Nonsingular Hasse principle for rings

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There have been numerous model theoretic studies in the last twenty years on fields with families of topologies. Most of them involved various local-global principles (LGP) analogous to the geometric Hasse principle.

In 1986, Rumely [7] proved that Hilbert's tenth problem had a positive answer for the ring $\tilde{\mathbf{Z}}$ of all algebraic integers. His approach involved a new kind of LGP, adapted to the context of rings. This paved the way to new results on the model theory of rings, and indeed several important theorems concerning decidability, quantifier elimination or model-completeness for various integral domains were obtained independently by van den Dries, Macintyre, Prestel and Schmid ([8], [9] and [5]) by using apparently different LGP's, related to Rumely's one.

In this paper we introduce another LGP for rings we call LGPH. It turns out to contain all the preceding ones as particular cases, and to remain first order (in the ordinary language of rings). On the other hand, the proof of the axiomatisability of the LGPH requires new material, wich was not needed by the above LGP's.

A motivation to introduce the LGPH is the remark that a ring A satisfying any of the other above mentioned LGP's has the property that the henselisation of any valuation ring containing A must have a separably closed fraction field, a condition wich is very restrictive (but it is precisely what makes the axiomatisation of those LGP's much easier than for the LGPH). On the other hand, Moret-Bailly extended widely the theorem of Rumely in [3], giving a lot of exemples of rings satisfying the LGPH but not this additionnal condition. The integral closure of $\mathbf{Z}$ in $\mathbf{Q}_p$ wich we note $\tilde{\mathbf{Z}} \cap \mathbf{Q}_p$ is an example of such a ring.

Prestel and Schmid proposed in [6] an axiomatisation of $\tilde{\mathbf{Z}} \cap \mathbf{R}$ and $\tilde{\mathbf{Z}} \cap \mathbf{Q}_p$ based on a variant of their LGP in [5]. Unfortunately, for the reason we have just mentionned, $\tilde{\mathbf{Z}} \cap \mathbf{Q}_p$ cannot satisfy this LGP as the p-adic valuation ring is henselian and has a fraction field wich is not separably closed. This is actually the only obstruction, and it appears that the methods used in [6] in order to axiomatize $\tilde{\mathbf{Z}} \cap \mathbf{Q}_p$ work quite well if we replace their LGP by the LGPH. We present this approach in the last section of this paper, where we obtain exactly the results announced in [6, theorems 3.2 to 3.4] and [6, corollary 3.5]. In particular we prove that the ring $\tilde{\mathbf{Z}} \cap \mathbf{Q}_p$ is decidable.

Let us introduce now a few facts and conventions we are going to use throughout this paper.

All our rings are commutative with 1. For any element a of a ring A we note $M_A(a)$ (resp. $M_A(a)^c$) the set of maximal ideals of A containing a (resp. not containing a), the subscript A is omitted whenever the context makes it superfluous. If A is a local ring then $m(A)$ denotes its unique maximal ideal. If A is integral then $\mathrm{Frac}(A)$ denotes its quotient field, and W_A is the set of valuation rings R such that $A \subseteq R \subseteq \mathrm{Frac}(A)$ and $m(R) \cap A$ is a maximal ideal of A.

Given a field K we denote by K^s (resp. K^a) any separable (resp. algebraic) closure of K. For any valuation ring R with fraction field K we note R^h the henselisation of R, and R^s (resp. R^a) the (unique) extension of R^h to K^s (resp. K^a).

For any x, y in K we note:

$$|x|_R \leq |y|_R \iff \exists u \in R,\ x = uy.$$

For any $x = (x_1, \ldots, x_m) \in K^m$ and any $y = (y_1, \ldots, y_n) \in K^n$ we note:

$$|x|_R \leq |y|_R \iff \max_{i \leq m} |x_i|_R \leq \max_{j \leq n} |y_j|_R.$$

One can easily deduce from the Taylor formula for polynomials that for any $x, y \in R^n$ and any $p \in R[X_1, \ldots, X_n]$:

$$|p(x) - p(y)|_R \leq |x - y|_R.$$

In order to make the typing easier, if $R = A_{\mathbf{p}}$, $A^h_{\mathbf{p}}$, $A^s_{\mathbf{p}}$ or $A^a_{\mathbf{p}}$ for some subring A and some prime ideal $\mathbf{p}$ of A, we write $|x|_{\mathbf{p}} \leq |\varepsilon|_{\mathbf{p}}$ in place of $|x|_R \leq |\varepsilon|_R$.

Our varieties are affine, geometrically reduced and geometrically irreducible. When we give a system of equations $f \in K[X]^m$ (with $X = (X_1, \ldots, X_n)$) for a variety V defined over a field K, we always mean that f generates a prime ideal of $K^a[X]$, but the proof will always be left to the reader, who may also look at [1]. A point of V in a K-algebra A is always an affine point, that is a zero of f in A^n, and $V(A)$ (resp. $V^\circ(A)$) denotes the set of points (resp. smooth points) of V with coordinates in A. A generic point of V over K is a point $x \in V(A)$ such that $K[x]$ is isomorphic to the coordinate ring $K[V]$ of V.

We will make an extensive use of the following notion, introduced by Prestel and Schmid [5]. A **r-domain** is an entire ring A together with a **radical relation** on A, that is a binary relation $\leq$ such that for every a, b, c in A:

1. $a \leq a$;
2. $a \leq b,\ b \leq c \Rightarrow a \leq c$;
3. $a \leq b \Rightarrow ac \leq bc$;
4. $1 \leq a$;
5. $a \leq b^2 \Rightarrow a \leq b$.

It is easy to check that if $\mathcal{P}$ is a nonempty set of prime ideals of A, the relation defined by:

$$a \leq_{\mathcal{P}} b \iff (\forall \mathbf{p} \in \mathcal{P},\ a \in \mathbf{p} \Rightarrow b \in \mathbf{p})$$

is a radical relation. Prestel and Schmid ([5], theorem 2.5) proved that every radical relation is of that kind. More precisely, they denoted by $\mathrm{Spec}(A, \leq)$ the set of prime ideals $\mathbf{p}$ of a such that:

$$a \leq b,\ a \in \mathbf{p} \Rightarrow b \in \mathbf{p}.$$

They proved that $\mathrm{Spec}(A, \leq)$ is nonempty, and:

$$a \leq b \iff (\forall \mathbf{p} \in \mathrm{Spec}(A, \leq),\ a \in \mathbf{p} \Rightarrow b \in \mathbf{p}).$$

Remark 0.1 Prestel and Schmid required their r-domains to satisfy the additional axiom $0 \not\leq 1$. It is easy to check that this condition is not necessary to have this result, the case $0 \leq 1$ corresponding to the radical relation defined by the empty set, so we will allow our radical relations to satisfy $0 \leq 1$ if necessary.

The main radical relation we will consider in a ring A is the relation $\leq_{\mathrm{Specmax}(A)}$ defined by the maximal spectrum of A. This radical relation will simply be noted $\leq_A$:

$$a \leq_A b \iff M_A(a) \subseteq M_A(b).$$

It is important to notice that this relation is definable in the language of rings:

$$a \leq_A b \iff \forall c \in A, ((b,c) = 1 \Rightarrow (a,c) = 1)$$

where $(x,y) = 1$ means that the ideal generated by x, y in A is A himself, in other words $A \models \exists u, \exists v (ux + vy = 1)$.

The residual upper semi-lattice associated to any radical relation on A is the set of all $P(a) := \{\mathbf{p} \in \mathrm{Spec}(A, \leq) \mid a \in \mathbf{p}\}$ with a ranging over A. Ordered by inclusion, $L(A, \leq)$ has a largest element $P(0)$, a least element $P(1)$ and is stable by finite union: $P(ab) = P(a) \cup P(b)$. If in addition A is Bezout (that is, any finitely generated ideal is principal) then $L(A, \leq)$ is also stable by finite intersection, hence a distributive lattice (we refer to [5] for this observation: if A is Bezout then for any a, b in A there is d in A such that $aA + bA = dA$, and for any such d, $P(d) = P(a) \cap P(b)$). The quotient ordered set canonically associated to the preorder $\leq$ is isomorphic to $L(A, \leq)$, hence $L(A, \leq)$ is interpretable in $(A, \leq)$ and $L(A, \leq_A)$ is interpretable in A. We note $L(A)$ for $L(A, \leq_A)$.

We note $\mathcal{L}_{\mathrm{rad}}$ the language of radical rings, that is the language of rings with an additional binary relation symbol for the radical relation.

1 The local-global principle

We say that a ring A satisfies the **local-global principle with respect to henselisations (LGPH)** if and only if it is integral domain and the following two conditions hold:

1. $\forall a \in A \setminus \{0\}$, $M(a)^c$ is compact.
2. Every V defined over the fraction field of A satisfies:

$$V^\circ(A) \neq \emptyset \quad \iff \quad \forall R \in W_A,\ V^\circ(R^h) \neq \emptyset.$$

This last condition is called the **geometric condition** of the LGPH. Notice that any field satisfies the LGPH.

Proposition 1.1 *A domain wich satisfies LGPH is Bezout.*

Proof: Let A be domain satisfying the LGPH, with quotient field K. For any $a, b \in A \setminus \{0\}$ the variety V defined over K by $(X(aU + bV) = a, Y(aU + bV) = b)$ is smooth, and has a point in every $R \in W_A$ (because R is Bezout). Hence the LGPH gives $(u, v, x, y) \in V(A)$ and we have then $d := au + bv$ wich divides both a and b, hence $aA + bA = dA$.

■

It follows that in the geometric condition of the LGPH we can replace W_A by the set of all the $A_{\mathbf{m}}$'s, with $\mathbf{m}$ ranging over the maximal spectrum of A.

Van den Dries and Macintyre [9] say that a ring A has the **block factorisation property** if for any nonzero a and any b in A, there exists a', a'' in A such that $a = a'a''$, $M(a') = M(a) \cap M(b)$ and $M(a'') = M(a) \setminus M(b)$. We are going to prove that the LGPH implies the block factorisation property.

Lemma 1.2 *Let A be a domain satisfying the LGPH, $a \in A \setminus \{0\}$ and $\mathbf{p}_1, \mathbf{p}_2$ two prime ideals of A. Suppose that none of $\mathbf{p}_1, \mathbf{p}_2$ contains the other. Then there exists $a_1, a_2 \in A$ such that $a = a_1 a_2$, $|a_1|_{\mathbf{p}_1} = |a_2|_{\mathbf{p}_2} = 1$.*

Proof: Fix $c_1 \in \mathbf{p}_1 \setminus \mathbf{p}_2$, $c_2 \in \mathbf{p}_2 \setminus \mathbf{p}_1$ and d a generator of $c_1 A + c_2 A$. Replacing c_1, c_2 by $c_1/d, c_2/d$ if necessary one can suppose c_1, c_2 relatively prime. We consider the variety V defined over $\mathrm{Frac}(A)$ by the equations $(A_1 A_2 = a, c_1 Y_1 + A_1 B_1 = 1, c_2 Y_2 + A_2 B_2 = 1)$. It is smooth, and has a point $(a_{1,R}, a_{2,R}, b_{1,R}, b_{2,R}, y_{1,R}, y_{2,R})$ in every $R \in W_A$: if $c_1 \notin m(R)$, $(a, 1, 0, 1, c_1^{-1}, 0) \in V(R)$, if not then as c_1 and c_2 are relatively prime $c_2 \notin m(R)$, hence $(1, a, 1, 0, 0, c_2^{-1}) \in V(R)$. The LGPH then gives a point $(a_1, a_2, b_1, b_2, Y_1, Y_2) \in V(A)$ and we have $a = a_1 a_2$, $a_1 \notin \mathbf{p}_1$ because $c_1 \in \mathbf{p}_1$ and $c_1 y_1 + a_1 b_1 = 1$, and $a_2 \notin \mathbf{p}_2$ by the same argument hence $|a_1|_{\mathbf{p}_1} = |a_2|_{\mathbf{p}_2} = 1$.
■

Proposition 1.3 *A domain satisfying the LGPH has the block factorisation property.*

Remark 1.4 It has to be noticed that until now we have used only the geometric condition of the LGPH. On the other hand, it is straightforward that an integral ring with the block factorisation property satisfies the non geometric condition of the LGPH. Hence the LGPH is equivalent to the block factorisation property plus the geometric condition.

Proof: Let A be a domain satisfying the LGPH, and $a, \alpha \in A \setminus \{0\}$. We are looking for a factorisation $a = bc$ with $M(b) = M(a) \cap M(\alpha)$ and $M(c) = M(a) \setminus M(\alpha)$. Let $\mathcal{M} := M(a) \setminus M(\alpha)$ and $\mathcal{N} := M(a) \cap M(\alpha)$. As a closed subset of $M(\alpha)^c$ (resp. of $\mathrm{Specmax}(A)$), $\mathcal{M}$ (resp. $\mathcal{N}$) is compact. (Note this is the first (and last) time we use the non geometric condition of the LGPH, in order to guarantee that $M(\alpha)^c$ is compact.) For any x in A, we note $\mathcal{M}(x) = \mathcal{M} \cap M(x)$ and similarly for $\mathcal{N}(x)$.

Let $\mathbf{n}$ be a fixed element of $\mathcal{N}$. For every $\mathbf{m} \in \mathcal{M}$, 1.2 gives a non zero $b_{\mathbf{n},\mathbf{m}}$ in A such that $b_{\mathbf{n},\mathbf{m}}$ divides a, $|b_{\mathbf{n},\mathbf{m}}|_{\mathbf{m}} = 1$ et $|b_{\mathbf{n},\mathbf{m}}|_{\mathbf{n}} = |a|_{\mathbf{n}}$. The intersection of all $\mathcal{M}(b_{\mathbf{n},\mathbf{m}})$ for $\mathbf{m}$ ranging over $\mathcal{M}$ is empty, hence there exist $\mathbf{m}_1, \ldots, \mathbf{m}_m$ in $\mathcal{M}$ such that

$$\mathcal{M}(b_{\mathbf{n},\mathbf{m}_1}) \cap \cdots \cap \mathcal{M}(b_{\mathbf{n},\mathbf{m}_m}) = \emptyset.$$

As A is Bezout, let $b_{\mathbf{n}} = \sum_{i=1}^{m} \lambda_i b_{\mathbf{n},\mathbf{m}_i}$ be a generator of the ideal generated by the $b_{\mathbf{n},\mathbf{m}_i}$'s for $i \leq m$. We have $a/b_{\mathbf{n}} = (a/b_{\mathbf{n},\mathbf{m}_1})(b_{\mathbf{n},\mathbf{m}_1}/b_{\mathbf{n}}) \in A$, hence $|b_{\mathbf{n}}|_{\mathbf{n}} \geq |a|_{\mathbf{n}}$. On the other hand:

$$|b_{\mathbf{n}}|_{\mathbf{n}} \leq \max_{1 \leq i \leq m} |\lambda_i b_{\mathbf{n},\mathbf{m}_i}|_{\mathbf{n}} \leq \max_{1 \leq i \leq m} |b_{\mathbf{n},\mathbf{m}_i}|_{\mathbf{n}} = |a|_{\mathbf{n}}.$$

It follows that $|b_{\mathbf{n}}|_{\mathbf{n}} = |a|_{\mathbf{n}}$. In addition for every $\mathbf{m}$ in $\mathcal{M}$ there is some $i \leq m$ such that $b_{\mathbf{n},\mathbf{m}_i} \notin \mathbf{m}$, and as $b_{\mathbf{n}}$ divides $b_{\mathbf{n},\mathbf{m}_i}$ it follows that $b_{\mathbf{n}} \notin \mathbf{m}$, that is $|b_{\mathbf{n}}|_{\mathbf{m}} = 1$.

For each $\mathbf{n}$ in $\mathcal{N}$ we set $c_{\mathbf{n}} := a/b_{\mathbf{n}}$. As $|c_{\mathbf{n}}|_{\mathbf{n}} = |a/b_{\mathbf{n}}|_{\mathbf{n}} = 1$, that is $c_{\mathbf{n}} \notin \mathbf{n}$, the $\mathcal{N}(c_{\mathbf{n}})$'s have an empty intersection, hence there are elements $\mathbf{n}_1, \dots, \mathbf{n}_n$ in $\mathcal{N}$ such that:

$$\mathcal{N}(c_{\mathbf{n}_1}) \cap \cdots \cap \mathcal{N}(c_{\mathbf{n}_n}) = \emptyset.$$

Let $c := \sum_{i=1}^{n} \mu_i c_{\mathbf{n}_i}$ be a generator of the ideal generated by the $c_{\mathbf{n}_i}$'s. Let $b := a/c$; as c divides $c_{\mathbf{n}_1}$ wich divides a, b belongs to A.

For each $\mathbf{n}$ in $\mathcal{N}$ there is an index $i \leq n$ such that $c_{\mathbf{n}_i} \notin \mathbf{n}$; and as c divides $c_{\mathbf{n}_i}$ it follows that $c \notin \mathbf{n}$, hence $M(c) \cap M(a) \cap M(\alpha)$ is empty. For every $\mathbf{m}$ in $\mathcal{M}$, $|c|_{\mathbf{m}} \geq |a|_{\mathbf{m}}$ because c divides a, but on the other hand:

$$|c|_{\mathbf{m}} \leq \max_{1 \leq i \leq n} |\mu_i c_{\mathbf{n}_i}|_{\mathbf{m}} \leq \max_{1 \leq i \leq n} |c_{\mathbf{n}_i}|_{\mathbf{m}} = \max_{1 \leq i \leq mn} |a/b_{\mathbf{n}_i}|_{\mathbf{m}} = |a|_{\mathbf{m}}$$

hence $|c|_{\mathbf{m}} = |a|_{\mathbf{m}}$. It follows that $|b|_{\mathbf{m}} = 1$, that is $b \notin \mathbf{m}$, hence $M(b) \cap M(a) \setminus M(b)$ is empty. Since $M(b) \cup M(c) = M(bc) = M(a)$, we conclude that $M(b) = M(a) \cap M(\alpha)$ and $M(c) = M(a) \setminus M(\alpha)$.

■

Recall that a **constructible subset** of the maximal spectrum of a ring A is any (finite) Boolean combination of susbets $M(a)$ with $a \in A$.

Lemma 1.5 *Let A be a ring having the block factorisation property. Then any constuctible subset of* Specmax(A) *is of the form $M(a)$ or $M(a)^c$ for some nonzero a in A.*

Proof: For any a, b in A we have $M(a) \cup M(b) = M(ab)$. If in addition A has the block factorisation property, then for any non zero a there are elements c, d such that $M(a) \setminus M(b) = M(c)$ and $M(a) \cap M(b) = M(d)$, hence an immediate induction shows that any constructible subset of Specmax(A) of the form $M(a)$ or $M(a)^c$ for some a in A, and we can suppose a non zero since $M(0) = M(1)^c$.

■

Let us say that a Prüfer domain A has the **block approximation proprety (BAP)** if for any disjoint constructible subsets $\mathcal{M}_1, \dots, \mathcal{M}_n$ of Specmax(A), for any $a_1, \dots, a_n$ in A and any nonzero $\varepsilon_1, \dots, \varepsilon_n$ in A there is $a \in A$ such that $|a - a_i|_{\mathbf{m}} \leq |\varepsilon_i|_{\mathbf{m}}$ for every $\mathbf{m} \in \mathcal{M}_i$ and every $i \leq n$.

Proposition 1.6 *A ring satisfying the LGPH has the block approximation property.*

Proof: Let A be a ring satisfying the LGPH, $\mathcal{M}_1, \dots, \mathcal{M}_n$ be a family of disjoint constructible subsets of Specmax(A), $a_1, \dots, a_n$ be any elements of A, and $\varepsilon_1, \dots, \varepsilon_n$ be nonzero elements of A. Using the block factorisation property and 1.5, one can find nonzero $\alpha_1, \dots, \alpha_n$ in A such that $M(\alpha_i) = \mathcal{M}_i \cap M(\varepsilon_i)$ for $i \leq n$.

Let V be the smooth variety defined over Frac(A) by the following equations:

$$\begin{cases} (X - a_i)V_i = \varepsilon_i U_i & i \leq n \\ V_i + \alpha_i T_i = 1 & i \leq n \end{cases}$$

For any $j \leq l$ and each maximal ideal $\mathbf{m} \in M(\alpha_j)$ take: $x_{\mathbf{m}} = a_j$, $t_{j,m} = u_{j,\mathbf{m}} = 0$, $v_{j,\mathbf{m}} = 1$ and for every $i \neq j$, $t_{i,\mathbf{m}} = \alpha_i^{-1}$ and $u_{i,\mathbf{m}} = v_{i,\mathbf{m}} = 0$. As $\alpha_i \notin \mathbf{m}$ for every $i \neq j$, $(a_{\mathbf{m}}, t_{\mathbf{m}}, u_{\mathbf{m}}, v_{\mathbf{m}})$ belongs to $V(A_{\mathbf{m}})$. This point also belongs to $V(A_{\mathbf{n}})$ for any

$\mathbf{n}$ out of $M(\alpha_1) \cup \cdots \cup M(\alpha_n)$. The LGPH then gives $(a, t, u, v) \in V(A)$ and for every $i \leq n$ and every $\mathbf{m} \in \mathcal{M}_i$, we have:
$|a - a_i|_{\mathbf{m}} \leq 1 = |\varepsilon_i|_{\mathbf{m}} = 1$ if $\mathbf{m} \notin M(\alpha_i)$ since $M(\varepsilon) \cap \mathcal{M}_i = M(\alpha_i)$,
$|a - a_i|_{\mathbf{m}} = |\varepsilon_i u_i / v_i|_{\mathbf{m}} \leq |\varepsilon_i|_{\mathbf{m}}$ if $\mathbf{m} \in M(\alpha_i)$ since $v_i + \alpha_i t_i = 1 \Rightarrow M(v_i) \cap M(\alpha_i) = \emptyset$.
■

Corollary 1.7 *If a ring A satisfies the LGPH, then any two $R_1 \neq R_2 \in W_A$ are independent (that is $R_1 R_2 = \mathrm{Frac}(A)$).*

Proof: Take α_1 in $m(R_1) \cap A \setminus m(R_2)$, and similarly for α_2.The block approximation property with $\mathcal{M}_1 = M(\alpha_1) \setminus M(\alpha_2)$ and $\mathcal{M}_2 = M(\alpha_2) \setminus M(\alpha_1)$ proves that R_1 and R_2 satisfy the usual strong approximation property, hence are independant.
■

2 Axiomatisation

Integral domains having the block approximation property share a number of nice properties, wich have been extensively studied in [1]. In this paper we focus on those wich allow us to prove that the LGPH is a first order property (in the language of rings).

We already noticed that the relation $a \leq b \iff M(a) \subseteq M(b)$ is (uniformally) definable, hence we can use here the formula $a \leq b$ as an abbreviation for the corresponding formula in the language of rings.

In the same way, we will write $(a, b) = d$ for the formula:

$$\exists u, v, a', b' \ (a = a'd \wedge b = b'd \wedge au + bv = d)$$

wich says that the ideal generated by a and b is principal, with generator d.

Any element of the fraction field of a Bezout domain A can be written u/v with $(u, v) = 1$. It follows that given any a, b and any non zero α in A, a divides b in every $A_{\mathbf{m}}$ such that $\mathbf{m} \in M(\alpha)$ if and only if there are u, v in A such that $(v, \alpha) = 1$ and $av = bu$, that is if and only if A models the following formula:

$$\mathit{DivLoc}(a, b, \alpha) \equiv \exists u, v \ [(v, \alpha) = 1 \wedge av = bu].$$

Proposition 2.1 *The class of Bezout domains having the block factorisation and the block approximation properties is elementary.*

Proof: A ring is Bezout if and only if it models $\forall a, b \exists d \ (a, b) = d$.

It has the block factorisation property if and only if it models:

$$\forall a, b, \ [a \neq 0 \rightarrow \exists a_1, a_2 \ (a = a_1 a_2 \wedge (a_1, b) = 1 \wedge a_2 \leq b)].$$

An immediate induction shows that a Bezout domain has the block approximation property if and only if it satisfies this property with only two disjoint constructible sets. We can even suppose each $\mathcal{M}_i \subseteq M(\varepsilon_i)$. If in addition the block factorisation property holds, by 1.5 those two constructible sets can be taken as $M(\alpha_1)$ and $M(\alpha_2)$ with $\alpha_1 A + \alpha_2 A = A$. Hence a Bezout domain having the block factorisation property has the block approximation property if and only if it models:

$$\forall \alpha_1, \alpha_2, a_1, a_2, \varepsilon_1, \varepsilon_2 \ (\varepsilon_1 \varepsilon_2 \neq 0 \wedge (\alpha_1, \alpha_2) = 1) \rightarrow \\ \exists a, \mathit{DivLoc}(\varepsilon_1, a - a_1, \alpha_1) \wedge \mathit{DivLoc}(\varepsilon_2, a - a_2, \alpha_2).$$

■

Lemma 2.2 *Let α be a nonzero element of a Bezout domain A having the block factorisation property. Let $\mathcal{M}_1, \ldots, \mathcal{M}_q$ be constructible subsets of* $\mathrm{Specmax}(A)$ *such that $M(\alpha) \subseteq \bigcup_{k=1}^q \mathcal{M}_k$. Let $\mathbf{m}_1, \ldots, \mathbf{m}_q$ be distinct maximal ideals of A such that $\mathbf{m}_k \in \mathcal{M}_k$ for every $k \leq q$.*
Then there exist non zero $\alpha_1, \ldots, \alpha_q$ such that for every $k \leq q$:

$$\mathbf{m}_k \in M(\alpha_k) \subseteq \mathcal{M}_k$$

and $(M(\alpha_k))_{k\leq q}$ is a partition of $M(\alpha)$.

Proof: For each $k \leq q$, by 1.5 there is β_k in A such that $M(\beta_k) = M(\alpha) \cap \mathcal{M}_k$.

We prove the result by induction on q. If $q = 1$, take $\alpha_1 = \alpha$ and the assertion is proved. Suppose $q > 1$ and it has been proved for $q - 1$.

The block factorisation property gives a nonzero α' such that $M(\alpha') = M(\alpha) \cap M(\gamma)$ where γ is the product of $\beta_1, \ldots, \beta_{q-1}$. Applying the induction hypothesis to α' and $\beta_1, \ldots, \beta_{q-1}$ gives $\alpha'_1, \ldots, \alpha'_{q-1}$ wich define a partition of $M(\alpha')$. The block factorisation property gives α'_q such that $M(\alpha'_q) = M(\alpha) \setminus M(\alpha')$. Then $M(\alpha'_q) \subseteq M(\beta_q)$, $(M(\alpha'_k))_{k\leq q}$ is a partition of $M(\alpha)$, and for every $k < q$ we have $\mathbf{m}_k \in M(\alpha'_k) \subseteq M(\beta_k)$.

If $\mathbf{m}_q \in M(\alpha'_q)$ it suffices to set $\alpha_k = \alpha'_k$ for every $k \leq q$ to finish the proof. If not, then $\mathbf{m}_q \in M(\alpha')$ hence there is $l < q$ such that $\mathbf{m}_q \in M(\alpha'_l)$. Take $\gamma \in \mathbf{m}_l \setminus \mathbf{m}_q$, by 1.5 there is γ_q in A such that $M(\gamma_q) = M(\beta_q) \cap M(\alpha'_l) \setminus M(\gamma)$. The reader may easily check that $\mathbf{m}_q$ belongs to $M(\alpha'_q\gamma_q)$ which is contained in $M(\beta_q)$, that $\mathbf{m}_l$ belongs to $M(\alpha'_l) \setminus M(\gamma_q)$ which is contained in $M(\beta_l)$, and that those two constructible sets form a partition of $M(\alpha'_l\alpha'_q)$. Hence setting $\alpha_q = \alpha'_q\gamma_q$, $\alpha_l =$ some element of A such that $M(\alpha_l) = M(\alpha'_l) \setminus M(\gamma_q)$, and $\alpha_k = \alpha'_k$ for every $k \neq l, q$ yields the conclusion (note that every α_k can be supposed non zero, as if $\alpha_k = 0$ then $k = q = 1$ and one can replace α_k by α).
■

Lemma 2.3 *Let A be a Bezout domain having the block approximation and the block factorisation properties. Fix a non zero α in A and $p_1, q_1, \ldots, p_r, q_r$ in $A[X]$ with $X = (X_1, \ldots, X_n)$. The following are equivalent:*

1. *For every $\mathbf{m} \in M(\alpha)$ there is $a_{\mathbf{m}} \in A^n_{\mathbf{m}}$ such that $|p_i(a_{\mathbf{m}})|_{\mathbf{m}} \leq |q_i(a_{\mathbf{m}})| \neq 0$ for every $i \leq r$.*
2. *There is $a \in A^n$ such that $|p_i(a)|_{\mathbf{m}} \leq |q_i(a)| \neq 0$ for every $\mathbf{m} \in M(\alpha)$ and every $i \leq r$.*

Proof: We prove only (1) $\Rightarrow$ (2, since the other direction is trivial.

For every $\mathbf{m}$ in $M(\alpha)$, since each $p_i(a_{\mathbf{m}})/q_i(a_{\mathbf{m}})$ belongs to $A_{\mathbf{m}}$ there is a non zero $v_{\mathbf{m}}$ in A such that $v_{\mathbf{m}} \notin \mathbf{m}$ and $v_{\mathbf{m}}p_i(a_{\mathbf{m}})/q_i(a_{\mathbf{m}}) \in A$ for every $i \leq m$. The family of all $M(v_{\mathbf{m}})^c$ covers $M(\alpha)$, hence by compactness of $M(\alpha)$ there are $\mathbf{m}_1, \ldots, \mathbf{m}_q$ in $M(\alpha)$ such that $M(\alpha) \subseteq \bigcup_{k\leq q} M(v_{\mathbf{m}_k})^c$. By 2.2 we can find $\alpha_1, \ldots, \alpha_q$ in $A \setminus \{0\}$ such that $M(\alpha_k) \subseteq M(v_{\mathbf{m}_k})^c$ for each $k \leq q$, and the $M(\alpha_k)$'s form a partition of $M(\alpha)$. The block approximation property then gives a in A^n such that for every $\mathbf{m}$ in $M(\alpha_k)$ and every $k \leq q$ we have $|a - a_{\mathbf{m}_k}|_{\mathbf{m}} \leq |h(a_{\mathbf{m}_k})|_{\mathbf{m}}$ with $h := \prod_{i\leq r} q_i$.

Fix any $\mathbf{m}$ in $M(\alpha)$, there is $k \leq q$ such that $\mathbf{m}$ belongs to $M(\alpha_{k\mathbf{m}})$. For every $i \leq r$, since $q_i \in A[X]$ the Taylor formula for polynomials yields:

$$|q_i(a) - q_i(a_{\mathbf{m}_k})|_{\mathbf{m}} \leq |a - a_{\mathbf{m}_k}|_{\mathbf{m}} < |q_i(a_{\mathbf{m}_k})|_{\mathbf{m}}$$

hence $|q_i(a)|_{\mathbf{m}} = |q_i(a_{\mathbf{m}_k})|_{\mathbf{m}} \neq 0$. The same argument yields to:

$$|p_i(a) - p_i(a_{\mathbf{m}_k})|_{\mathbf{m}} \leq |a - a_{\mathbf{m}_k}|_{\mathbf{m}} \leq |q_i(a_{\mathbf{m}_k})|_{\mathbf{m}} = |q_i(a)|_{\mathbf{m}}.$$

Since $M(\alpha_{\mathbf{m}_k} \subseteq M(v_{\mathbf{m}_k})^c$ we have $p_i(a_{\mathbf{m}_k})/q_i(_{\mathbf{m}_k}) \in v_{\mathbf{m}_k}^{-1}A \subseteq A_{\mathbf{m}_k}$, hence $|p_i(a_{\mathbf{m}_k})|_{\mathbf{m}} \leq |q_i(a_{\mathbf{m}_k})|_{\mathbf{m}}$ and the conclusion follows:

$$|p_i(a)|_{\mathbf{m}} \leq \max(|p_i(a) - p_i(a_{\mathbf{m}_k})|_{\mathbf{m}}, |p_i(a_{\mathbf{m}_k})|_{\mathbf{m}}) \leq |q_i(a)|_{\mathbf{m}}.$$

■

Let us recall this well known fact: a valuation ring R is dense in its henselisation if and only if for every $f \in R[X]$, every non zero ε in R and every x in R such that $f(x) \in m(R)$ and $f'(x) \notin m(R)$, there is an element y in R such that $|f(y)|_R \leq |\varepsilon|_R$. This is always true if the valuation group can be embedded in $\mathbf{R}_+^*$, but not in the general case.

Proposition 2.4 *Let A be a Bezout domain having the block factorisation and block approximation properties. Then the following are equivalent:*

1. *Every R in W_A is dense in its henselisation.*
2. *For every $f \in A[X]$, every a and every nonzero ε in A there is an element b in A such that $|f(b)|_{\mathbf{m}} \leq |\varepsilon|_{\mathbf{m}}$ for every $\mathbf{m}$ in $M(f(a)) \setminus M(f'(a))$.*

Remark 2.5 The reader may notice that a Prüfer domain A having the block factorisation is dense in each $R \in W_A$. Indeed, if $x = u/v \in R$ with $R = A_{\mathbf{m}}$, $u, v \in A$ and $v \notin \mathbf{m}$, for any nonzero ε the block approximation property gives a in A such that:

$$\begin{cases} |a-u|_{\mathbf{n}} \leq |\varepsilon|_{\mathbf{n}} & \text{for every } \mathbf{n} \in M(v)^c, \\ |a|_{\mathbf{n}} \leq |v|_{\mathbf{n}} & \text{for every } \mathbf{n} \in M(v). \end{cases}$$

Hence v divides a in A (because $A = \bigcap W_A$ and $|a|_{\mathbf{n}} \leq |v|_{\mathbf{n}}$ for every maximal ideal of A, hence a/v belongs to every $R \in W_A$) and as $\mathbf{m} \in M(v)^c$ we have $|b - x|_{\mathbf{m}} \leq |\varepsilon|_{\mathbf{m}}$ with $b := a/v \in A$.

Proof: Suppose that (2) above holds, and fix $R = A_{\mathbf{m}}$ in W_A. Let g be any polynomial in $R[X]$, suppose there is x in R such that $g(x) \in m(R)$ and $g'(x) \notin m(R)$, and fix an arbitrary δ in $R \setminus \{0\}$. Since A is dense in R there exists $a \in A$ and $f \in A[X]$ such that $f - g \in \delta R[X]$, $f(a) \in \mathbf{m}$ and $f'(a) \notin \mathbf{m}$. Write $\delta = \varepsilon/u$ with $\varepsilon, u \in A \setminus \{0\}$ and $u \notin \mathbf{m}$. By (2) there is b in A such that $|f(b)|_{\mathbf{m}} \leq |\varepsilon|_{\mathbf{m}} = |\delta|_{\mathbf{m}}$ hence $|g(b)|_{\mathbf{m}} = |f(b) + (g(b) - f(b))|_{\mathbf{m}} \leq |\delta|_{\mathbf{m}}$ wich proves that R is dense in R^h.

Conversely, suppose (1) holds, and let f, a be as in (2). The block factorisation property gives a non zero α in A such that $M(\alpha) = M(f(a)) \setminus M(f'(a))$. Fix some non zero ε in A, for each $\mathbf{m} \in M(\alpha)$ (1) gives $a_{\mathbf{m}} \in A_{\mathbf{m}}$ such that $|f(a_{\mathbf{m}})|_{\mathbf{m}} \leq |\varepsilon|_{\mathbf{m}}$, hence 2.3 proves (2).

■

Corollary 2.6 *The class of Bezout domains A having both the block approximation and block factorisation properties, and such that every R in W_A is dense in its henselisation, is an elementary class. It is recursively axiomatisable.*

Proof: By 2.1 it is sufficient to axiomatise this class modulo the theory of Bezout domains having the block factorisation and block approximation properties, but by 2.4 it is axiomatised by the following family of axioms, with $d \in \mathbf{N}^*$:

$$\forall c_0, \dots, c_d \forall \varepsilon \forall a \left[\varepsilon \neq 0 \to \exists b, u, v, d \, (f(b)v = \varepsilon u \wedge (v, f(a)) = d \wedge d \leq f'(a)) \right]$$

Indeed, observe that $(v, f(a)) = d \wedge d \leq f'(a)$ is equivalent to saying that $M(v)$ is disjoint from $M(f(a)) \setminus M(f'(a))$.

■

Proposition 2.7 *If a ring A satisfies the LGPH, then A is dense in $A^h_{\mathbf{m}}$ for every $\mathbf{m} \in \mathrm{Specmax}(A)$.*

This result will be strengthened in the next section.

Proof: By remark 2.5 we know that A is dense in $A_{\mathbf{m}}$ hence it is sufficient to prove that $A_{\mathbf{m}}$ is dense in $A^h_{\mathbf{m}}$, and for this we are going to use criterion 2.4. Let f be a polynomial with one variable and coefficients in A, fix an element a of A and a non zero ε in A. If $f(a) = 0$ there is nothing to prove, if not the block factorisation property gives a non zero α in A such that $M(\alpha) = M(f(a)) \setminus M(f'(a))$.

Let us considere the variety V defined over $\mathrm{Frac}(A)$ by the equations $\varepsilon Y = Zf(X)$ and $\alpha U + Z = 1$. For every $\mathbf{m} \in M(\alpha)$ since by construction $f(a) \in \mathbf{m}$ and $f'(a) \notin \mathbf{m}$, Hensel's lemma gives $x_{\mathbf{m}} \in A^h_{\mathbf{m}}$ such that $f(x_{\mathbf{m}}) = 0$. Taking $y_{\mathbf{m}} = u_{\mathbf{m}} = 0$ and $z_{\mathbf{m}} = 1$ then gives a point of V with coordinates in $A^h_{\mathbf{m}}$. For every maximal ideal $\mathbf{m}$ of A not in $M(\alpha)$, $x_{\mathbf{m}} = a$, $y_{\mathbf{m}} = z_{\mathbf{m}} = 0$, $u_{\mathbf{m}} = \alpha^{-1}$ defines a point of V with coordinates in $A_{\mathbf{m}}$.

Since V is smooth the LGPH provides a point $(x, y, z, u) \in V(A)$ such that for every $\mathbf{m} \in M(\alpha)$, $z \notin \mathbf{m}$ since $\alpha u + z = 1$ hence:

$$|f(x)|_{\mathbf{m}} = |\varepsilon y z^{-1}|_{\mathbf{m}} \leq |\varepsilon|_{\mathbf{m}}$$

and the conclusion follows by (2) of 2.4.

■

The following result is supposed to be well known, although we never heard about any published reference on it, except in the case of absolute values wich has been proved by Lang. The interested reader will find in [1, §I.5] a simple proof valid for any valuation ring and based on non standard arguments.

Lemma 2.8 *Let R be a henselian valuation ring, and $f = (f_1, \dots, f_m) \in R[X]^m$ with $X = (X_1, \dots, X_n)$ be a system of equations for a variety V defined over $F := \mathrm{Frac}(R)$. Let Δ be a $(n - \dim V)$-subdeterminant of the Jacobian matrix of f, and let μ be a nonzero element of R. Suppose that for any nonzero ε in R there is $a_\varepsilon \in R^n$ such that $|\Delta(a_\varepsilon)|_R \geq |\mu|_R$ and $|f_i(a_\varepsilon)|_R \leq |\varepsilon|_R$ for every $i \leq m$.*
Then V has a smooth point with coordinates in R.

Lemma 2.9 *Let A be a Prüfer domain with fraction field K, and for every maximal ideal $\mathbf{n}$ of A let $K_{\mathbf{n}}$ be the fraction field of $A^h_{\mathbf{n}}$. Fix a maximal ideal $\mathbf{m}$ of A and x in $K^n_{\mathbf{m}}$. Let us suppose that $A_{\mathbf{m}}$ is dense in $A^h_{\mathbf{m}}$.*
Then for every nonzero ε in A there is a constructible subset $\mathcal{M}$ of $\mathrm{Specmax}(A)$ and y in K^n such that:

1. *$\mathbf{m} \in \mathcal{M}$ and $|x - y|_{\mathbf{m}} < |\varepsilon|_{\mathbf{m}}$.*

2. For every $\mathbf{n} \in \mathcal{M}$ there is a K-conjugate $x_{\mathbf{n}}$ of x in $K_{\mathbf{n}}^n$ such that $|x_{\mathbf{n}} - y|_{\mathbf{n}} < |\varepsilon|_{\mathbf{n}}$.

If in addition A has the block factorisation property one can choose $\mathcal{M} = M(\alpha)$ for some non zero $\alpha \in \mathbf{m}$. Moreover, if x belongs to $(A_{\mathbf{m}}^h)^n$ and A is dense in $A_{\mathbf{m}}$, then y and $x_{\mathbf{n}}$ can be taken in $(A_{\mathbf{n}}^h)^n$ for every $\mathbf{n} \in \mathcal{M}$.

Until the end of this paper, we will refer to this result as to the **accessibility lemma**.
Proof: We prove it first with $n = 1$. Fix some nonzero ε in K. Let f be the minimal polynomial of x over K, let u in A be a non zero element such that $g := uf \in A[X]$. By hypothesis K is dense in $K_{\mathbf{m}}$ hence there exists y in K^n such that:

$$|x - y|_{\mathbf{m}} < |\varepsilon|_{\mathbf{m}}. \tag{1}$$

As $K_{\mathbf{m}}$ is a separable extension of K, $g'(x)$ is non zero hence by continuity of polynomial functions we can choose y so that:

$$|g(y)|_{\mathbf{m}} < \min\left(|g'(y)^2|_{\mathbf{m}}, |\varepsilon g(y)|_{\mathbf{m}}\right). \tag{2}$$

Let a, b, u, v in A such that $u, v \notin \mathbf{m}$, $g(y)/g'(y)^2 = a/u$ and $g(y)/\varepsilon g'(y) = c/v$. By (2) we have $a, b \in \mathbf{m}$ hence:

$$\mathbf{m} \in \mathcal{M} := M(a) \cap M(b) \setminus (M(u) \cup M(v)). \tag{3}$$

For every $\mathbf{n}$ in $\mathcal{M}$ we have $|g(y)|_{\mathbf{n}} < |g'(y)^2|_{\mathbf{n}}$ and $|g(y)|_{\mathbf{n}} < |\varepsilon g(y)|_{\mathbf{n}}$Hensel-Rychlik's theorem then gives $x_{\mathbf{n}} \in K_{\mathbf{n}}$ such that $g(x_{\mathbf{n}}) = 0$, hence $x_{\mathbf{n}}$ is a K-conjugate of x over K, and:

$$|x_{\mathbf{n}} - y|_{\mathbf{n}} = \left|\frac{g(y)}{g'(y)}\right|_{\mathbf{n}} < |\varepsilon|_{\mathbf{n}}$$

wich proves this case.

Let now n take any value in $\mathbf{N}^*$. The primitive element theorem gives $\theta \in K_{\mathbf{m}}$ such that $K[x] = K[\theta]$. Let $p \in K[T]^n$ be such that $x = p(\theta)$, and $u \in A \setminus \{0\}$ such that $q := up \in A[T]^n$. Eventually fix any non zero ε in K. Let $t \in K$, $\mathcal{M}$ a construtible subset of $\mathrm{Specmax}(A)$ given by the above case. Let $y := p(t)$ and for any $\mathbf{n} \in \mathcal{M}$ let $\theta_{\mathbf{n}} \in K_{\mathbf{n}}$ be a K-conjugate of θ such that $|\theta_{\mathbf{n}} - t|_{\mathbf{n}} < |u\varepsilon|_{\mathbf{n}}$, and $x_{\mathbf{n}} := p(\theta_{\mathbf{n}})$. By construction $x_{\mathbf{n}}$ is K-conjugate of x and we have:

$$|x_{\mathbf{n}} - y|_{\mathbf{n}} = |u^{-1}|_{\mathbf{n}} |q(\theta_{\mathbf{n}}) - q(t)|_{\mathbf{n}} \leq |u^{-1}|_{\mathbf{n}} |\theta_{\mathbf{n}} - t|_{\mathbf{n}} < |\varepsilon|_{\mathbf{n}}.$$

This ends the proof of the main statement. For the two additional remarks, let us observe that 1.5 and (3) above give a non zero α in A such that $\mathcal{M} = M(\alpha)$. If we suppose that A is dense in $A_{\mathbf{m}}$ and $x \in (A_{\mathbf{m}}^h)^n$ then t can be taken so that $y \in A_{\mathbf{m}}^n$ hence there is $s \in A \setminus \mathbf{m}$ such that $sy \in A^n$. If we take ε in A and replace $\mathcal{M}$ by $\mathcal{N} = \mathcal{M} \setminus M(s)$, for every $\mathbf{n} \in \mathcal{N}$ we have $y \in A_{\mathbf{n}}$ and $|x_{\mathbf{n}} - y| < |\varepsilon|_{\mathbf{n}} \leq 1$ hence $x_{\mathbf{n}} \in (A_{\mathbf{m}}^h)^n$.
■

Proposition 2.10 *Let A be a Bezout domain having the block factorisation and block approximation properties. Suppose that every R in W_A is dense in its henselisation. Let V be a variety defined over $K := \mathrm{Frac}(A)$ and let $f = (f_1, \ldots, f_m) \in A[X]^m$ with $X = (X_1, \ldots, X_n)$ be a system of equations for V. For any non zero α in A the following are equivalent:*

1. *$V^\circ(A^h_{\mathbf{m}}) \neq \emptyset$ for every $\mathbf{m}$ in $M(\alpha)$.*

2. *There is a $(n - \dim V)$-subdeterminant of the Jacobian matrix of f and a non zero element μ of A with the property that for every non zero ε in A there exists $a \in A^n$ such that for every $\mathbf{m}$ in $M(\alpha)$:*

$$\begin{cases} |\Delta(a)|_{\mathbf{m}} \geq |\mu|_{\mathbf{m}} \\ |f(a)|_{\mathbf{m}} \leq |\varepsilon|_{\mathbf{m}} \end{cases}$$

Proof: (2) $\Rightarrow$ (1) Suppose (2) holds and fix $\mathbf{m}$ in $M(\alpha)$. For any non zero δ in $A^h_{\mathbf{m}}$ by density there is ε in A such that $|\varepsilon|_{\mathbf{m}} = |\delta|_{\mathbf{m}}$ hence (2) implies $V^\circ(A^h_{\mathbf{m}}) \neq \emptyset$ by 2.8.

(1) $\Rightarrow$ (2) Conversely, suppose (1) holds: for each $\mathbf{m} \in M(\alpha)$ there is $x_{\mathbf{m}}$ in $V(A^h_{\mathbf{m}})$ and a submatrix of size $n - \dim V$ of the Jacobian matrix of f whose determinant $\Delta_{\mathbf{m}}(x_{\mathbf{m}})$ is non zero. Let Δ be any of those determinants, and fix $\mathbf{m}$ in $M(\alpha)$. Since $\Delta \notin I(V)$ the implicit function theorem gives a point $x'_{\mathbf{m}}$ in $V(A^h_{\mathbf{m}})$ such that $\Delta(x'_{\mathbf{m}}) \neq 0$. Replacing $x_{\mathbf{m}}$ by $x'_{\mathbf{m}}$ if necessary, we can suppose that $\Delta_{\mathbf{m}} = \Delta$, in other words $\Delta_{\mathbf{m}}$ does not depend on $\mathbf{m}$. The next step is to reduce to a finite number of local points, in order to construct something like a common lower bound for the $\Delta(x_{\mathbf{m}})$'s.

Let K denote the fraction field of A. For each $\mathbf{m}$ in $M(\alpha)$ let $\gamma_{\mathbf{m}}$ be a non zero element of $\mathbf{m}$ given by the accessibility lemma 2.9: for any $\mathbf{n}$ in $M(\gamma_{\mathbf{m}})$ there is a K-conjugate of $x_{\mathbf{m}}$ in $(A^h_{\mathbf{n}})^n$. Replacing $A^h_{\mathbf{n}}$ by some K-isomorphic image we can suppose $x_{\mathbf{m}}$ itself belongs to $(A^h_{\mathbf{n}})^n$. Let $\beta_{\mathbf{m}} \in A \setminus \mathbf{m}$ such that $M(\beta_{\mathbf{m}}) = M(\alpha) \setminus M(\gamma_{\mathbf{m}})$. The family of all $M(\beta_{\mathbf{m}})^c$'s covers $M(\alpha)$ as $\mathbf{m}$ ranges over $M(\alpha)$. By compactness there are $\mathbf{m}_1, \ldots, \mathbf{m}_q$ in $M(\alpha)$ such that $M(\alpha) \subseteq \bigcup_{k=1}^q M(\beta_{\mathbf{m}_k})^c$. By 2.2 there is a partition $(M(\alpha_1), \ldots, M(\alpha_q))$ of $M(\alpha)$ such that:

$$M(\alpha_k) \subseteq M(\alpha) \setminus M(\beta_{\mathbf{m}_k}) = M(u_{\mathbf{m}}) \cap M(v_{\mathbf{m}})^c \cap M(s_{\mathbf{m}})^c. \qquad (*)$$

We may observe that we have already reduced to a finite number of local points. Indeed for every $\mathbf{m}$ in $M(\alpha)$ there is $k \leq q$ such that $x_{\mathbf{m}_k} \in V(A^h_{\mathbf{m}})$.

For each $k \leq q$ let d_k, e_k be nonzero elements integral over A such that $\Delta(x_{\mathbf{m}_k}) = d_k/e_k$, and μ_k be the norm of d_k over A (that is the product of all K-conjugates of d_k). Eventually let $\mu' := \prod_{k \leq q} \mu_k \in A \setminus \{0\}$ and use the block factorisation property to write $\mu' = \mu''\mu$ with $M(\mu) = M(\mu') \cap M(\alpha)$ and $M(\mu'') = M(\mu') \setminus M(\alpha)$, hence $|\mu|_{\mathbf{m}} = |\mu'|_{\mathbf{m}}$ for every $\mathbf{m} \in M(\alpha)$. It is immediate that $|\Delta(x_{\mathbf{m}_k})|_{\mathbf{m}} \geq |d_k|_{\mathbf{m}} \geq |\mu|_{\mathbf{m}}$ for every $k \leq q$ and every $\mathbf{m}$ in $M(\alpha_k)$.

It remains to check that (2) holds with this μ. Fix any non zero ε in A, for each $\mathbf{m}$ in $M(\alpha)$ there is $k \leq q$ such that $\mathbf{m}$ belongs to $M(\beta_k)$. The density of A in $A^h_{\mathbf{m}}$ gives $a_{\mathbf{m}} \in A^n$ such that $|a_{\mathbf{m}} - x_{\mathbf{m}_k}|_{\mathbf{m}} \leq |\varepsilon|_{\mathbf{m}}$. We can even choose $a_{\mathbf{m}}$ so that $|\Delta(a_{\mathbf{m}})|_{\mathbf{m}} \geq |\mu|_{\mathbf{m}}$. Moreover since $f \in A[X]^m$ we have:

$$|f(a_{\mathbf{m}})|_{\mathbf{m}} \leq |a_{\mathbf{m}} - x_{\mathbf{m}_k}|_{\mathbf{m}} \leq |\varepsilon|_{\mathbf{m}}.$$

Then 2.3 gives $a \in A^n$ such that $|\Delta(a)|_m \geq |\mu|_{\mathbf{m}}$ and $|f(a)|_{\mathbf{m}} \leq |\varepsilon|_{\mathbf{m}}$ for every $\mathbf{m} \in M(\alpha)$.

■

We can now set the result wich was the aim of this section.

Theorem 2.11 *LGPH is a recursively axiomatisable property (in the language of rings).*

Proof: By 2.6 it it sufficient to axiomatise the LGPH modulo the theory, noted T for this proof, of Bezout domains A having the block factorisation and the block approximation properties, and such that each R in W_A is dense in its henselisation. By 1.4 we only have to deal with the geometric condition.

Since the theory of algebraically closed fields has the elimination of quantifiers, t is well known that there is a formula $Var_{m,n,e,d}(x)$ in the language of rings such that for any integral domain A and any tuple c (whose length N is a function of m, n and e) of coefficients in A defining a system $f_c = (f_1, \ldots, f_m)$ of polynomials with n indeterminates and degree less than or equal to e, we have $A \models Var_{m,n,e,d}(c)$ if and only if f is a system of equations of a variety V_c defined over $\mathrm{Frac}(A)$ with $\dim V_c = d$. The set of $(n-d)$-subdeterminants of the jacobian matrix of f_c is defined uniformly in c, we denote it by $\mathcal{J}$.

For any model A of T and any $c \in A^N$ such that $A \models Var_{m,n,e,d}(c)$, 2.10 shows that V_c has a smooth point in R^h for every $R \in W_A$ if and only if A models the following formula (with parameters c):

$$\begin{aligned} ZH(c) \equiv \forall \alpha,\ \alpha \neq 0 \to \Big[\exists \mu,\ \mu \neq 0 \wedge \bigvee_{\Delta_c \in \mathcal{J}} \forall \varepsilon,\ 0 \neq \varepsilon \to \\ \exists a = (a_1, \ldots, a_n)\ DivLoc(\Delta_c(a), \mu, \alpha) \wedge \bigwedge_{i=1}^{m} DivLoc(\varepsilon, f_i(a), \alpha) \Big]. \end{aligned}$$

It follows that a ring A satisfies the LGPH if and only if it is a model of T and of the following family of formulas:

$$\begin{aligned} \forall c = (c_1, \ldots, c_N)\ Var_{m,n,e,d}(c) \wedge ZH(c) \to \\ \exists a = (a_1, \ldots, a_n)\ f_c(a) = 0 \wedge \bigvee_{\Delta_c \in \mathcal{J}} \Delta_c(a) \neq 0. \end{aligned}$$

■

3 Extending LGPH

Proposition 3.1 *Let A be a ring satisfying LGPH and $\mathbf{m}$ a fixed maximal ideal of A. Let V be a variety defined over $\mathrm{Frac}(A)$, and for any maximal ideal $\mathbf{n}$ of A suppose that V has a smooth point $x_{\mathbf{n}}$ with coordinates in $A^h_{\mathbf{n}}$.*
Then $V^\circ(A)$ is dense in $V^\circ(A^h_{\mathbf{m}})$ (with respect to the topology induced by the valuation). In particular given any polynomials $h_0, h_1, \ldots, h_r$ with coefficients in A such that $h_0 \notin I(V)$, the LGPH provides smooth points a of V in A such that:

$$\begin{cases} h_0(a) \neq 0 \\ h_i(a) \in \mathbf{m} \iff h_i(x_{\mathbf{m}}) \in m(A^h_{\mathbf{m}}), \quad i \leq r \end{cases}$$

Until the end of this paper, we will refer to 3.1 as to the **density property**.

Proof: Fix any non zero ε in A, we can assume $\varepsilon \in \mathbf{m}$. Let α be a non zero element of $\mathbf{m}$ given by the accessibility lemma 2.9: there is y in K^n such that $|x_{\mathbf{m}} - y|_{\mathbf{m}} \leq |\varepsilon|_{\mathbf{m}}$ and for every $\mathbf{n}$ in $M(\alpha)$ there is a K-conjugate $y_{\mathbf{m},\mathbf{n}}$ of $x_{\mathbf{m}}$ in $(A^h_{\mathbf{n}})^n$ such that $|y_{\mathbf{m},\mathbf{n}} - y|_{\mathbf{n}} \leq |\varepsilon|_{\mathbf{n}}$. Replacing if necessary $A^h_{\mathbf{n}}$ by some K-isomorphic image we can assume $y_{\mathbf{m},\mathbf{n}} = x_{\mathbf{m}} \in V^\circ(A^h_{\mathbf{n}})$. The block factorisation property gives $\varepsilon', \varepsilon''$ in A such that $\varepsilon = \varepsilon'\varepsilon''$, $M(\varepsilon') = M(\varepsilon) \cap M(\alpha)$ and $M(\varepsilon'') = M(\varepsilon) \setminus M(\alpha)$. Since $|\varepsilon|_{\mathbf{m}} = |\varepsilon'|_{\mathbf{m}}$, by replacing if necessary ε by ε_1, we can suppose $M(\varepsilon) \subseteq M(\alpha)$.

Now let $f = (f_1, \ldots, f_m) \in K[X]$ with $X = (X_1, \ldots, X_n)$ be a system of equations of V, and consider the variety V' defined over K by:

$$\begin{cases} f(X) = 0 \\ X - y = \varepsilon U \end{cases}$$

with $U = (U_1, \ldots, U_n)$ a set of new variables. A point (x, y) of V' is easily seen to be smooth if and only if x is a smooth point of V. For every maximal ideal $\mathbf{n}$, if $\mathbf{n} \in M(\alpha)$ then $(x_{\mathbf{m}}, \varepsilon^{-1}(x_{\mathbf{m}} - y))$ is a smooth point of V' whose coordinates belong to $A^h_{\mathbf{n}}$, as $x_{\mathbf{m}} \in V^\circ(A^h_{\mathbf{n}})$ and $|x_{\mathbf{m}} - y|_{\mathbf{n}} \leq |\varepsilon|_{\mathbf{n}}$. If not, since $M(\varepsilon) \subseteq M(\alpha)$ we have $\varepsilon^{-1} \in A_{\mathbf{n}}$ hence $(x_{\mathbf{n}}, \varepsilon^{-1}(x_{\mathbf{n}} - y)) \in V^\circ(A^h_{\mathbf{n}})$. The LGPH then gives $(x, y) \in V'^\circ(A)$, hence $x \in V^\circ(A)$ and $|x - y|_{\mathbf{m}} = |\varepsilon u|_{\mathbf{m}} \leq |\varepsilon|_{\mathbf{m}}$. The conclusion then follows:

$$|x - x_{\mathbf{m}}|_{\mathbf{m}} \leq \max\left(|x - y|_{\mathbf{m}}, |x_{\mathbf{m}} - y|_{\mathbf{m}}\right) \leq |\varepsilon|_{\mathbf{m}}.$$

The additional assertion on the h_i's is then an easy consequence, by continuity of polynomial fonctions and the fact that $m(A^h_{\mathbf{m}})$ is closed open in $A^h_{\mathbf{m}}$. The reader has only to remember that as h_0 does not belong to $I(V)$, the implicit function theorem in $A^h_{\mathbf{m}}$ gives points $x'_{\mathbf{m}}$ in $V^\circ(A^h_{\mathbf{m}})$ arbitrarily close to $x_{\mathbf{m}}$ such that $h_0(x'_{\mathbf{m}}) \neq 0$. Replacing $x_{\mathbf{m}}$ by such an $x'_{\mathbf{m}}$ if necessary, we can suppose that $h_0(x_{\mathbf{m}}) \neq 0$, and the conclusion follows by density.

■

Ershov introduced in [2] a very general notion of regular closure, wich contains all the ordinary local-global principles for fields. When W is a family of valuation rings with fraction field K, Ershov says that K is regularly closed with respect to $\mathcal{F}_W := \{\mathrm{Frac}(R^h) \mid R \in W\}$ if and only if for every variety V defined over K:

$$V^\circ(K) \neq \emptyset \iff \forall F \in \mathcal{F},\ V^\circ(F) \neq \emptyset.$$

The following corollary makes clearer the relation between the LGPH and the relative regular closure.

Corollary 3.2 *Let A be an integral ring with fraction field K.*
If A satisfies the LGPH, then K is regularly closed with respect to $\mathcal{F}_{W_A}$.

Proof: Let V be a variety defined over K having a smooth point in $\mathrm{Frac}(R^h)$ for every $R \in W_A$. Then the homogenisation V^* of V has a smooth point in R^h for every $R \in W_A$, hence in A by the LGPH. Moreover the density property allows us to choose a smooth point $(a_1, \ldots, a_n, u)$ in $V^*(A)$ such that $u \neq 0$ and $(a_1/u, \ldots, a_n/u)$ belongs to $V^\circ(K)$ and the conclusion follows.

■

The converse is true in some cases, in particular when W_A is the family of all p-valuation rings of K. However it is false in general: let R be the valuation ring of rational fonctions in one variable over $\mathbf{Q}$ with no pole at 0. Let A be the integral closure of R in the algebraic closure of $\mathrm{Frac}(R)$. As $\mathrm{Frac}(A)$ is algebraically closed, trivially $\mathrm{Frac}(A)$ is regularly closed with respect to $\mathcal{F}_{W_A} = \{\mathrm{Frac}(R)\}$. But it has been proved that A does not satisfy the geometric condition of the LGPH (see [9, 5.5]).

Theorem 3.3 *Let A be a ring satisfying LGPH, with fraction field K. Every R^h ($R \in W_A$) is endowed with the radical relation induced by its maximal spectrum, and so is A.*

Let $f = (f_1, \ldots, f_m) \in A[X]^m$ with $X = (X_1, \ldots, X_n)$ be a system of equations of a variety V defined over K, and g a fixed element of $A[X] \setminus I(V)$. For every $i \in \{1, \ldots, u\}$ and every $j \in \{1, \ldots, v\}$ fix some p_i, p'_i, q_j, q'_j in $A[X]$ with the p_i's not belonging to $I(V)$. Suppose that V has a smooth point $x_{\mathbf{m}}$ with coordinates in $A^h_{\mathbf{m}}$ for every maximal ideal $\mathbf{m}$ of A, such that for every $i \leq u$:

$$A^h_{\mathbf{m}} \models p_i(x_{\mathbf{m}}) \leq p'_i(x_{\mathbf{m}}) \qquad (i \leq u)$$

Suppose in addition that for each $j \leq v$ there is a maximal ideal $\mathbf{m}(j)$ such that:

$$A^h_{\mathbf{m}(j)} \not\models q_j(x_{\mathbf{m}(j)}) \leq q'_j(x_{\mathbf{m}(j)}).$$

Then V has a smooth point a with coordinates in A such that $g(a) \neq 0$, $A \models p_i(x) \leq p'_i(x)$ for every $i \leq u$ and $A \not\models q_j(x) \leq q'_j(x)$ for every $j \leq v$.

Proof: We are going to use several times the following simple argument:

(∗) For any valuation ring R and any $a, b \in R^n$ such that $|a-b|_R < 1$, for any $h \in R[X]$ we have $|h(a) - h(b)|_R < 1$ hence $h(a) \in m(R)$ if and only if $h(b) \in m(R)$.
In particular for any $h, h' \in R[X]$ we have $h(a) \leq_R h'(a)$ if and only if $h(b) \leq_R h'(b)$.

Let p be the product of the p_i's for $i \leq u$ and let $a_0 \in V^\circ(A)$ be given by the density property so that $\alpha := p(a_0) \neq 0$. By hypothesis there are distinct maximal ideals $\mathbf{m}_1, \ldots, \mathbf{m}_r$ of A and a partition $J_1, \ldots, J_r$ of $\{1, \ldots, v\}$ such that for every $k \in \{1, \ldots, r\}$ and every $j \in J_k$:

$$A^h_{\mathbf{m}_k} \not\models q_j(x_{\mathbf{m}_k}) \leq q'_j(x_{\mathbf{m}_k}).$$

Muliplying α if necessary by a non zero element of $\mathbf{m}_k$ for each $k \leq r$, we can suppose that $M(\alpha)$ contains $\{\mathbf{m}_1, \ldots, \mathbf{m}_r\}$.

For every $\mathbf{m} \in M(\alpha)$ let $\gamma_{\mathbf{m}}$ be a non zero element of $\mathbf{m}$ given by the accessibility lemma 2.9: there is $b_{\mathbf{m}}$ in A^n such that $|x_{\mathbf{m}} - b_{\mathbf{m}}|_{\mathbf{m}} \leq |\alpha|_{\mathbf{m}}$ and for every $\mathbf{n}$ in $M(\gamma_{\mathbf{m}})$ there is a K-conjugate $x'_{\mathbf{m},\mathbf{n}}$ of $x_{\mathbf{m}}$ in $(A^h_{\mathbf{n}})^n$ such that $|x'_{\mathbf{m},\mathbf{n}} - b_{\mathbf{m}}|_{\mathbf{n}} \leq |\alpha|_{\mathbf{n}}$. Replacing if necessary $A^h_{\mathbf{n}}$ by some K-isomorphic image we can assume $x_{\mathbf{m}} = x'_{\mathbf{m},\mathbf{n}} \in V^\circ(A^h_{\mathbf{n}})$.

By 1.5 there is for each $\mathbf{m} \in M(\alpha)$ a non zero $\beta_{\mathbf{m}}$ in A such that:

$$M(\beta_{\mathbf{m}}) = M(\alpha) \setminus \left[M(\gamma_{\mathbf{m}}) \cap \bigcap_{i \leq u} M(p'_i(b_{\mathbf{m}})) \cap M(p_i(b_{\mathbf{m}}))^c \right].$$

Since $A^h_{\mathbf{m}} \models \bigwedge_{i \leq u} p_i(x_{\mathbf{m}}) \leq p'_i(x_{\mathbf{m}})$ we have $\mathbf{m} \in M(\alpha) \cap M(\beta_{\mathbf{m}})^c$. Moreover for every $\mathbf{n} \in M(\alpha) \cap M(\beta_{\mathbf{m}})^c$ by construction $A^h_{\mathbf{n}} \models \bigwedge_{i \leq u} p_i(b_{\mathbf{m}}) \leq p'_i(b_{\mathbf{m}})$, and since $|x_{\mathbf{m}} - b_{\mathbf{m}}|_{\mathbf{n}} < 1$ the argument (∗) applies:

$$A^h_{\mathbf{n}} \models \bigwedge_{i \leq u} p_i(x_{\mathbf{m}}) \leq p'_i(x_{\mathbf{m}}).$$

The $M(\beta_{\mathbf{m}})^c$'s cover $M(\alpha)$ hence a finite subfamily of them, wich we can suppose to include $M(\beta_{\mathbf{m}_k})^c$ for every $k \leq r$, covers $M(\alpha)$. Say $M(\alpha) \subseteq \bigcup_{k \leq s} M(\beta_{\mathbf{m}_k})^c$ with $s \geq r$. We can suppose all those $\mathbf{m}_k$'s to be distinct, hence 2.2 provides a partition $(M(\alpha_k))_{k \leq s}$ of $M(\alpha)$ such that $\mathbf{m}_k \in M(\alpha_k) \subseteq M(\alpha) \cap M(\beta_{\mathbf{m}_k})^c$ for every $k \leq s$.

The block approximation property gives a point b in A^n such that for every $k \le s$ and every $\mathbf{m} \in M(\alpha_k)$ we have $|b - b_{\mathbf{m}_k}|_{\mathbf{m}} \le |\alpha|_{\mathbf{m}}$. Since by construction $|b_{\mathbf{m}_k} - x_{\mathbf{m}_k}|_{\mathbf{m}} \le |\alpha|_{\mathbf{m}}$ it follows that:

$$|b - x_{\mathbf{m}_k}|_{\mathbf{m}} \le |\alpha|_{\mathbf{m}}. \tag{1}$$

Since $p(x_{\mathbf{m}_k}) \ne 0$ the block approximation property also gives non zero δ_i in A such that for $i \le u$, every $k \le s$ and every $\mathbf{m} \in M(\alpha_k)$:

$$|\delta_i|_{\mathbf{m}} \le |p_i(x_{\mathbf{m}_k})|_{\mathbf{m}}. \tag{2}$$

Use the block factorisation property to write $\delta_i = \delta_i' \delta_i''$, with $M(\delta_i') = M(\delta_i) \cap M(\alpha)$ and $M(\delta_i'') = M(\delta_i) \setminus M(\alpha)$. For every $k \le s$ and every $\mathbf{m} \in M(\alpha_k)$ we then have $|\delta_i'|_{\mathbf{m}} = |\delta_i|_{\mathbf{m}}$. Replacing δ_i by δ_i' if necessary, we can suppose $M(\delta_i) \subseteq M(\alpha)$.

Let V' be the variety defined over K by the following equations, with Y a n-tuple of new variables:

$$\begin{cases} f(X) = 0 \\ \alpha Y = X - b, \\ p_i(X) Z_i = \delta_i, \quad i \le u \end{cases}$$

A point (x, y, z) of V' is smooth if and only if x is a smooth point of V.

For every $\mathbf{m}$ in $M(\alpha)$ there is $k \le s$ such that $\mathbf{m} \in M(\alpha_k)$, hence $x_{\mathbf{m}_k}$ is a smooth point of $V(A^h_{\mathbf{m}})$ such that $|b - x_{\mathbf{m}_k}|_{\mathbf{m}} \le |\alpha|_{\mathbf{m}}$ by (1) and $|\delta_i|_{\mathbf{m}} \le |p_i(x_{\mathbf{m}_k})|_{\mathbf{m}}$ by (2). Taking $x = x_{\mathbf{m}_k}$, $y = \alpha^{-1}(x_{\mathbf{m}_k} - b)$, and $z_i = \delta_i / p_i(x_{\mathbf{m}_k})$ for $i \le u$ then gives a smooth point of V' with coordinates in $A^h_{\mathbf{m}}$.

For every $\mathbf{m}$ in $M(\alpha)^c$ by construction α and the $p_i(a_0)$'s are invertible in $A_{\mathbf{m}}$ hence taking $x = a_0$, $y = \alpha^{-1}(a_0 - b)$ and $z_i = \delta_i / p_i(a_0)$ gives a smooth point of V' with coordinates in $A^h_{\mathbf{m}}$.

The LGPH then gives a smooth point (x, y, z) of V' with coordinates in A such that $g(x) \ne 0$. Hence x is a smooth point of V with coordinates in A such that $g(x) \ne 0$.

Fix any $i \le u$. Since $p_i(x)$ divides δ_i it follows that $M(p_i(x)) \subseteq M(\alpha)$. For every $\mathbf{m} \in M(p_i(a))$ there is then some $k \le s$ such that $\mathbf{m} \in M(\alpha_k)$.

$$|x - x_{\mathbf{m}_k}|_{\mathbf{m}} \le \max\left(|x - b|_{\mathbf{m}}, |b - x_{\mathbf{m}_k}|_{\mathbf{m}}\right) \le |\alpha|_{\mathbf{m}} < 1.$$

Since $A_{\mathbf{m}} \models p_i(x_{\mathbf{m}}) \le p_i'(x_{\mathbf{m}})$ the argument $(*)$ then implies that $A_{\mathbf{m}} \models p_i(x) \le p_i'(x)$ hence $p_i'(x) \in \mathbf{m}$. This proves that $p_i(x) \le_A p_i'(x)$ for every $i \le u$.

Fix now any $j \le v$; there is $k \le r$ such that j belongs to J_k hence $A^h_{\mathbf{m}_k} \not\models q_j(x_{\mathbf{m}_k}) \le q_j'(x_{\mathbf{m}_k})$. Since $|x - x_{\mathbf{m}_k}|_{\mathbf{m}} < 1$ the argument $(*)$ implies that $A^h_{\mathbf{m}_k} \not\models q_j(x) \le q_j'(x)$. It follows that $q_j(x) \in \mathbf{m}_k$ and $q_j'(x) \notin \mathbf{m}_k$ hence $A \not\models q_j(x) \le q_j'(x)$.

■

4 The p-adic case

Let us recall from the literature some facts about formally p-adic fields (see [4]). Let p be a fixed prime number, a **p-adically closed** field is a field elementary equivalent to some finite extension of the p-adic field $\mathbf{Q}_p$. Such a field K is the fraction field of a unique henselian valuation ring R, definable in K, whose residue field is a finite extension of the field $\mathbf{F}_p$ with p elements, and whose maximal ideal is generated by an element π such that $|p|_R = |\pi^e|_R$ for some $e \in \mathbf{N}^*$. The ring R is called the **holomorphy ring** of K.

The complete theory of any p-adically closed field is recursively axiomatisable and model-complete. It follows that if $K \subseteq K'$ are elementary equivalent p-adically closed fields with holomorphy ring R, R' then $R' \cap K = R$, a remark that we will often use in what follows.

Until the end of this section let F be a fixed finite extension of $\mathbf{Q}_p$, and denote by T_F its complete theory. Similarly we let TH_F (resp. $TH_{F,\forall}$) denote the complete (resp. universal) theory of the holomorphy ring of F (in the language of rings). It is well known that given any model K of T_F, a subfield K_0 of K is a model of T_F if and only of K is a regular extension of K_0. In particular $\mathrm{Alg}(K)$ the subfield of algebraic elements of K is a model of T_F.

Prestel and Schmid [5] called $\mathbf{r}_\circ$**-domains** those r-domains $(A, \leq)$ for wich $(0) \in \mathrm{Spec}(A, \leq)$, that is $(A, \leq) \models \forall a,\ 0 \leq a \rightarrow a = 0$. Let $T(F)$ denote the theory of $\mathrm{r}_\circ$-domains $(A, \leq)$ such that $A \models TH_{F,\forall}$. It is a universal theory and *a fortiori* an inductive theory, hence any model of $T(F)$ embeds in an existentially closed model.

Theorem 4.1 *Existentially closed models $(A, \leq)$ of $T(F)$ are exactly those which satisfy the following conditions:*

1. *A satisfies the LGPH;*

2. $\mathrm{Frac}(A)$ *models T_F;*

3. *$L(A)$ is dense;*

4. *The radical relation of $(A, \leq)$ is induced by* $\mathrm{Specmax}(A)$.

They form an elementary class, whose theory $T(F)^*$ is model-complete: it is the model companion of $T(F)$.

This theorem, like all the results wich are proved in this section, has almost been proved by Prestel and Schmid in [6]. Here "almost" means that their local-global principle was not the LGPH, but if we replace it by the LGPH all the arguments used in [6] (wich were actually taken from [5]) carry over with only little change. In what follows we refer the reader to [5] for the easiest steps and rewrite the main steps.

The reader may have noticed that the second assertion of 4.1 ($T(F)^*$ is well defined) has already been proved: as $\mathrm{Frac}(A)$ and $L(A)$ are interpretable in A, and the radical relation $\leq_A$ is definable in A, it is clear that conditions 2 to 4 are first order, and we have proved in 2.11 that the LGPH (condition 1) is recursively axiomatisable.

The arguments used in [5, theorem 3.3] apply here to prove that any existentially closed model of $T(F)$ is a model of $T(F)^*$. For example, let us check that any existentially closed model of $T(F)$ satisfies the LGPH.

Proof: Let $(A, \leq)$ be an existentially closed model of $T(F)$, and K its fraction field. Let V be a variety defined over K having a smooth point x_R with coordinates in R^h for every $R \in W$. Let x_V be a generic point of V over K, and $A' = A[x_V]$. For any non zero $\mathbf{p} \in \mathrm{Spec}(A, \leq)$, let $\mathbf{m}$ be a maximal ideal of A containing $\mathbf{p}$, and let R in W_A be such that $m(R) \cap A = \mathbf{m}$. Choose an elementary extension R' of R^h with transcendence degree over K greater than or equal to $\dim V$. Then the implicit function theorem applied to the smooth point x_R in R' gives a point $x'_R \in V(R')$ such that the transcendence degree of $K(x'_R)$ over K is $\dim V$, hence x'_R is a generic point of V over K. Since $A[x'_R]$ and $A[x_V]$ are isomorphic over A, we can identify them and consider that $A' \subseteq R'$. There is a prime ideal $\mathbf{q}'$ in R' such that $\mathbf{q}' \cap A = \mathbf{p}$ ($\mathbf{q}'$ is the ideal generated by $\mathbf{p}$ in R'). Let $\mathbf{p}' := \mathbf{q}' \cap A'$, endow A' with the radical relation defined by $\mathcal{P}' := \{\mathbf{p}' \mid \mathbf{p} \in \mathrm{Spec}(A, \leq) \setminus \{(0)\}\} \cup \{(0)\}$. As by construction the restriction map $\mathbf{p}' \mapsto \mathbf{p}' \cap A$ sends $\mathcal{P}'$ onto $\mathrm{Spec}(A, \leq)$, it follows that $(A', \leq)$ is an extension of $(A, \leq)$ (see [5, lemma 2.7]) hence $(A, \leq)$ is existentially closed in $(A', \leq)$. Since V has a smooth point with coordinates in A' (namely x_V) it then follows that $V^\circ(A) \neq \emptyset$, proving that the LGPH holds in A.

■

It remains to prove that $T(F)^*$ is model-complete. Let us recall the following basic properties of r-domains.

Lemma 4.2 *Let $(A, \leq) \subseteq (A', \leq)$ with A, A' Bezout.*

1. *For every $\mathbf{p}' \in \mathrm{Spec}(A', \leq)$, $\mathbf{p}' \cap A$ belongs to $\mathrm{Spec}(A, \leq)$.*
2. *There is a natural lattice-embedding of $L(A, \leq)$ into $L(A', \leq)$. We will use it to identify $L(A, \leq)$ as a sub-lattice of $L(A', \leq)$.*
3. *Given another extension $(A'', \leq)$ of $(A, \leq)$ with A'' Bezout and an order-isomorphism σ between $L(A', \leq)$ and $L(A'', \leq)$ whose restriction to $L(A, \leq)$ is identity, define:*
$$\hat{\sigma} : \mathbf{p}'' \mapsto \{a' \in A' \mid \mathbf{p}'' \in \sigma(P(a'))\}.$$
Then $\hat{\sigma}$ is a bijection between $\mathrm{Spec}(A'', \leq)$ and $\mathrm{Spec}(A', \leq)$, and $\mathbf{p}'' \cap A = \hat{\sigma}(\mathbf{p}'') \cap A$ for every $\mathbf{p}'' \in \mathrm{Spec}(A'', \leq)$.

Proof: (1) is straightforward, (2) is [5, lemma 2.10 (e)] and (3) is essentially [5, proposition 2.8].

■

Since a ring A satisfying the LGPH has the block factorisation property, it follows that for any non zero a and any b in A there is c in A such that $M(a) \setminus M(b) = M(c)$. The argument used in the proof of [5, lemma 3.5 (viii)] then proves that:

$$\mathrm{Spec}(A, \leq_A) \subseteq \mathrm{Specmax}(A) \cup \{(0)\}.$$

It follows that for any a, b in A with $a \neq 0$ there is c in A such that $P(a) \setminus P(b) = P(c)$ hence $L(A, \leq_A)$ is a relatively complemented lattice (that is a distributive lattice L such that for any element α of L except maybe the maximal element, the sublattice of elements of of L smaller than α is a Boolean algebra).

The following result can be proved along the lines of [5, page 194]. For a complete proof see [1, Annexe A].

Lemma 4.3 *The theory of dense relatively complemented lattices is complete and model-complete (it is actually $\aleph_0$-categorical).*

We can now finish the proof of 4.1 by proving that $T(F)^*$ is model-complete. Notice that for any model $(A, \leq)$ of $T(F)^*$, from $A \models TH_{F,\forall}$ and $\mathrm{Frac}(A) \models T_F$ we infer that the holomorphy ring R of $\mathrm{Frac}(A)$ contains A. Moreover $R \in W_A$ since $A/(m(R) \cap A)$ is a subring of the finite field $R/m(R)$, hence $A/(m(R) \cap A)$ is equal to the field itself, that is $m(R) \cap A$ is a maximal ideal of A.

Proof: (of 4.1) Let $(A,\leq)\subseteq(A',\leq)$ be two models of $T(F)^*$. Let K, K' be their fraction fields, and $R_\circ$, $R'_\circ$ their holomorphy rings. Since T_F is model-complete and $R'_\circ$ (resp. $R_\circ$) is definable in K (resp. K') we have $R'_\circ\cap K=R_\circ$, and $R'_\circ$ is an elementary extension of $R_\circ$. Let $\varphi(x)$ be a quantifier free formula in $\mathcal{L}_{\rm rad}$ with parameters in A and n free variables, such that $(A',\leq)\models\exists x\varphi(x)$. We have to find a in A^n such that $\varphi(a)$ holds in $(A,\leq)$.

As $(A,\leq)$ and $(A',\leq)$ are both $\mathrm{r}_\circ$-domains, they have the following property:

$$a\leq b\iff b=0\ \text{ or }\ 0\neq a\leq b.$$

We can therefore suppose $\varphi(x)$ of the following kind:

$$\varphi(x)\equiv\bigwedge_{i=1}^{m}f_i(x)=0\wedge g(x)\neq 0\wedge\bigwedge_{i=1}^{u}0\neq p_i(x)\leq p_i'(x)\wedge\bigwedge_{i=1}^{v}q_i(x)\not\leq q_i'(x)$$

with f_i,g,p_i,p_i',q_i,q_i' in $A[X]$ $(X=(X_1,\dots,X_n))$.

Let $a'\in A'^n$ be such that $(A',\leq)\models\varphi(a')$. As K' is a regular extension of K, a' is the generic point of a variety V defined over K. As for any $f\in I(V)$ and any point x of V we have $f(x)=0$, it is sufficient to find a point $a\in V(A)$ such that $g(a)\neq 0$, $p_i(a)\leq p_i'(a)$ for $i\leq u$ and $q_j(a)\not\leq q_j'(a)$ for $j\leq v$. In order to achieve this goal we are going to use 3.3, hence it is worthwhile to notice that since $g(a')$ and the $p_i(a')$'s are non zero, neither g nor the p_i's belong to the ideal of V.

Since the theory of dense relatively complemented lattices is model complete by 4.3, $L(A)\preceq L(A')$ hence they have some ultrapowers wich are isomorphic over $L(A)$. As $L(A)$ is interpretable in A, the residual lattice of an ultrapower of A is itself an ultrapower of $L(A)$, and the same holds for A'. Hence the residual lattices of some elementary extensions of $(A,\leq)$ and $(A',\leq)$ are isomorphic over $L(A)$. Let $(A'',\leq)$ and $(A''',\leq)$ be such elementary extensions of $(A,\leq)$ and $(A',\leq)$ respectively. Replacing $(A',\leq)$ by $(A''',\leq)$ if necessary we can suppose there is an order-isomorphism σ from $L(A')$ to $L(A'')$ whose restriction to $L(A)$ is identity.

Let K'' be the fraction field of A'', and $R''_\circ$ its holomorphy ring: recall that since $(A'',\leq)$ is an elementary extension of $(A,\leq)$ it is a model of $T(F)^*$, hence $K''\models T_F$ and $R''_\circ\cap K=R_\circ=R'_\circ$. The bijection $\hat\sigma$ between $\mathrm{Spec}(A'',\leq)$ and $\mathrm{Spec}(A',\leq)$ given by 3 of 4.2 sends obviously the zero ideal of A'' to the zero ideal of A', hence it induces a bijection between $\mathrm{Specmax}(A'')$ and $\mathrm{Specmax}(A')$. Moreover $\mathbf{m}''\cap A=\hat\sigma(\mathbf{m}'')\cap A$ for every maximal ideal $\mathbf{m}''$ of A''. We can compose $\hat\sigma$ with the permutation of of $\mathrm{Specmax}(A')$ wich exchanges $\hat\sigma(m(R''_\circ)\cap A'')$ with $m(R'_\circ)\cap A'$, and fixes every other maximal ideals of A'. Since we have:

$$\hat\sigma(m(R''_\circ)\cap A'')\cap A=m(R''_\circ)\cap A=m(R_\circ)\cap A=m(R'_\circ)\cap A,$$

it follows that $\mathbf{m}''\cap A=(\tau\circ\hat\sigma(\mathbf{m}''))\cap A$ for every maximal ideal $\mathbf{m}''$ of A''. Replacing $\hat\sigma$ by $\tau\circ\hat\sigma$ if necessary, we can then suppose that $\hat\sigma$ sends $m(R''_\circ)\cap A$ to $m(R'_\circ)\cap A'$.

Fix any $\mathbf{m}''\in\mathrm{Specmax}(A'')$ and let $R'':=A_{\mathbf{m}''}$, $R':=A'_{\hat\sigma(\mathbf{m}'')}$ and $R:=R''\cap K=R'\cap A$. We can consider R^h as a common subring of R'^h and R''^h.

If $\mathbf{m}''\neq m(R''_\circ)\cap A''$ then R'' and $R''_\circ$ are independent by 1.7, hence R''^h has an algebraically closed fraction field (it is a well-known property of henselian fields that the henselisation of any valuation ring independent from the henselian one has a separably closed fraction field). The same holds for R'^h and since the theory of non trivial valuation rings with algebraically closed fraction fields has quantifier elimination it follows that

R''^h and R'^h are elementary equivalent over R^h (that is in the language of rings enriched with a constant symbol for each element of R^h).

If $\mathbf{m}'' = m(R''_\circ) \cap A$, by construction $R^h \preceq R'^h$ and $R^h \preceq R''^h$ as all three are models of TH_F, hence R'^h and R''^h are elementary equivalent over R^h (in this case $R = R^h$ but we prefer to write R^h in order to set a statement uniform in R, and similarly for R', R'').

Let $\{h_1, \dots, h_r\}$ be an enumeration of all the p_i, p'_i, q_j, q'_j. Let $\Omega_{R'}$ be the set of indices $i \le r$ such that $h_i(a')$ belongs to $m(R')$, and denote by $\Omega^c_{R'}$ its complement in $\{1, \dots, r\}$. As a' is a generic point of V it is a smooth point, hence there is a determinant Δ with an appropriate size to caracterise smooth points of V such that $\Delta(a') \ne 0$. The henselisation R'^h models $\exists x \varphi_{R'}(x)$ where $\varphi_{R'}(x)$ is the following formula, with parameters in R^h:

$$\bigwedge_{i=1}^{r} f_i(x) = 0 \wedge \Delta(x) \ne 0 \wedge \bigwedge_{i \in \Omega_{R'}} \forall y_i,\ y_i h_i(x) \ne 1 \wedge \bigwedge_{i \in \Omega^c_{R'}} \exists y_i,\ y_i h_i(x) = 1.$$

Since R''^h is elementary equivalent to R'^h over R^h, the same holds in R''^h hence there is $x_{\mathbf{m}''} \in V^\circ(R''^h)$ such that for every $i \le r$:

$$h_i(x_{\mathbf{m}''}) \in m(R''^h) \iff h_i(a') \in m(R'^h).$$

In particular, $A''^h_{\mathbf{m}''} \models p_i(x_{\mathbf{m}''}) \le p'_i(x_{\mathbf{m}''})$ for every $i \le u$, and for every $j \le v$ there is $\mathbf{n}'' \in \mathrm{Specmax}(A'')$ such that $A''^h_{\mathbf{n}''} \models q_i(x_{\mathbf{n}''}) \not\le q'_i(x_{\mathbf{m}''})$. Theorem 3.3 then gives a point $a'' \in V(A'')$ such that $g(a'') \ne 0$, $p_i(a'') \le p'_i(a'')$ for $i \le u$ and $q_j(a'') \not\le q'_j(a'')$ for $j \le v$, hence $(A'', \le) \models \exists x \varphi(x)$. As $(A, \le) \preceq (A'', \le)$ the same holds in $(A, \le)$ wich ends the proof.

■

In [5, p 195] Prestel and Schmid defined the **algebraic part** $\mathrm{Alg}(A)$ of an integral domain A to be the subring of elements of A wich are algebraic over the subfield of $\mathrm{Frac}(A)$. Along the lines of [5, theorem 4.5] one can deduce from 4.1:

Corollary 4.4 *Two models of $T(F)^*$ are elementary equivalent if and only if their algebraic parts are isomorphic.*

We can now conclude that the ring $\tilde{\mathbf{Z}} \cap \mathbf{Q}_p$ is decidable by setting $F = \mathbf{Q}_p$ in the following statement.

Corollary 4.5 *The ring $\tilde{\mathbf{Z}} \cap F$ is decidable.*
Moreover the complete theory of $(\tilde{\mathbf{Z}} \cap F, \le_{\tilde{\mathbf{Z}} \cap F})$ is decidable and model complete, it is axiomatised by $T(F)^*$ and the following family of axioms:
$\forall x, f(x) \ne 0$, *for every nonmonic irreducible f in $\mathbf{Z}[X]$.*

Proof: Let $A := \tilde{\mathbf{Z}} \cap F$, since the radical relation defined by the maximal spectrum is definable in the ring, the decidability of A follows from the decidability of $(A, \le_A)$. Let T denote the above theory, that is $T(F)^*$ plus the condition on nonmonic polynomials.

Moret-Bailly has proved in [3] that A models the LGPH. The arguments given in [6, proposition 2.1] apply here and prove that $L(A)$ is dense, hence $(A, \le_A)$ models $T(F)^*$. The additional conditions on nonmonic poynomials are obviously satisfied by A hence $(A, \le_A)$ models T.

Conversely any model $(A', \le)$ of T has a fraction field K' wich models T_F hence $\mathrm{Alg}(K') \simeq \mathrm{Alg}(F) \simeq \mathrm{Frac}(A)$. From the conditions on monic polynomials we infer that any element of $\mathrm{Alg}(A')$ is integral over $\mathbf{Z}$, and conversely since A' is integrally closed $\mathrm{Alg}(A')$ contains a copy of the integral closure of $\mathbf{Z}$ in $\mathrm{Alg}(K')$, that is a copy of A, hence $\mathrm{Alg}(A') \simeq A = \mathrm{Alg}(A)$. By 4.4 we conclude that $(A', \leq) \equiv (A, \leq)$.

Since $T(F)^*$ is model-complete so is T, and since T is given by a recursive family of axioms the conclusion follows.

■

Remark 4.6 We have tried in this section to follow closely the lines of Prestel and Schmid, because it is the shorter way to prove 4.5. From 4.5 it is immediate that for any non zero integer n the ring $(\tilde{\mathbf{Z}} \cap \mathbf{Q}_p)[1/n]$ is decidable. But a more general approach such as in [1, chapitre 5] gives necessary and sufficient condition of decidability for a larger class of rings between $\tilde{\mathbf{Z}} \cap \mathbf{Q}_p$ and its fraction field.
Let us mention here which result is actually proved with the local-global principle and the method of [5] in place of the LGPH: namely, that for any non zero integer n such that p divides n the ring $(\tilde{\mathbf{Z}} \cap \mathbf{Q}_p)[1/n]$ is decidable.

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