = \{w \in K_n \mid w \sqsubseteq a\}$. In this proof, "successor" and "predecessor" are always meant in $(K_n, \leqslant)$.

Clearly, the set of atoms of $\mathcal{F}_n$ is $\emptyset$-definable. For example, they are exactly the elements whose support is a singleton. The unique element of the support of an atom a will be called w_a. Let a be an atom, and β the valuation of w_a. Consider the set of all elements of K_n of the form $w_{\beta',\{w_a\}}$ for $\beta' \subset \beta$. It has $2^{|\beta|} - 1$ elements and is $\{a\}$-definable because it consists of all elements $w \in K_n$ which are successors of w_a without other predecessors. Therefore for any k, the set $A_k := \big\{a \;\big|\; a \text{ atom and } |\mathrm{val}(w_a)| = k\big\}$ is $\emptyset$-definable.

Let a_i be the atom with $w_{a_i} = w_{\{P_i\},\emptyset}$. First we remark that the set of atoms

$$B_i := \big\{a \;\big|\; a \text{ atom and } P_i \in \mathrm{val}(w_a)\big\}$$

is $\{a_i\}$-definable, because this is exactly a_i together with the set of those atoms a such that the point w_a has two common successors with w_{a_i} without other predecessors, namely $w_{\emptyset,\{w_a,w_{a_i}\}}$ and $w_{\{P_i\},\{w_a,w_{a_i}\}}$. Now $A_1 = \{a_1, \ldots, a_n\}$ is a finite $\emptyset$-definable set. Therefore, to prove the proposition, it is sufficient to show that $\mathrm{supp}_{\bigsqcup}(g_i)$ is $\{a_i\}$-definable (uniformly in i) .

Claim: $\mathrm{supp}_{\bigsqcup}(g_i)$ consists of all points $v \in K_n$ satisfying the following first order conditions:

(1) v has a successor in $(K_n, \leqslant)$ that has no other predecessor than v;
(2) either $v \geqslant w_{a_i}$ or v has two common successors with w_{a_i} that have no other predecessors.

Proof: For the inclusion "$\subseteq$", note first that any element $v \in \mathrm{supp}_{\bigsqcup}(g_i)$ has the successor $w_{\emptyset,\{v\}_\downarrow}$ that has no other predecessor. Then, if $v \in \mathrm{supp}_{\bigsqcup}(g_i)$ is not above w_{a_i}, then there are the two elements $w_{\emptyset,\{v,w_{a_i}\}_\downarrow}$ and $w_{\{P_i\},\{v,w_{a_i}\}_\downarrow}$ satisfying (2).

For the converse inclusion, we first notice that no point with valuation $\emptyset$ can satisfy condition (1) since the Kripke model is reduced. If $v \geqslant w_{a_i}$, then the valuation of v is either $\emptyset$ or $\{P_i\}$; the former is excluded by (1). If $v \ngeqslant w_{a_i}$ and $P_i \notin \mathrm{val}(v)$, then there is only one common successor with w_{a_i} without other predecessors, namely $w_{\emptyset,\{v,w_i\}_\downarrow}$, contradicting (2). □

Definition We call *pre-generators* of $\mathcal{F}_n$ the atoms $a_1, \ldots, a_n$ such that $|\mathrm{val}(a_i)| = 1$, and we will fix them for the remaining of this section. (With the notation of the previous proof, the pre-generators are the elements of A_1.)

The proof of the theorem shows in particular that in a dense sub-algebra of $\widehat{\mathcal{F}_n}$ containing g_i, the corresponding pre-generator a_i is interdefinable with g_i. On the one hand, g_i is the unique element having the $\{a_1\}$-definable set $\mathrm{supp}_{\bigsqcup}(g_i)$ as its support; on the other hand, a_i is the unique element in the $\{g_i\}$-definable set $A_1 \cap \mathrm{supp}_{\bigsqcup}(g_i)$.

Corollary 6.2 *If H is dense in $\widehat{\mathcal{F}_n}$, then the Kripke model $\mathfrak{K}_n$ is interpretable in H with parameters $a_1, \ldots, a_n$.*

Proof: We have already seen in the proof of Theorem 6.1 that the partial ordering $(K_n, \leqslant)$ is $\emptyset$-definable and that the support $\mathrm{supp}_{\bigsqcup}(g_i)$ is $\{a_i\}$-definable. Now the points $x \in K_n$ with valuation $\{P_i \mid i \in I\}$ are definable as those satisfying the formula that expresses $\bigwedge_{i=1}^{n} \big(x \in \mathrm{supp}_{\bigsqcup}(g_i) \iff i \in I\big)$. □

Corollary 6.3 (Grigolia [Gr3]) *$\mathcal{F}_n$ has only one set of free generators, and hence* $\mathrm{Aut}(\mathcal{F}_n) = \mathrm{Sym}(n)$.

Proof: Any set of free generators of $\mathcal{F}_n$ has size n (because there are 2^n atoms). Assume $\mathcal{F}_n$ is freely generated by $g_1, \ldots, g_n$ and $h_1, \ldots, h_n$. Then $g_i \mapsto h_i$ extends to an automorphism of $\mathcal{F}_n$, which has to leave the $\emptyset$-definable set $\{g_1, \ldots, g_n\}$ invariant. By definition of $\mathcal{F}_n$ as the free algebra, any permutation of the free generators extends uniquely to an automorphism of $\mathcal{F}_n$. □

Corollary 6.4 *If H is dense in $\widehat{\mathcal{F}_n}$, then $\mathrm{Aut}(H) \leqslant \mathrm{Sym}(n)$. If H is in addition setwise invariant under $\mathrm{Aut}(\widehat{\mathcal{F}_n})$, as for example $\widehat{\mathcal{F}_n}$ and $\breve{\mathcal{F}}_n$, then $\mathrm{Aut}(H) = \mathrm{Sym}(n)$.*

Proof: The metric on $\mathcal{F}_n$ is invariant under $\mathrm{Aut}(\mathcal{F}_n)$, hence every automorphism is continuous and therefore extends uniquely to the completion $\widehat{\mathcal{F}_n}$. Let H be dense in $\widehat{\mathcal{F}_n}$. As any automorphism of H permutes the $\emptyset$-definable set $\{a_1, \ldots, a_n\}$, we get a map $\mathrm{Aut}(H) \to \mathrm{Sym}(\{a_1, \ldots, a_n\})$. Let α be in the kernel, i.e. fixing $a_1, \ldots, a_n$ pointwise. We have to show that α is the identity. Now α fixes the Kripke model $\mathfrak{K}_n$ interpreted in H as in 6.2. But every element of H is interdefinable with a subset of K_n, namely its support. Therefore α has to be the identity.

Conversely, any automorphism of $\widehat{\mathcal{F}_n}$ restricts to H if H is invariant, so $\mathrm{Aut}(H) = \mathrm{Sym}(n)$ in this case. $\breve{\mathcal{F}}_n$ is invariant as being generated by all the $\prod$-irreducibles. □

Let dcl_T and acl_T stand for the definable and model theoretic algebraic closure in the theory T, see e.g. [Ho].

Corollary 6.5 *If H is dense in $\widehat{\mathcal{F}_n}$, then*

$$\mathcal{F}_n \cap H \subseteq \mathrm{dcl}_{\mathrm{Th}(H)}(a_1, \ldots, a_n) \subseteq \mathrm{acl}_{\mathrm{Th}(H)}(\emptyset).$$

Proof: Corollary 6.2 allows us, over the parameters $a_1, \ldots, a_n$, to define the supports of the generators of $\mathcal{F}_n$. Now every element of $\mathcal{F}_n$ is a term in the generators. This implies that the support of every element x in $\mathcal{F}_n$ is $\{a_1, \ldots, a_n\}$-definable (cf. Remark 4.5). If x is also in H, then x is $\{a_1, \ldots, a_n\}$-definable as the unique element having its support. The second inclusion is clear as the a_i are algebraic over $\emptyset$ (for example as the elements of the finite $\emptyset$-definable sets of atoms). □

In particular, $\mathcal{F}_n \subseteq \mathrm{acl}_{\mathrm{Th}(\widehat{\mathcal{F}_n})}(\emptyset)$.

Question 6.6 Does equality hold?

6.2 Comparing theories

What can be said about the first order theories of $\widehat{\mathcal{F}_n}, \mathcal{F}_n$ and $\breve{\mathcal{F}}_n$?

As n is coded in the number of atoms, which are first order definable, we get that $H_n \not\equiv H_m$ if $n \neq m$, H_n is dense in $\widehat{\mathcal{F}_n}$ and H_m dense in $\widehat{\mathcal{F}_m}$. More precisely, this proves a difference in the $\forall\exists$-theories. Bellissima's Corollary 3.2 in [Be] gives a better result, namely $(\mathcal{F}_n)_\forall \neq (\mathcal{F}_m)_\forall$ for $n \neq m$, due to an "identity", i.e. a positive universal formula. With Proposition 6.10, it follows that $(H_n)_\forall \neq (H_m)_\forall$ for H_n, H_m as above.

Comparing $\widehat{\mathcal{F}_n}, \mathcal{F}_n$ and $\breve{\mathcal{F}}_n$ with the same n, we have to distinguish the case $n = 1$ where $\breve{\mathcal{F}}_1 = \mathcal{F}_1 = \widehat{\mathcal{F}_1}$ from the case $n > 1$ where the three algebras are pairwise not isomorphic: $\widehat{\mathcal{F}_n}$, has size continuum, whereas $\mathcal{F}_n$ and $\breve{\mathcal{F}_n}$ are countable; $\mathcal{F}_n$ is finitely generated, but $\breve{\mathcal{F}_n}$ is not (Fact 3.6).

Concerning elementary equivalence and similar concepts, Theorem 6.1 yields the following results:

Proposition 6.7 *$\mathcal{F}_n$ embeds in every model of its theory ("$\mathcal{F}_n$ is an algebraic prime model"). No proper dense sub-algebra of $\mathcal{F}_n$ is elementarily equivalent to $\mathcal{F}_n$.*

Proof: The generators form a finite $\emptyset$-definable set, thus they belong to every model. Every element of $\mathcal{F}_n$ is a term in the generators, and the theory of $\mathcal{F}_n$ knows which terms describe the same element in $\mathcal{F}_n$ and which not. Therefore $\mathcal{F}_n$ is a substructure of every model.

If H is a dense sub-algebra of $\mathcal{F}_n$, then H has the same atoms as $\mathcal{F}_n$ and interprets the partial ordering $(K_n, \leqslant)$ of the Kripke model in the same way as $\mathcal{F}_n$. Now $\mathcal{F}_n$ satisfies the formula saying that there are elements $g_1, \ldots, g_n$ such that their supports in K_n are defined from the atoms in A_1 as in the proof of Theorem 6.1. Thus any elementarily equivalent dense sub-algebra has to contain the generators. □

With Grigolia's result that no proper sub-algebra of $\mathcal{F}_n$ is isomorphic to $\mathcal{F}_n$, the first part of the proposition immediately implies

Corollary 6.8 *No proper sub-algebra of $\mathcal{F}_n$ can be elementarily equivalent to $\mathcal{F}_n$.*

We do not know whether $\mathcal{F}_n$ is also an elementary prime model of its theory (i.e. embeds elementarily into every model of its theory), and we do not know whether the free Heyting algebra is an elementary substructure of its completion.

Proposition 6.9 *If $\mathcal{F}_n \equiv \widehat{\mathcal{F}_n}$, then $\mathcal{F}_n \preccurlyeq \widehat{\mathcal{F}_n}$.*

Proof: If $\mathcal{F}_n \equiv \widehat{\mathcal{F}_n}$, then Theorem 6.1 implies

$$(\mathcal{F}_n, g_1, \ldots, g_n) \equiv (\widehat{\mathcal{F}_n}, g_{\sigma(1)}, \ldots, g_{\sigma(n)})$$

for some $\sigma \in \mathrm{Sym}(n)$. With Corollary 6.4 we then get

$$(\mathcal{F}_n, g_1, \ldots, g_n) \equiv (\widehat{\mathcal{F}_n}, g_1, \ldots, g_n).$$

Finally the result follows from Corollary 6.5 and the interdefinability of a_i and g_i. □

Proposition 6.10 *If H is dense in $\widehat{\mathcal{F}_n}$, then $H \preccurlyeq^+_\forall \widehat{\mathcal{F}_n}$, which means that both algebras satisfy the same positive universal $\mathcal{L}_{HA}$-formulae with parameters in H.*

Proof: It is clear that if $\widehat{\mathcal{F}_n}$ satisfies a universal formula, then also H. Assume $\widehat{\mathcal{F}_n} \models \exists \bar{x}\, \varphi(\bar{x}, \bar{a})$ where φ is a negative quantifier-free formula with parameters $\bar{a}$ from H. Then φ can be put in the form $\bigwedge_i \bigvee_j \tau_{ij}(\bar{x}, \bar{a}) \neq 1$ for $\mathcal{L}_{HA}$-terms τ_{ij}. Each term defines a continuous function (see Theorem 4.10), and as points are closed, $\tau_{ij}(\bar{x}, \bar{a}) \neq 1$ defines an open set. Thus $\varphi(\bar{x}, \bar{a})$ defines an open set in $\widehat{\mathcal{F}_n}^l$ where l is the length of $\bar{x}$. If this open set is non-empty as the formula above asserts, then the intersection with the dense subset H^l is also non-empty. □

In the language of universal algebra, $H \preccurlyeq_{\forall}^{+} H'$ means that H satisfies the same identities as H in the language with constants for all element of H. In particular, the proposition provides a proof of Lemma 4.6 in [Be].[9]

Corollary 6.11 *If H is dense in $\widehat{\mathcal{F}_n}$, then every $\sqcap$-irreducible element of H remains $\sqcap$-irreducible in $\widehat{\mathcal{F}_n}$.*

PROOF: An element $u \in H$ is $\sqcap$-irreducible iff the positive universal formula $\forall x\, (x \sqcup u = u \vee (x \sqcup u) \to u = u)$ holds in H. □

Thus the $\sqcap$-irreducible elements of a dense sub-algebra are exactly the co-principal sets. For $\sqcup$-irreducible elements, the situation is different: the corresponding result of the corollary holds (see Corollary 7.2), but there are more $\sqcup$-irreducibles than just the principle sets, and in general not all the $\sqcup$-irreducibles of $\widehat{\mathcal{F}_n}$ are in a dense sub-algebra.

Remark 6.12 $\widehat{\mathcal{F}_n}$, as the profinite limits of the finite Heyting algebras $\mathcal{F}_n^d$, can be embedded in a pseudo-finite Heyting algebra, namely in a nontrivial ultraproduct of the $\mathcal{F}_n^d$ via $x \mapsto (\pi_d(x))_{d\in\omega} \in \big(\prod_{d\in\omega} \mathcal{F}_n^d\big)/\mathcal{U}$. Hence $\widehat{\mathcal{F}_n}$ (and hence every dense sub-algebra) satisfies the universal theory of all finite Heyting algebras.

Similarly, if H is a dense sub-algebra of $\widehat{\mathcal{F}_n}$, we can map $\widehat{\mathcal{F}_n}$ in an ultrapower of H, via

$$\begin{aligned} \widehat{\mathcal{F}_n} \to H^{\mathcal{U}}, \quad & x \mapsto \big(\sigma_d^{\min}(\pi_d(x))\big)_{d\in\omega} \\ \text{or via} \quad & x \mapsto \big(\sigma_d^{\max}(\pi_d(x))\big)_{d\in\omega} \end{aligned}$$

with the sections $\sigma_d^{\min}, \sigma_d^{\max}$ as in Remark 4.16. If $\mathcal{U}$ is a non-trivial ultrafilter on ω, then these are $\{0, \sqcap, \sqcup, \sqsubseteq\}$-embeddings and $\{0, 1, \sqcap, \to, \sqsubseteq\}$-embeddings respectively. Thus H has the same universal theory as $\widehat{\mathcal{F}_n}$ in any of the two languages: $\{0, \sqcap, \sqcup, \sqsubseteq\}$ and $\{0, 1, \sqcap, \to, \sqsubseteq\}$.

7 Further remarks and open problems

7.1 Open problems

Problem 1 *Is it possible to characterise the subsets of K_n that are in $\mathcal{F}_n$? in $\breve{\mathcal{F}}_n$?*

By Fact 5.3, every Heyting algebra embeds into a topological Heyting algebra. Therefore, the universal theory of all topological Heyting algebras equals $(T_{HA})_\forall$. In particular, on the quantifier-free level one can compute in the theory of Heyting algebras as if one were in an arbitrary topological space.

Problem 2 *Does T_{HA} equal the theory of all topological Heyting algebras? I.e. does any $\mathcal{L}_{HA}$-sentence which holds in all lattices of open sets of topologies hold in all Heyting algebras?*

Problem 3 *Is $\mathcal{F}_n \preccurlyeq \widehat{\mathcal{F}_n}$? Does $\mathcal{F}_n$ eliminate quantifiers in a reasonable language?*

[9] The proof of Lemma 4.6 in [Be] uses Lemma 4.5, which contains a mistake: The hypothesis must be $a_i \cap H_{\alpha,n} = b_i \cap H_{\alpha,n}$. Otherwise (with w_i as in figure 1 p.156 of [Be]) for $\alpha = 1$, $a_0 = \{w_0, w_2\}$, $b_0 = \{w_0, w_1, w_2\}$ and $p(x) = (x \to 0) \to 0$ one gets a counterexample, as $a_0 \cap \mathrm{Lev}_{1,1} = b_0 \cap \mathrm{Lev}_{1,1} = \{w_2\}$, but $p(a_0) = a_0$ and $p(b_0) = 1$, thus $p(b_0) \cap \mathrm{Lev}_{1,1} = \{w_2, w_3\}$. But the proof of Lemma 4.6 works with this weaker version of Lemma 4.5.

7.2 The $\sqcup$-irreducible elements

Fact 3.5, Lemmas 4.1, 4.14 and Corollary 6.11 completely determine the $\bigsqcup$-, $\bigsqcap$- and $\sqcap$-irreducible elements of $\widehat{\mathcal{F}_n}$ and its dense sub-algebras. Now we are going to characterise the $\sqcup$-irreducible elements. For $n = 1$, the principal sets and 1 are the only $\sqcup$-irreducibles; for $n > 1$, there are more infinite $\sqcup$-irreducibles.

Proposition 7.1 *If H is dense in $\widehat{\mathcal{F}_n}$, then $X \in H$ is $\sqcup$-irreducible iff for all (incomparable) $w_0, w_1 \in X$ there exists an element $w \in X$ with $w \geqslant w_0$ and $w \geqslant w_1$, i.e. X as a subset of $(K_n, \leqslant)$ is upward filtering.*

PROOF: If X is a proper union of $X_0, X_1 \in H$, choose $w_i \in X_i \setminus X_{1-i}$. Then they are incomparable and have no common larger element w in X. Conversely, let $w_0, w_1 \in X$ and define $X_i := X \cap \{w_i\}^{\uparrow\complement}$. Then X_i is a proper subset of X, $X_i \in H$ because the co-principle sets are in H by Lemma 4.14, and $X_0 \cup X_1 = X \cap \big(\{w_0\}^{\uparrow\complement} \cup \{w_1\}^{\uparrow\complement}\big) = X \setminus \big(\{w_0\}^{\uparrow} \cap \{w_1\}^{\uparrow}\big)$. If X is $\sqcup$-irreducible, then $X_0 \cup X_1 \neq X$, and there is $w \in X \cap \{w_0\}^{\uparrow} \cap \{w_1\}^{\uparrow}$. □

Corollary 7.2 *If H is dense in $\widehat{\mathcal{F}_n}$, then a $\sqcup$-irreducible element of H remains $\sqcup$-irreducible in $\widehat{\mathcal{F}_n}$.*

Proposition 7.3 *For $n > 1$, there are continuum many $\sqcup$-irreducibles in $\widehat{\mathcal{F}_n}$.*

PROOF: There exists an infinite antichain $(z_i)_{i\in\omega}$ in K_n (Fact 3.7), and for any proper subset I of ω, the set $Z_I := \bigcap_{i\in I}\{z_i\}^{\uparrow\complement}$ is $\sqcup$-irreducible by Proposition 7.1: for $w_0, w_1 \in Z_I$ and $j \notin I$, there is a common larger element $w_{\emptyset,\{w_0,w_1,z_j\}_\downarrow}$. □

In particular, not every $\sqcup$-irreducible element of $\widehat{\mathcal{F}_n}$ is in $\mathcal{F}_n$. Also it follows from this proof that all co-principal sets are $\sqcup$-irreducible (Theorem 3.1 in [Be]), because any element of K_n is part of a two-element antichain.

It is easy to check that the element $\bigcup_{i\in\omega}\{z_i\}_\downarrow$ of $\widehat{\mathcal{F}_n}$ is not a finite union of $\sqcup$-irreducible elements. In contrast to this, Urquhart (Theorem 3 in [Ur]) has shown that every element of $\mathcal{F}_n$ is a finite union of $\sqcup$-irreducible elements.

Question 7.4 Is $\widehat{\mathcal{F}_n}$, or more generally any dense sub-algebra, generated by its $\sqcup$-irreducible elements?

Proposition 7.5 *Any intersection of some of the free generators of $\mathcal{F}_n$ is $\sqcup$-irreducible in both, $\mathcal{F}_n$ and $\widehat{\mathcal{F}_n}$.*

PROOF: Consider $[\![P_1]\!] \cap \cdots \cap [\![P_k]\!]$, i.e. all points whose valuation includes $P_1, \ldots, P_k$. For any two such points w_0, w_1, either they are comparable and the larger one is a common larger element, or they are incomparable, and then

$$w = w_{\{P_1,\ldots,P_k\},\{w_0,w_1\}_\downarrow} \in [\![P_1]\!] \cap \cdots \cap [\![P_k]\!]$$

is a common larger element. □

It follows that if $n \geqslant 2$, then any intersection of at most $n-1$ of the generators is an example of a $\sqcup$-irreducible element that is neither $\bigsqcup$- nor $\sqcap$-irreducible. The intersection of all generators is the atom $\{w_{\{P_1,\ldots,P_n\},\emptyset}\}$.

7.3 Approximations of $\breve{\mathcal{F}}_n$

Let, as in the proof of Theorem 4.4 in [Be], $B_{n,d}$ be the sub-algebra of $\breve{\mathcal{F}}_n$ generated by all principal sets $\{w\}_\downarrow$ with w of foundation rank $\leqslant d$; and let $C_{n,d}$ be the sub-algebra of $\breve{\mathcal{F}}_n$ generated by all co-principal sets $\{w\}^{\uparrow\complement}$ with w of foundation rank $\leqslant d$. Recall from the proof of Lemma 4.14 that

$$\{w\}^{\uparrow\complement} = \{w\}_\downarrow \to \Big(\bigsqcup_{v<w}\{v\}_\downarrow\Big) \quad \text{and} \quad \{w\}_\downarrow = \bigsqcap\Big\{\{v\}^{\uparrow\complement} \mid v \text{ minimal} \notin \{w\}_\downarrow\Big\},$$

hence we get

$$C_{n,d} \;\subseteq\; B_{n,d} \;\subseteq\; C_{n,d+1} \;\subseteq\; \cdots \bigcup_{d\in\omega} B_{n,d} \;=\; \bigcup_{d\in\omega} C_{n,d} \;=\; \breve{\mathcal{F}}_n.$$

If $n > 1$, the inclusions are all strict: Bellissima has shown that $B_{n,d}$ can't separate points $w_{\beta,Y}, w_{\beta',Y} \in K_n^{d+1} \setminus K_n^d$ with $\beta \neq \beta'$, but the set $\{w_{\beta,Y}\}^{\uparrow\complement}$ in $C_{n,d+1}$ does. The proof for the second sort of inclusion is similar, but even easier: For $w = w_{\{P_i\},Y} \in K_n$ of foundation rank d, the set $\{w\}_\downarrow \in B_{n,d}$ separates w from $w' := w_{\emptyset,\{w\}_\downarrow}$. On the other hand, $C_{n,d}$ can't separate between w and w': This is clear for the generators and clearly preserved under $\sqcup$ and $\sqcap$, and it is not hard to see that it is also preserved under $\to$.

7.4 Cantor–Bendixson analysis

$\mathcal{F}_1$ only consists of finite elements and 1. Thus Lemma 4.12 implies that the metric topology is the one-point compactification of a countable discrete set; all points are isolated, i.e. have Cantor–Bendixson rank 0, except the maximum with rank 1.

Proposition 7.6 *For $n > 1$, $\widehat{\mathcal{F}_n}$ has infinitely many points of rank 1. They are the maximal elements of sub-lattices that look similar to $\mathcal{F}_1$. The elements of higher rank form a perfect subset.*

Proof (sketchy): Let $U_a^d := \{x \in \widehat{\mathcal{F}_n} \mid x \cap K_n^d = a\}$ be a basic open set. By an *extension* of a, we mean an $x \in U_a^d$, and by a *k-extension*, we mean an extension x by adding k new points of the Kripke model. One can check that there are only the following three possibilities:

(A) For some k, there is no k-extension of a. Then U_a^d consists of finitely many finite sets.

(B) For each k, there are exactly two k-extensions of a. Then the extensions of a form a copy of K_1, i.e. the Kripke model for one free generator. Therefore U_a^d contains infinitely many finite sets and exactly one infinite set, which thus is an element of Cantor–Bendixson rank 1.

(C) There at least three 1-extensions of a. Then the number of k-extensions of a increases with k. In this case, there are infinitely many elements in U_a^d of rank > 1 (for each big enough $l > 1$, take a point $w \in K_n^l \setminus K_n^{l-1}$ appearing in some extension of a and then consider the maximal extension of a omitting this point). $\square$

7.5 Order Topologies

One might wonder how the metric topologies on $\mathcal{F}_n$ and $\widehat{\mathcal{F}_n}$ relate to topologies induced by the partial order $\sqsubseteq$. There are at least three topologies that one might consider on a partially ordered set $(X, \leqslant)$:

- the topology $\mathcal{O}_\downarrow(X, \leqslant)$ of decreasing sets;
- the topology $\mathcal{O}^\uparrow(X, \leqslant)$ of increasing sets;
- the "order topology" $\mathcal{O}(X, \leqslant)$ generated by the generalised open intervals $(a, b) := \{x \in X \mid a < x < b\}$ as a sub-basis, where $a = -\infty$ and $b = \infty$ are allowed.

Proposition 7.7 **(a)** *The trace on $\mathcal{F}_n$ of the increasing and decreasing topology on $\widehat{\mathcal{F}_n}$ is the corresponding topology on $\mathcal{F}_n$.*

(b) *For both $\mathcal{F}_n$ and $\widehat{\mathcal{F}_n}$, the order topology contains $\mathcal{O}_\downarrow$. Neither the order topology nor the metric topology contains $\mathcal{O}^\uparrow$.*

(c) *For $n = 1$, the metric topology on $\mathcal{F}_1$ equals the order topology. For $n > 1$ and $\mathcal{F}_n$ as well as $\widehat{\mathcal{F}_n}$, the metric topology is incomparable with the order topology and does not contain $\mathcal{O}_\downarrow$.*

PROOF: (a) The intersection of an in-/de-creasing set of $\widehat{\mathcal{F}_n}$ with $\mathcal{F}_n$ is an in-/de-creasing set of $\widehat{\mathcal{F}_n}$.

(b) First statement: every proper decreasing set a has successors $a \cup \{w\}$ where w is an element of the Kripke model of minimal foundation rank among those not in a. Either there is a unique such successor, then $a = \big(-\infty, a \cup \{w\}\big)$, or there are at least two such points w_1, w_2 and then $a = \big(-\infty, a \cup \{w_1\}\big) \cap \big(-\infty, a \cup \{w_2\}\big)$.

Second statement: 1 is an isolated point in $\mathcal{O}^\uparrow$, but neither in the order topology, as it does not have predecessors, nor in the metric topology.

(c) First statement: First we show that the finite elements are isolated in the order topology. Let a be finite. If $a = 0$, then $\{a\} = (-\infty, c)$ for some successor c of 0. If $a \neq 0$ has two distinct predecessors b_1, b_2, then choose a successor c of a and then $\{a\} = (b_1, c) \cap (b_2, c)$. Otherwise $a \neq 0$ has a unique predecessor b, that is a is principal. But then a is not co-principal, hence has two distinct successors c_1, c_2 and $\{a\} = (b, c_1) \cap (b, c_2)$. As remarked in the proof of (a), 1 is not isolated in the order topology, therefore the order topology on $\mathcal{F}_1$ is the same as the metric topology: discrete on the finite elements and 1 is a compactifying point.

Second statement: Consider a metric neighbourhood $\pi_i^{-1}(x)$ containing a co-principal set $\{w\}^{\uparrow\complement}$ with w of foundation rank less than i. A neighbourhood of $\{w\}^{\uparrow\complement}$ in the order topology contains one of the form $\big(a_1, (\{w\}^{\uparrow\complement})^+\big) \cap \cdots \cap \big(a_k, (\{w\}^{\uparrow\complement})^+\big)$, and because $\{w\}^{\uparrow\complement}$ does not have predecessors, such a neighbourhood always contains elements b with $\pi_i(b) = x \cup \{w\}$.

For the converse and the third statement, consider the decreasing set generated by $[\![P_1]\!]$. It is also open in the order topology as it equals $\big(-\infty, [\![P_1]\!] \cup \{w_{\{P_2\},\emptyset}\}\big) \cap \big(-\infty, [\![P_1]\!] \cup \{w_{\emptyset,\emptyset}\}\big)$. But it is not open in the metric topology, because its element $[\![P_1]\!]$ does not contain a metric open neighbourhood: for every finite part $a_i := \pi_i([\![P_1]\!])$ the set $a_i \cup \{w_{\emptyset,a_i}\}$ is not in $[\![P_1]\!]$, but in $\pi_i^{-1}(a_i)$. (Note that the existence of $w_{\emptyset,a_i}$ needs $n > 1$). $\square$

Question 7.8 Is the order topology on $\mathcal{F}_n$ the trace of the order topology on $\widehat{\mathcal{F}_n}$?

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