FORMAL STRUCTURE OF DIRECT IMAGE OF SOME $\mathcal{D}$-MODULES

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Abstract. We compute invariants associated with the formal irregular part of some geometric $\mathcal{D}$-modules, it being the cohomology modules of the direct image of $\mathcal{D}$-modules of exponential type. We describe which exponentials appear in the formal decomposition and we compute the rank and the characteristic polynomial of the monodromy of each factor in the decomposition. In particular, we construct the Newton polygon of these modules.

1. Introduction

In this article, we compute invariants associated with the formal irregular part of some geometric $\mathcal{D}$-modules.

A first example of an irregular $\mathcal{D}$-module is a module of exponential type. It consists in twisting a regular holonomic $\mathcal{D}$-module by the exponential of a meromorphic function. This means that if $\bigtriangledown$ is the connection associated with the regular holonomic $\mathcal{D}$-module, $\bigtriangledown + dg$ is the connection associated with the $\mathcal{D}$-module twisted by e^g, where g is a meromorphic function. The behaviour of this module along the poles of g prevents it from being regular. Definitions of modules of exponential type can be given in the algebraic, analytic or formal setting.

These modules of exponential type are important for the study of the $\mathbb{C}[t]\langle\partial_t\rangle$-modules (or $\mathbb{C}\{t\}\langle\partial_t\rangle$-modules). Indeed, the formalization of any $\mathcal{D}$-modules in one variable can be expressed as the direct sum of a regular $\mathbb{C}[[t]]\langle\partial_t\rangle$-module and a purely irregular $\mathbb{C}[[t]]\langle\partial_t\rangle$-module (cf. [7]). Then, after a convenient ramification, the formal irregular part can be written as the direct sum of formal modules of exponential type.

In this article, we are interested in the formal irregular part of the $\mathcal{D}$-modules which are obtained as the direct image of modules of exponential type.

Let B_1 be a small disc centered at 0 in $\mathbb{C}$, $p_1 : B_1 \times \mathbb{P}^1 \to B_1$ and $p_2 : B_1 \times \mathbb{P}^1 \to \mathbb{P}^1$ be the two canonical projections. We regard p_2 as a meromorphic function on $B_1 \times \mathbb{P}^1$. The first result concerns the formal decompostion of the $\mathcal{D}_{B_1,0}$-module $\mathcal{H}^0 p_{1+}(\mathcal{M}e^{p_2})_0$, where $\mathcal{M}$ is any regular holonomic $\mathcal{D}_{B_1\times\mathbb{P}^1}$-module.

In [12], the irregularity number of $\mathcal{H}^0 p_{1+}(\mathcal{M}e^{p_2})$ at 0 is computed using the characteristic cycle $\mathrm{Cch}(\mathcal{M})$ of $\mathcal{M}$ in the neighbourhood of $(0,\infty)$. In this article, we show how to recover the Newton polygon of the $\mathcal{D}_{B_1,0}$-module $\mathcal{H}^0 p_{1+}(\mathcal{M}e^{p_2})_0$ using $\mathrm{Cch}(\mathcal{M})$. Let us describe this construction.

Key words and phrases. Formal structure, direct image of $\mathcal{D}$-modules.

Let $\mathrm{Cch}(\mathcal{M})$ be the characteristic cycle of $\mathcal{M}$ in some neighbourhood $B = B_1^{'} \times B_2 \subset B_1 \times \mathbb{P}^1$ of $(0, \infty)$. As $\mathcal{M}$ is holonomic, if B is small enough, $\mathrm{Cch}(\mathcal{M})$ takes the form: $mT^*_B B + m_1 T^*_{B_1^{'} \times \{\infty\}} B + m_2 T^*_{\{0\} \times B_2} B + mT^*_{(0,\infty)} B + \sum_{\ell \in \Lambda} m_\ell T^*_{Z_\ell} B$, where Z_ℓ are germs at $(0, \infty)$ of irreducible curves distincts from $B_1^{'} \times \{\infty\}$ and $\{0\} \times B_2$.

Let Q be the second quadrant of $\mathbb{R}^2$ (i.e. the set of (u, v) in $\mathbb{R}^2$ such that $u \leq 0$ and $v \geq 0$). For $\ell \in \Lambda$, let us set $\rho_\ell = \frac{q_\ell}{p_\ell}$, where p_ℓ is the intersection multiplicity at $(0, \infty)$ of Z_ℓ with $\{0\} \times B_2$ and q_ℓ is the intersection multiplicity at $(0, \infty)$ of Z_ℓ with $B_1^{'} \times \{\infty\}$. We denote by N_ℓ the convex hull in $\mathbb{R}^2$ of $Q \cup ((m_\ell p_\ell, m_\ell q_\ell) + Q)$ (it is a polygon with one edge of slope ρ_ℓ and height $m_\ell q_\ell$).

THEOREM 1.1. *After a convenient translation, the Newton polygon of the $\mathcal{D}_{B_1,0}$-module $\mathcal{H}^0 p_{1+}(\mathcal{M} e^{p_2})_0$ is the convex hull of $\sum_{\ell \in \Lambda} N_\ell$.*

In particular, the slopes of this Newton polygon are in $\{\rho_\ell,\ \ell \in \Lambda\}$ and the height of the edge of slope ρ is $h_\rho = \sum_{\{\ell \in \Lambda | \rho_\ell = \rho\}} m_\ell q_\ell$.

We also give more details about the formal irregular part of the $\mathcal{D}_{B_1,0}$-module $\mathcal{H}^0 p_{1+}(\mathcal{M} e^{p_2})_0$. After a convenient ramification, the formal irregular part of $\mathcal{H}^0 p_{1+}(\mathcal{M} e^{p_2})_0$ can be written as $\oplus_{\beta(1/\tau) \in \Gamma} M_\beta e^{\beta(1/\tau)}$, where τ is a local coordinate in the neighbourhood of 0, $\beta(1/\tau)$ is an element of $1/\tau.\mathbb{C}[1/\tau]$ and M_β are regular holonomic $\mathbb{C}[[\tau]]\langle \partial_\tau \rangle$-modules such that $M_\beta[\frac{1}{\tau}] = M_\beta$. In Theorem 5.2, we describe explicitly which exponentials appear in this decomposition using the germs Z_ℓ. We also compute the rank of M_β and the characteristic polynomial of the monodromy associated with M_β.

A motivation for these computations is the question whether every formal $\mathbb{C}[[t]]\langle \partial_t \rangle$-modules can be expressed as the formalization of modules of the type $\mathcal{H}^0 p_{1+}(\mathcal{M} e^{p_2})_0$, where $\mathcal{M}$ is regular holonomic. More generally, we can ask whether every $\mathbb{C}\{t\}\langle \partial_t \rangle$-modules are isomorphic to a module of the type $\mathcal{H}^0 p_{1+}(\mathcal{M} e^{p_2})_0$.

The second result concerns the direct image of modules of exponential type in the algebraic setting. Let U be a smooth affine variety, $f, g : U \to \mathbb{C}$ be two regular functions and $\mathcal{M}$ be a regular holonomic $\mathcal{D}_U^{\mathrm{alg}}$-module. We compute invariants (the same as below) associated with the formal irregular part of the $\mathcal{D}_{\mathbb{C}}^{\mathrm{alg}}$-module $\mathcal{H}^k f_+(\mathcal{M} e^g)$ at its irregular singularities. As we are also interested in the behaviour at infinity of this cohomology module, we will consider its extension to $\mathbb{P}^1$. Let j be the inclusion of $\mathbb{C}$ in $\mathbb{P}^1$ and i be the inclusion of $\mathbb{C}^2$ in $\mathbb{P}^1 \times \mathbb{P}^1$. We will describe the invariants associated with the formal irregular part of the module $\mathcal{H}^k j_+ f_+(\mathcal{M} e^g)$ using the module $\mathcal{H}^k i_+(f, g)_+(\mathcal{M})$.

In the case where $f, g : \mathbb{C}^2 \to \mathbb{C}$ are two polynomial maps which are algebraically independent and $\mathcal{M}$ is the sheaf $\mathcal{O}_{\mathbb{C}^2}^{\mathrm{alg}}$ of regular functions on $\mathbb{C}^2$, the irregularity number at $c \in \mathbb{P}^1$ of $\mathcal{H}^0 j_+ f_+(\mathcal{O}_{\mathbb{C}^2}^{\mathrm{alg}} e^g)$ can be expressed only in terms of the geometry of the map (f, g) (cf. [11]). We can also describe the exponential factor which appear in the formal irregular part of $\mathcal{H}^0 j_+ f_+(\mathcal{O}_{\mathbb{C}^2}^{\mathrm{alg}} e^g)$ at $c \in \mathbb{P}^1$ (cf. Remark 6.4).

Let W be the maximal Zariski open subset of $\mathbb{C}^2$ such that $(f, g) : (f, g)^{-1}(W) \to W$ is a covering. Let $\mathcal{C}$ be the complement of W in $\mathbb{C}^2$.

Let Z_ℓ, $\ell \in \Lambda$, be the germs at (c,∞) of the local irreducible branches of $\overline{\mathcal{C} \setminus (\{c\} \times \mathbb{C})}$, where $^-$ means that we consider the Zariski closure in $\mathbb{P}^1 \times \mathbb{P}^1$. Then, the exponential factors can be expressed using a parametrization of the germs of curves Z_ℓ. If $c \in \mathbb{C}$, let $s \to (c + s^{p_\ell}, \alpha_\ell(1/s) + \delta(s))$ be a parametrization of Z_ℓ, where $\alpha_\ell(1/s) \in 1/s.\mathbb{C}[1/s]$ and $\delta(s) \in \mathbb{C}\{s\}$. Let N be the least common multiple of the p_ℓ. After the ramification $\tau \to \tau^N$, the exponential factors in the formal irregular part are $\exp\left(\alpha_\ell(1/\xi s^{N/p_\ell})\right)$, where $\xi^{p_\ell} = 1$. If $c = \infty$, the same result holds, up to the change of $c + s^{p_\ell}$ by $1/s^{p_\ell}$ in the parametrization of the Z_ℓ.

If $c \in \mathbb{C}$, we can also prove that the Z_ℓ's are the germs at (c,∞) of the local irreducible branches of $\overline{\Delta \setminus \{c\} \times \mathbb{C}}$, where Δ is the discriminant locus of f and g. At infinity, we have to consider $\overline{\Delta}$ and another curve which comes from the fact that the map (f,g) is not necessarily proper ouside of Δ.

In Section 3, we give a procedure to compute the formal irregular part of a $\mathbb{C}[t]\langle\partial_t\rangle$-module. After an analytization and a ramification, it consists in twisting by an exponential and finding the regular part of the module that we have obtained. While computing the formal regular part of a $\mathbb{C}\{t\}\langle\partial_t\rangle$-module, we will use the nearby cycles module.

In Section 4, we give the main result concerning the computation of the rank and the characteristic polynomial of the monodromy of a special nearby cycles module. These computations will be helpful in Section 5. During the proof, we will need some local computations which are stated in the appendix (Section 7).

In Section 5, we state the result about the formal irregular part of the module $\mathcal{H}^0 p_{1+}(\mathcal{M}e^{p_2})_0$. This theorem enables us to compute (in Section 6) the formal irregular part of the module $\mathcal{H}^k j_+ f_+(\mathcal{M}e^g)_c$.

2. Notations and conventions

If X is a smooth analytic (resp. algebraic) variety over $\mathbb{C}$, $\mathcal{D}_X$ (resp. $\mathcal{D}_X^{\text{alg}}$) denotes the sheaf of analytic (resp. algebraic) differential operators on X. Denote by $\mathcal{O}_X$ (resp. $\mathcal{O}_X^{\text{alg}}$) the sheaf of holomorphic (resp. regular) functions on X.

Given $\mathcal{M}$ a $\mathcal{D}_X^{\text{alg}}$-module, the analytization of $\mathcal{M}$ is denoted by $\mathcal{M}^{\text{an}}$.

We adopt the convention that the analytic de Rham complex and the direct image complex of a $\mathcal{D}_X$-module (or $\mathcal{D}_X^{\text{alg}}$-module) are concentrated in non-positive degree.

If $\mathcal{M}$ is a $\mathbb{C}\{t\}\langle\partial_t\rangle$-module (or $\mathbb{C}[t]\langle\partial_t\rangle$-module), the formalization of $\mathcal{M}$ is denoted by $\widehat{\mathcal{M}}$.

If $\alpha : X \to \mathbb{P}^1$ is a meromorphic function with poles along Y and $\mathcal{M}$ is a $\mathcal{D}_X$-module, we denote by $\mathcal{M}[*Y]e^\alpha$ the $\mathcal{O}_X$-module $\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{O}_X[*Y]e^\alpha$ provided with the structure of $\mathcal{D}_X$-module given by the tensor product. We can also define the twist by an exponential in the algebraic or formal setting.

If $\mathcal{C}^\bullet$ is a bounded complex of $\mathbb{C}$-vector spaces, we denote by $\chi(\mathcal{C}^\bullet)$ the Euler characteristics of $\mathcal{C}^\bullet$, $\chi(\mathcal{C}^\bullet) = \sum_{k\in\mathbb{Z}}(-1)^k \dim_{\mathbb{C}}(H^k\mathcal{C}^\bullet)$.

If $T : \mathcal{C}^\bullet \to \mathcal{C}^\bullet$ is an endomorphism of bounded complexes of $\mathbb{C}$-vector spaces, we denote by $\chi_T(\lambda) = \prod_{k\in\mathbb{Z}} \chi_k(\lambda)^{(-1)^k}$ the zeta function of T, where $\chi_k(\lambda)$ is the characteristic polynomial of $T : H^k\mathcal{C}^\bullet \to H^k\mathcal{C}^\bullet$.

3. Formal decomposition and nearby cycles modules

3.1. **Formal decomposition of $\mathcal{D}$-modules in one variable.** Let $\widehat{M}$ be a holonomic $\mathbb{C}[[t]]\langle\partial_t\rangle$-module. This module can be decomposed as the direct sum of a formal regular module and some formal modules of exponential type. Indeed:

THEOREM 3.1 (Formal decomposition theorem)**.** *After a convenient ramification (i.e. the change of $\widehat{M}$ by $\gamma^*(\widehat{M})$, where $\gamma : \tau \to t = \tau^N$), there exists a unique formal decomposition:*

$$\gamma^*(\widehat{M}) = N_R \oplus \big(\oplus_{\alpha(1/\tau)\in\Gamma} N_\alpha e^{\alpha(1/\tau)}\big),$$

where N_R, N_α are regular holonomic $\mathbb{C}[[\tau]]\langle\partial_\tau\rangle$-modules, $N_\alpha = N_\alpha[\frac{1}{\tau}]$ and Γ is a finite subset of $1/\tau.\mathbb{C}[1/\tau]$.

We will call N_R the formal regular part of $\gamma^*(\widehat{M})$ and $\oplus_{\alpha(1/\tau)\in\Gamma}N_\alpha e^{\alpha(1/\tau)}$ its formal irregular part.

This theorem is proved for linear differential systems in [15] and [4]. Another proof in [7] uses cyclic vector Lemma (cf. Theorem 1.2 p. 43 for meromorphic connections and Theorem 2.3 p. 51 for holonomic modules).

REMARK 3.2. *If M is a $\mathbb{C}[t]\langle\partial_t\rangle$-module (or a $\mathbb{C}\{t\}\langle\partial_t\rangle$-module), we can apply this theorem to its formalization $\widehat{M}$. It gives the formal decompostion of M. It is obvious that a $\mathbb{C}[t]\langle\partial_t\rangle$-module has the same formal decomposition as its analytization.*

REMARK 3.3. *While describing the formal irregular part of a $\mathbb{C}[t]\langle\partial_t\rangle$-module M, we will use the following procedure.*

- *We consider the analytization of the module ($M \to M^{\mathrm{an}}$).*
- *We ramify the module if it is necessary ($M^{\mathrm{an}} \to \gamma^*(M^{\mathrm{an}})$).*
- *We twist by an exponential. Let $\alpha(1/\tau) \in 1/\tau.\mathbb{C}[1/\tau]$. Then $e^{\alpha(1/\tau)}$ appears in the formal irregular part of $\gamma^*(M^{\mathrm{an}})$ if and only if the formal regular part of $\gamma^*(M^{\mathrm{an}})[\frac{1}{\tau}]e^{-\alpha(1/\tau)}$ is not trivial. With the notations of Theorem 3.1, this regular part is exactly N_α.*

Then we need a technique to separate the regular part and the formal irregular part of a $\mathcal{D}$-module. This technique uses the nearby cycles modules (cf. Section 3.2).

In Sections 5 and 6, we will need a result about the commutation of the direct image functor and the twist by an exponential. Let us state this lemma:

LEMMA 3.4. *Let X, Z be smooth analytic varieties and $p : X \times Z \to X$ be a proper projection. Let $\mathcal{M}^\bullet$ be a bounded complex of $\mathcal{D}_{X\times Z}$-modules with holonomic cohomology. Let $\alpha : X \to \mathbb{P}^1$ be a meromorphic function with poles along a hypersurface Y of X. Then we have an isomorphism of $\mathcal{D}_X$-modules: $p_+(\mathcal{M}^\bullet)[*Y]e^\alpha = p_+(\mathcal{M}^\bullet[*(Y \times Z)]e^{\alpha\circ p})$.*

Proof. According to the way out lemma (cf. [8] p. 240), we are led to prove this in the case of a single module $\mathcal{M}$. As p is a proper projection, we have:

$$\begin{aligned}
p_+(\mathcal{M}[*(Y\times Z)]e^{\alpha\circ p}) &= Rp_*(\mathrm{DR}_{X\times Z/X}(\mathcal{M}[(Y\times Z)]e^{\alpha\circ p})),\\
&= Rp_*(\mathrm{DR}_{X\times Z/X}(\mathcal{M})\otimes_{p^{-1}(\mathcal{O}_X)} p^{-1}(\mathcal{O}_X[*Z]e^{\alpha})),\\
&= p_+(\mathcal{M})[*Z]e^{\alpha}.
\end{aligned}$$

□

3.2. **Specialization and nearby cycles modules.** In this section, we recall the notion of specialization along a smooth hypersurface and of nearby cycles module. For a complete account of this subject, we refer to [1], [6], [9], [3] and [5].

Let X be a smooth analytic variety and Y be a smooth hypersurface of X. Denotes by $V_k\mathcal{D}_X$ the V-filtration of $\mathcal{D}_X$ along Y and let $\mathrm{gr}^V\mathcal{D}_X$ be the graded sheaf of rings $\oplus_{k\in\mathbb{Z}}V_k\mathcal{D}_X/V_{k-1}\mathcal{D}_X$.

We denote by E the element of $\mathrm{gr}_0^V\mathcal{D}_X$ which is locally defined by the class of $t\partial_t$, where (x,t) is a local system of coordinates on X such that the equation of Y is $t=0$. We remark that $(\mathrm{gr}_0^V\mathcal{D}_X)_{|Y}$ is isomorphic to $\mathcal{D}_Y[E]$.

Let $\mathcal{M}$ be a Y-specializable $\mathcal{D}_X$-module. We denote by $\{V_\beta\mathcal{M}\}_{\beta\in\mathbb{C}}$ the canonical V-filtration of $\mathcal{M}$ along Y. We define a total order on $\mathbb{C}$ by the lexicographical order on $\mathbb{R}+i\mathbb{R}$. Let $V_{<\beta}=\cup_{\alpha<\beta}V_\alpha$.

DEFINITION 3.5. *We denotes by $\mathrm{gr}_\beta^V\mathcal{M}$ the quotient $V_\beta\mathcal{M}/V_{<\beta}\mathcal{M}$.*

- *The specialization of $\mathcal{M}$ along Y is the $\mathrm{gr}^V\mathcal{D}_X$-module $\mathrm{sp}_Y(\mathcal{M}) = \oplus_{\beta\in\mathbb{C}}\mathrm{gr}_\beta^V(\mathcal{M})$.*
- *The nearby cycles module of $\mathcal{M}$ along Y is the $\mathcal{D}_Y$-module $\Psi_Y(\mathcal{M}) = \oplus_{-1\leq\beta<0}\mathrm{gr}_\beta^V(\mathcal{M})$. It comes equipped with an endomorphism of monodromy $T:\Psi_Y(\mathcal{M})\to\Psi_Y(\mathcal{M})$, induced by the morphism $\exp(-2i\pi E)$.*

REMARK 3.6. *The specialization along a hypersurface is an exact functor from the category to Y-specializable $\mathcal{D}_X$-modules and the category of monodromic $\mathrm{gr}^V\mathcal{D}_X$-modules.*

In some special cases, we can work with the specialization or with the module of nearby cycles. Indeed:

PROPOSITION 3.7. *If Y is defined by a global equation $f=0$ and $\mathcal{M}=\mathcal{M}[\frac{1}{f}]$, giving the $\mathrm{gr}^V\mathcal{D}_X$-module $\mathrm{sp}_Y(\mathcal{M})$ is equivalent to giving the $\mathcal{D}_Y$-module $\Psi_Y(\mathcal{M})$ equipped with its monodromy.*

We recall that holonomic $\mathcal{D}_X$-modules are Y-specializable, for all smooth hypersurfaces Y of X. One can also define the specialization of a complex of $\mathcal{D}_X$-modules which has holonomic cohomology. One way to define it (cf. [9] p. 236) is to realize this complex by a complex of holonomic $\mathcal{D}_X$-modules and to apply the specialization to each term of the complex. Another approach is given in [3] (a generalisation to complexes with coherent cohomology). It has the advantage to have a good behaviour under direct image functor (cf. theorem 9.4.1 of [3]). Let us states a particular case of this theorem that we will need in Sections 5 and 6.

Let X, Z be two smooth analytic varieties. Let $\Pi : X \to Z$ be a proper holomorphic map. Let Y be a smooth hypersurface of Z such that $\Pi^{-1}(Y)$ is also a smooth hypersurface of X.

We denote by $\mathcal{D}^{\mathrm{alg}}_{X\to Z} = \mathrm{gr}^V \mathcal{O}_X \otimes_{\Pi^{-1}(\mathrm{gr}^V \mathcal{O}_Z)} \Pi^{-1}(\mathrm{gr}^V \mathcal{D}_Z)$. With this transfert module, we can define a new direct image functor $\overline{\Pi}_+$ between the category of complexes of $\mathrm{gr}^V \mathcal{D}_X$-modules and the category of complexes of $\mathrm{gr}^V \mathcal{D}_Z$-modules. If $M^\bullet$ is a complex of $\mathrm{gr}^V \mathcal{D}_X$-modules,

$$\overline{\Pi}_+(M^\bullet) = R\Pi_*(\mathcal{D}^{\mathrm{alg}}_{X\leftarrow Z} \otimes^{\mathbb{L}}_{\mathrm{gr}^V \mathcal{D}_X} M^\bullet).$$

Let $\mathcal{M}^\bullet$ be a complex of $\mathcal{D}_X$-modules with holonomic cohomology. Then $\mathcal{M}^\bullet$ is $\Pi^{-1}(Y)$-specializable, $\Pi_+(\mathcal{M}^\bullet)$ is Y-specializable and theorem 9.4.1 of [3] says:

THEOREM 3.8. $\mathrm{sp}_Y(\Pi_+\mathcal{M}^\bullet) = \overline{\Pi}_+(\mathrm{sp}_{\Pi^{-1}(Y)} \mathcal{M}^\bullet)$.

REMARK 3.9. *Theorem 3.8 enables us to state a result of commutation between direct image and nearby cycles functor. Indeed, if $\widetilde{\Pi} : \Pi^{-1}(Y) \to Y$ is the restriction of Π to $\Pi^{-1}(Y)$, there exists an isomorphism of $\mathcal{D}_Y$-modules $\mathcal{H}^k\widetilde{\Pi}_+(\Psi_{\Pi^{-1}(Y)}\mathcal{M}^\bullet) = \Psi_Y(\mathcal{H}^k\Pi_+\mathcal{M}^\bullet)$ which respect the monodromies (cf. Theorem (4.8-1) of* [9] *for modules and Proposition 9-2-5 of* [3] *for complexes).*

3.3. **Regular part of a $\mathbb{C}\{t\}\langle\partial_t\rangle$-module and nearby cycles module.** For $\mathcal{D}$-modules in one variable, considering the specialization enables us to compute the formal regular part of a module. Indeed, let M be a $\mathbb{C}\{t\}\langle\partial_t\rangle$-module. According to the formal decomposition theorem, we know that there exists two $\mathbb{C}[[t]]\langle\partial_t\rangle$-modules $M^{'}$ and $M^{''}$ such that $\widehat{M} = M^{'} \oplus M^{''}$, where $M^{'}$ is regular and $M^{''}$ is purely irregular.

PROPOSITION 3.10. *The formal regular part $M^{'}$ of M is isomorphic to $\mathrm{sp}_0(M) \otimes_{\mathbb{C}[t]} \mathbb{C}[[t]]$.*

(cf. Example 5-2-1 of [3])

REMARK 3.11. *While studying the formal irregular part of a $\mathbb{C}\{t\}\langle\partial_t\rangle$-module M, we have to determine each N_α. According to Remark 3.3 and Proposition 3.10, it is sufficient to compute $\mathrm{sp}_0(\gamma^*(M^{\mathrm{an}})[\frac{1}{\tau}]e^{-\alpha(1/\tau)})$.*

In this article, we want to compute the rank of N_α and the characteristic polynomial of its monodromy. As $\gamma^*(M^{\mathrm{an}})[\frac{1}{\tau}]e^{-\alpha(1/\tau)}$ is localised by τ, the rank of N_α is the dimension of the $\mathbb{C}$-vector space $\Psi_0(\gamma^*(M^{\mathrm{an}})[\frac{1}{\tau}]e^{-\alpha(1/\tau)})_0$ and the characteristic polynomial of its monodromy is equal to the one on $\Psi_0(\gamma^*(M^{\mathrm{an}})[\frac{1}{\tau}]e^{-\alpha(1/\tau)})_0$.

We will use to a large extent the complex of nearby cycles and its relation with the nearby cycles module. Indeed, in the regular case, the de Rham functor commute with the nearby cycles functor (cf. Theorem 4.10-1 of [9]):

THEOREM 3.12. *Let $\mathcal{M}$ be a regular holonomic $\mathcal{D}_X$-module. Let Y be a smooth hypersurface in X defined by a global equation $f = 0$. Then $\mathrm{DR}_Y(\Psi_Y(\mathcal{M}))$ is isomorphic to $R\Psi_f(\mathrm{DR}_X(\mathcal{M}))$ and the monodromy on $\Psi_Y(\mathcal{M})$ correspond by this isomorphism to the monodromy on the complex $R\Psi_f(\mathrm{DR}_X(\mathcal{M}))$.*

4. On nearby cycles module of certain $\mathcal{D}_{\mathbb{C}^2}$-modules of exponential type

In the next section, we willl need a computation of the characteristic polynomial of the monodromy of some nearby cycles modules. Some of these computations are stated in the appendix. In this section, we give the local computation needed in the next section.

Let (x, y) be coordinates on $\mathbb{C}^2$. Let $\mathcal{M}$ be a regular holonomic $\mathcal{D}_{\mathbb{C}^2}$-module and let $\alpha(1/x) \in 1/x.\mathbb{C}[1/x]$. We want to study the nearby cycles module $\Psi_{x=0}(\mathcal{M}[\frac{1}{xy}]e^{1/y-\alpha(1/x)})$.

Notation 4.1.
- *Let* $\mathrm{Cch}(\mathcal{M})$ *be the characteristic cycle of* $\mathcal{M}$ *in the neighbourhood of* $(0,0)$. *As* $\mathcal{M}$ *is holonomic,* $\mathrm{Cch}(\mathcal{M})$ *takes the form* $rT^*_{\mathbb{C}^2}\mathbb{C}^2 + n_x T^*_{x=0}\mathbb{C}^2 + n_y T^*_{y=0}\mathbb{C}^2 + nT^*_{(0,0)}\mathbb{C}^2 + \sum_{\ell\in\Lambda} n_\ell T^*_{Y_\ell}\mathbb{C}^2$, *where* Y_ℓ *are germs at* $(0,0)$ *of irreducible curves of* $\mathbb{C}^2$ *distincts from* $\{x = 0\}$ *and* $\{y = 0\}$. *In this section, we will assume that* **the intersection multiplicity at** $(0,0)$ **of** Y_ℓ **and of** $\{x = 0\}$ **is equal to 1**.
- *Let* $\Lambda_\alpha \subset \Lambda$ *be the set of* ℓ *such that there exists a parametrization of* Y_ℓ, $s \to (s, \delta(s))$, *where* $\delta(s) \in \mathbb{C}\{s\}$ *and such that* $1/\delta(s) = \alpha(1/s) + \gamma(s)$, *with* $\gamma(s) \in \mathbb{C}\{s\}$.
- *For* $\ell \in \Lambda_\alpha$, *we consider the complex of* $\mathcal{D}_{\mathbb{C}^2}$*-modules* $R\Gamma_{Y_\ell}(\mathcal{M})$. *The cohomology modules of this complex have support in* Y_ℓ *and are non trivial only in degree* 0 *and* 1. *Moreover, we have the exact sequence:* $0 \to R^0\Gamma_{Y_\ell}(\mathcal{M}) \to \mathcal{M} \to \mathcal{M}[*Y_\ell] \to R^1\Gamma_{Y_\ell}(\mathcal{M}) \to 0$. *Then, we have two local systems on* $Y_\ell \setminus \{(0,\infty)\}$*: for* $i = 0, 1$, *let* $\mathcal{L}^\ell_i = \mathrm{DR}(R^i\Gamma_{Y_\ell}(\mathcal{M}))_{|Y_\ell\setminus\{(0,\infty)\}}$. *As* Y_ℓ *is smooth, we can define* $\chi^i_\ell(\lambda)$, $i = 0, 1$, *the characteristic polynomial of the monodromy of* $\mathcal{L}^i_\ell$ *around the point* $(0,\infty)$. *Denote by* $\chi_\ell(\lambda)$ *the quotient* $\chi^1_\ell(\lambda)/\chi^0_\ell(\lambda)$. *As* Y_l *is smooth and transverse to* $\{x = 0\}$, *it consists in the zeta function of the monodromy on* $R\Psi_x(\mathrm{DR}\ \ R\Gamma_{Y_\ell}\mathcal{M}[+1])_{(0,0)}$.

THEOREM 4.2. *The Euler characteristic of the complex of* $\mathbb{C}$*-vector spaces* $DR\ \Psi_{x=0}(\mathcal{M}[\frac{1}{xy}]e^{1/y-\alpha(1/x)})_{(0,0)}$ *is equal to* $-\sum_{\ell\in\Lambda_\alpha} n_\ell$. *Moreover the zeta function of the monodromy on* $\mathrm{DR}\ \Psi_{x=0}(\mathcal{M}[\frac{1}{xy}]e^{1/y-\alpha(1/x)})_{(0,0)}$ *is equal to* $\prod_{\ell\in\Lambda_\alpha}\chi_\ell(\lambda)^{-1}$.

Proof. Let $f(x, y) = 1/y - \alpha(1/x)$. In the situation of Theorem 4.2, we can not compute directly the data we are looking for as in the appendix. The problem is that f is a rational map at $(0,0)$. Therefore we have to find a good resolution of the indeterminacy of f.

Resolution of the indeterminacy:

LEMMA 4.3. *There exists a resolution* $\pi : \mathbb{X} \to \mathbb{C}^2$ *(actually it is the finite composition of some blow up of points) such that:*

1. $f \circ \pi$ *is well-defined everywhere on* $\mathbb{X}$.
2. *Let* $E = \pi^{-1}(0,0)$ *be the exceptional locus of* π. *For all irreducible component* Z *of* E, $f_{|Z}$ *is constant equal to* ∞ *except for one irreducible component* Z_d *on which* f *is surjective.*
3. Z_d *intersects* $\overline{E \setminus Z_d}$ *in only one point which we will denote by* P.

(Loc P) *In the neighbourhood of* P*, we can choose local coordinates* (u,v) *on* $\mathbb{X}$ *such that:*
- $u=0$ *is the equation of* Z_d*,*
- $v=0$ *is the equation of the other component of* E*,*
- $\pi_1(u,v)=uv^\lambda$*,* $\lambda\in\mathbb{N}^*$*,*
- $f\circ\pi(u,v)=h_1(u,v)/v$*,* h_1 *invertible.*

(Loc Q) *In the neighbourhood of* $Q\in Z_d\setminus\{P\}$*, we can choose local coordinates* (u,v) *on* $\mathbb{X}$ *such that:*
- $u=0$ *is the equation of* Z_d*,*
- $\pi_1(u,v)=u$*,*
- $f\circ\pi(u,v)=a+v$*,* $a\in\mathbb{C}$*.*

(4) The singular support of $\pi^*(\mathcal{M})$ *has normal crossing in the neighbourhood of any point of* $\overline{E\setminus Z_d}$*.*

Proof. Let us begin by proving the first three points. Let us set $\alpha(1/x)=\beta(x)/x^k$, where $\beta(x)\in\mathbb{C}[x]$ and $\beta(0)\neq 0$. Then we have $f(x,y)=(x^k-y\beta(x))/(x^ky)$.

As $(0,0)$ is the indeterminacy point of f, we blow it up. In the chart $x=st$, $y=t$, f is well-defined and is equal to ∞ on the exceptional locus. In the second chart $x=s$, $y=st$, f becomes $(s^{k-1}-t\beta(s))/(s^kt)$.

After $(k-1)$ similar blow up, we obtain a resolution $\pi:\mathbb{X}\to\mathbb{C}^2$ such that $f\circ\pi$ is well-defined everywhere on $\mathbb{X}$ except at a point I and it takes the value ∞ on the exceptional locus except at I. In the neighbourhood of I, we can choose local coordinates (s,t) such that $\pi_1(s,t)=s$ and $f\circ\pi(s,t)=(-t\beta(s)+s\gamma(s))/(s^kh(t))$, where $h(t)$ is invertible and $\gamma(s)$ is a polynomial such that $\deg_s\gamma(s)\leq k-1$.

Then we blow I up. In the chart $s=uv$, $t=v$, f is well-defined at $(0,0)$ and is equal to ∞. In the chart $s=u$, $t=uv$, f is well-defined and takes the value ∞ on the exceptional locus except at a point $\widetilde{I}$ which we have to blow up. In the neighbourhood of this point, we can choose local coordinates (u,v) such that $\pi_1(u,v)=u$ and $f\circ\pi(u,v)=(-v\beta(u)+u\widetilde{\gamma}(u))/(u^{k-1}\widetilde{h}(u,v))$, where $\widetilde{h}(u,v)$ is invertible and $\widetilde{\gamma}(u)$ is a polynomial such that $\deg_u\widetilde{\gamma}(u)\leq k-2$.

After $(k-1)$ more similar blow up, we obtain the first three points of the lemma. For the last one, we have to do additional blow up. □

Now, we replace the exponential of a rational map by the exponential of a meromorphic map using the lemma:

LEMMA 4.4. $\mathcal{M}[\frac{1}{xy}]e^f=\pi_+(\pi^*(\mathcal{M})[*\pi^{-1}(xy=0)]e^{f\circ\pi})$.

Proof. • π is an isomorphism out of E and $\mathcal{M}[\frac{1}{xy}]e^f$ is holonomic. Then there exists an isomorphism (cf. proposition 7.4.5 of [8]):

$$\mathcal{M}[\frac{1}{xy}]e^f=\pi_+\mathbb{L}\pi^*(\mathcal{M}[\frac{1}{xy}]e^f).$$

- We have also: $\mathbb{L}\pi^*(\mathcal{M}[\frac{1}{xy}]e^f)[*E]=\pi^*(\mathcal{M}[\frac{1}{xy}]e^f)[*E]$.
- Then
$$\begin{aligned}\mathcal{M}[\tfrac{1}{xy}]e^f &= \pi_+(\pi^*(\mathcal{M}[\tfrac{1}{xy}]e^f)[*E]),\\ &= \pi_+(\pi^*(\mathcal{M})[*\pi^{-1}(xy=0)]e^{f\circ\pi}).\end{aligned}$$

□

Reduction to the local computations of the appendix: Now we consider the complex DR $\Psi_{x=0}(\pi_+(\pi^*(\mathcal{M})[*\pi^{-1}(xy=0)]e^{f\circ\pi}))$. First we want to apply the theorem of commutation between nearby cycles and direct image functor. However, as $\pi^{-1}(x=0)=\pi_1^{-1}(0)$ is not smooth, we can not do it directly. In this case, we usually consider the embedding by the graph of π_1:

$$\begin{array}{ccc} \mathbb{X} & \xrightarrow{\pi} & \mathbb{C}^2 \\ i \downarrow & & \downarrow j \\ \mathbb{X}\times\mathbb{C} & \xrightarrow{\widehat{\pi}} & \mathbb{C}^2\times\mathbb{C}, \end{array}$$

where $i=(id_{\mathbb{X}},\pi_1)$, $\widehat{\pi}=(\pi,id_{\mathbb{C}})$ and $j(x,y)=(x,y,x)$.

Let us set $\mathcal{P}=\pi^*(\mathcal{M})[*\pi^{-1}(xy=0)]$.

LEMMA 4.5. *There exists an isomorphism of complexes of* $\mathbb{C}$*-vector spaces which respects the monodromies:*
DR $\Psi_{x=0}(\pi_+(\mathcal{P}e^{f\circ\pi}))_{(0,0)}=R\Gamma(E\times\{0\},\text{DR }\Psi_{\mathbb{X}\times\{0\}}(i_+(\mathcal{P}e^{f\circ\pi})))[+1]$.

Proof. We consider the two following diagrams:

$$\begin{array}{ccc} \{0\}\times\mathbb{C} & \subset & \mathbb{C}^2 \\ \downarrow \widehat{j} & & \downarrow j \\ \mathbb{C}^2\times\{0\} & \subset & \mathbb{C}^2\times\mathbb{C} \end{array}\quad,\quad \begin{array}{ccc} \mathbb{X}\times\{0\} & \subset & \mathbb{X}\times\mathbb{C} \\ \downarrow \widetilde{\pi} & & \downarrow \widehat{\pi} \\ \mathbb{C}^2\times\{0\} & \subset & \mathbb{C}^2\times\mathbb{C} \end{array}\quad.$$

Then we have the following isomorphisms of complexes of $\mathbb{C}$-vector spaces with respect to the monodromies:

$$\begin{aligned} &\text{DR }\Psi_{x=0}(\pi_+(\mathcal{P}e^{f\circ\pi}))_{(0,0)}=\\ &\quad=R\widehat{j}_*(\text{DR }\Psi_{x=0}(\pi_+(\mathcal{P}e^{f\circ\pi})))_{(0,0,0)},\\ &\quad=\text{DR }\widehat{j}_+\Psi_{x=0}(\pi_+(\mathcal{P}e^{f\circ\pi}))_{(0,0,0)}[+1],\\ &\quad=\text{DR }\Psi_{\mathbb{C}^2\times\{0\}}(j_+\pi_+(\mathcal{P}e^{f\circ\pi}))_{(0,0,0)}[+1],\ \text{(cf. Remark 3.9)}\\ &\quad=\text{DR }\Psi_{\mathbb{C}^2\times\{0\}}(\widehat{\pi}_+i_+(\mathcal{P}e^{f\circ\pi}))_{(0,0,0)}[+1],\\ &\quad=\text{DR }\widetilde{\pi}_+\Psi_{\mathbb{X}\times\{0\}}(i_+(\mathcal{P}e^{f\circ\pi}))_{(0,0,0)}[+1],\ \text{(cf. Remark 3.9)}\\ &\quad=R\widetilde{\pi}_*(\text{DR }\Psi_{\mathbb{X}\times\{0\}}(i_+(\mathcal{P}e^{f\circ\pi})))_{(0,0,0)}[+1],\\ &\quad=R\Gamma(E\times\{0\},\text{DR }\Psi_{\mathbb{X}\times\{0\}}(i_+(\mathcal{P}e^{f\circ\pi})))[+1]. \end{aligned}$$

□

Use of the local computations of the appendix: In order to compute the Euler characteristics of $R\Gamma(E\times\{0\},\text{DR }\Psi_{\mathbb{X}\times\{0\}}(i_+(\mathcal{P}e^{f\circ\pi})))$ and the zeta function of its monodromy, we will use the Mayer-Vietoris theorem and the local computations of the appendix. Let us first remark:

LEMMA 4.6. DR $\Psi_{\mathbb{X}\times\{0\}}(i_+(\mathcal{P}e^{f\circ\pi}))$ *has support in* $Z_d\times\{0\}$.

Proof. Let $p\in E\setminus Z_d$. We remark that $\mathcal{P}$ is regular and that the singular support of $\mathcal{P}$ in the neighbourhood of p is a normal crossing. We have two cases to consider:

(1) If p is the intersection point of two irreducible components of E, there exists some local coordinates (u,v,w) on $\mathbb{X}\times\mathbb{C}$ in the neighbourhood of $(p,0)$ such that:
   - $\pi_1(u,v)=u^m v^n$, $m,n\geq 1$.
   - $f\circ\pi(u,v)=1/u^k v^l$, $k,l\geq 1$.

   Then $\mathrm{DR}\ \Psi_{\mathbb{X}\times\{0\}}(i_+(\mathcal{P}e^{f\circ\pi}))_{(p,0)}=\mathrm{DR}\ \Psi_{w=0}(i_+(\mathcal{P})_{(p,0)}e^{1/u^k v^l})$, where $i:\mathbb{C}^2\to\mathbb{C}^3$ is defined by $i(u,v)=(u,v,u^m v^n)$. According to Lemma 7.3 1, this is equal to 0.

(2) If p is not an intersection point of two irreducible components of E, there exists some local coordinates (u,v) on $\mathbb{X}$ in the neighbourhood of p such that:
   - $\pi_1(u,v)=u^m$, $m\geq 1$.
   - $f\circ\pi(u,v)=1/u^k v^l$, $k\geq 1$ and $l\geq 0$.

   Then $\mathrm{DR}\ \Psi_{\mathbb{X}\times\{0\}}(i_+\mathcal{P}e^{f\circ\pi})_{(p,0)}=\mathrm{DR}\ \Psi_{u^m=0}(\mathcal{P}_p e^{1/u^k v^l})[-1]$. It is equal to 0 according to Lemma 7.1.

□

Finally, we have to compute the Euler characteristics of the complex $R\Gamma(Z_d\times\{0\},\mathrm{DR}\ \Psi_{\mathbb{X}\times\{0\}}(i_+(\mathcal{P}e^{f\circ\pi})))$ and the zeta function of its monodromy.

Notation 4.7.
- *Let $\widetilde{Y_\ell}$ be the strict transform of Y_ℓ.*
- *For all $\ell\in\Lambda$, let $\chi_\ell(\lambda)$ be the zeta function of the monodromy on $R\Psi_x\ \mathrm{DR}(R\Gamma_{Y_\ell}\mathcal{M}[+1])_{(0,0)}$.*

  If P_ℓ is the intersection point of Z_d and $\widetilde{Y_\ell}$ and $u=0$ is the equation of Z_d in the neighbourhood of P_ℓ, $\chi_\ell(\lambda)$ is also the zeta function of the monodromy on $R\Psi_u\ \mathrm{DR}(R\Gamma_{\widetilde{Y_\ell}}\mathcal{P}[+1])_{P_\ell}$.
- *Let $j:(\mathbb{C}^*)^2\setminus(\underset{\ell\in\Lambda}{\cup}Y_\ell)\to\mathbb{C}^2$ and $\chi_r(\lambda)$ be the zeta function of the monodromy on $R\Psi_x\ \mathrm{DR}(Rj_*j^{-1}\mathcal{M})_{(0,0)}$.*

  If $p\in Z_d$ and $u=0$ is the equation of Z_d in the neighbourhood of P_ℓ, $\chi_r(\lambda)$ is also the zeta function of the monodromy on $R\Psi_u\ \mathrm{DR}(R\tilde{j}_*\tilde{j}^{-1}\mathcal{P})_{P_l}$, where $\tilde{j}:\mathbb{X}\setminus(\underset{\ell\in\Lambda}{\cup}\widetilde{Y_\ell}\cup\pi^{-1}(xy=0))\to\mathbb{X}$.

Then we have to consider the following two cases:

(1) In the neighbourhood of P, according to Lemma 4.3,
$$\mathrm{DR}\ \Psi_{\mathbb{X}\times\{0\}}(i_+(\mathcal{P}e^{f\circ\pi}))_{(p,0)}=\mathrm{DR}\ \Psi_{w=0}(i_+(\mathcal{P})_{(p,0)}e^{1/v}),$$
   with $i(u,v)=(u,v,uv^\lambda)$. Moreover, the singular support of $\mathcal{P}$ in the neighbourhood of P is a normal crossing. Then according to Lemma 7.3 2, its Euler characteristics is r and the zeta function of its monodromy is $\chi_r(\lambda)$.

(2) In the neighbourhood of $Q\in Z_d\setminus\{P\}$, according to Lemma 4.3,
$$\mathrm{DR}\ \Psi_{\mathbb{X}\times\{0\}}(i_+\mathcal{P}e^{f\circ\pi})_{(Q,0)}=\mathrm{DR}\ \Psi_{u=0}(\mathcal{P})_Q[-1].$$
   Then according to Lemma 7.5, the Euler characteristics of this complex is $\sum_{\widetilde{Y_\ell}\cap Z_d=\{Q\}}n_\ell-r$ and the zeta function of its monodromy is $\left(\prod_{\widetilde{Y_\ell}\cap Z_d=\{Q\}}\chi_\ell(\lambda)\right)/\chi_r(\lambda)$.

According to the Mayer-Vietoris lemma, we obtain that the Euler characteristics of $R\Gamma(Z_d \times \{0\}, \mathrm{DR}\ \Psi_{\mathbb{X}\times\{0\}}(i_+\mathcal{P}e^{f\circ\pi}))$ is $\sum_{\widetilde{Y_\ell}\cap Z_d\neq\emptyset} n_\ell$ and the zeta function of its monodromy is $\sum_{\widetilde{Y_\ell}\cap Z_d\neq\emptyset} \chi_\ell(\lambda)$.

We have finished if we prove the lemma:

LEMMA 4.8. *Let $\ell \in \Lambda$. $\widetilde{Y}_\ell \cap Z_d \neq \emptyset$ if and only if $\ell \in \Lambda_\alpha$.*

Proof. First, according to the choice of the resolution, $\widetilde{Y}_\ell \cap Z_d \neq \emptyset$ if and only if $\widetilde{Y}_\ell \cap (Z_d \setminus \{P\}) \neq \emptyset$ (Lemma 4.3 4). Let (u, v) be some coordinates on Z_d in the neighbourhood of Q. According to lemma 4.3 3, we can choose these coordinates such that $f \circ \pi(u, v) = a + v$ and $\pi_1(u, v) = u$, with $a \in \mathbb{C}$.

Let $\widetilde{Y}_\ell$ which intersects Z_d. As the intersection multiplicity of Y_ℓ and $\{x = 0\}$ is 1 and $\pi_1(u, v) = u$, the intersection multiplicity of $\widetilde{Y}_\ell$ and Z_d is also 1. Then:

$\widetilde{Y}_\ell \cap Z_d \neq \emptyset$,

$\Longleftrightarrow \exists\delta(s) \in \mathbb{C}\{s\}$ such that $a + v = \delta(u)$ is an equation of $\widetilde{Y}_\ell$,

$\Longleftrightarrow \exists\delta(s) \in \mathbb{C}\{s\}$ such that $f \circ \pi(u, v) = \delta \circ \pi_1(u)$ is an equation of $\widetilde{Y}_\ell$,

$\Longleftrightarrow \exists\delta(s) \in \mathbb{C}\{s\}$ such that $f(x, y) = \delta(x)$ is an equation of Y_ℓ,

$\Longleftrightarrow \exists\delta(s) \in \mathbb{C}\{s\}$ such that $y = 1/\alpha(1/x) + \delta(x)$ is an equation of Y_ℓ,

$\Longleftrightarrow \exists\delta(s) \in \mathbb{C}\{s\}$ such that $s \to (s, 1/\alpha(1/s) + \delta(s))$ is a parametrization of Y_ℓ,

$\Longleftrightarrow \ell \in \Lambda_\alpha$.

□

□

5. On the formal decomposition of some geometric $\mathcal{D}$-modules

In this section, we state a result about the formal irregular part of some geometric $\mathcal{D}$-modules in one variable.

Let B_1 be a small disc centered at 0 in $\mathbb{C}$. Let $p_1 : B_1 \times \mathbb{P}^1 \to B_1$ and $p_2 : B_1 \times \mathbb{P}^1 \to \mathbb{P}^1$ be the two canonical projections. We identify $\mathbb{P}^1$ to $\mathbb{C} \cup \{\infty\}$ and regards p_2 as a meromorphic function.

We consider the complex of $\mathcal{D}_{B_1}$-modules $p_{1+}(\mathcal{M}e^{p_2})$, where $\mathcal{M}$ is a regular holonomic $\mathcal{D}_{B_1\times\mathbb{P}^1}$-module such that $\mathcal{M} = \mathcal{M}[*(B_1 \times \{\infty\})]$.

Before describing its formal irregular part, let us begin by some notations:

Notation 5.1.
- *Consider* $\mathrm{Cch}(\mathcal{M})$ *the characteristic cycle of* $\mathcal{M}$ *in some neighbourhood* $B = B_1' \times B_2 \subset B_1 \times \mathbb{P}^1$ *of* $(0, \infty)$. *As* $\mathcal{M}$ *is holonomic, if* B *is small enough, it takes the form:* $\mathrm{Cch}(\mathcal{M}) = mT^*_B B + m_1 T^*_{B_1'\times\{\infty\}} B + m_2 T^*_{\{0\}\times B_2} B + mT^*_{(0,\infty)} B + \sum_{\ell\in\Lambda} m_\ell T^*_{Z_\ell} B$, *where* Z_ℓ *are germs at* $(0, \infty)$ *of irreducible curves distincts from* $B_1' \times \{\infty\}$ *and* $\{0\} \times B_2$.
- *For all* $\ell \in \Lambda$, *let* p_ℓ *be the intersection multiplicity at* $(0, \infty)$ *of* Z_ℓ *with* $\{0\} \times \mathbb{P}^1$. *Denote by* N *the least common multiple of* p_ℓ, $\ell \in \Lambda$, *and let* $\gamma : \widetilde{B_1} \to B_1$ *defined by* $\gamma(\tau) = \tau^N$.

- *For $\ell \in \Lambda$, let $s \to (s^{p_\ell}, \alpha_\ell(1/s) + \delta(s))$ be a parametrization of Z_ℓ, where $\alpha_\ell(1/s) \in 1/s.\mathbb{C}[1/s]$ and $\delta(s) \in \mathbb{C}\{s\}$. Let $k_\ell = N/p_\ell$.*
- *Let Γ be the set of $\alpha_\ell(1/\xi\tau^{k_\ell}) \in 1/\tau.\mathbb{C}[1/\tau]$, where $\xi^{p_\ell} = 1$.*
- *For $\beta(1/\tau) \in \Gamma$, we denote by Λ_β the set of $\ell \in \Lambda$ such that there exists $\xi \in \mathbb{C}$, $\xi^{p_\ell} = 1$, which satisfies $\beta(1/\tau) = \alpha_\ell(1/\xi\tau^{k_\ell})$.*
- *For $\ell \in \Lambda$, we consider the complex of $\mathcal{D}_{B_1 \times \mathbb{P}^1}$-modules $R\Gamma_{Z_\ell}(\mathcal{M})$. The cohomology modules of this complex have support in Z_ℓ and are non trivial only in degree 0 and 1. Moreover, we have the exact sequence : $0 \to R^0\Gamma_{Z_\ell}(\mathcal{M}) \to \mathcal{M} \to \mathcal{M}[*Z_\ell] \to R^1\Gamma_{Z_\ell}(\mathcal{M}) \to 0$. We denote by $\chi^i_\ell(\lambda)$, $i = 0, 1$, the zeta function of the monodromy on $R\Psi_{p_1}(\mathrm{DR}\ \ R^i\Gamma_{Z_\ell}(\mathcal{M}))_{(0,\infty)}$. Let us set $\chi_\ell(\lambda) = \chi^1_\ell(\lambda)/\chi^0_\ell(\lambda)$. In other words, it consists in the zeta function of the monodromy on $R\Psi_{p_1}(\mathrm{DR}\ \ R\Gamma_{Z_\ell}(\mathcal{M})[+1])_{(0,\infty)}$.*

THEOREM 5.2. *The formal irregular part of $\gamma^*(\mathcal{H}^0 p_{1+}(\mathcal{M}e^{p_2}))_0$ takes the form $\oplus_{\beta(1/\tau)\in\Gamma} M_\beta e^{\beta(1/\tau)}$, where M_β are regular holonomic $\mathbb{C}[[\tau]]\langle\partial_\tau\rangle$-modules, such that $M_\beta = M_\beta[\frac{1}{\tau}]$.*

Moreover, the rank of M_β is equal to $\sum_{\ell\in\Lambda_\beta} m_\ell$ and the characteristic polynomial of the monodromy of M_β is $\prod_{\ell\in\Lambda_\beta} \chi_\ell(\lambda)$.

Proof. • **Ramification:** The choice of the ramification enables us to reduce the proof of Theorem 5.2 to the case where all the p_ℓ's are equal to 1. It is the non ramified case.

Let $\widetilde{\gamma} = (\gamma, id) : \widetilde{B_1} \times \mathbb{P}^1 \to B_1 \times \mathbb{P}^1$. By applying γ to $\mathcal{H}^0 p_{1+}(\mathcal{M}e^{p_2})$, we will repace $\mathcal{M}$ by $\widetilde{\gamma}^*(\mathcal{M})$. Indeed, as γ is a finite morphism, γ^* is an exact functor and $\gamma^* p_{1+} = p_{1+}\widetilde{\gamma}^*$. Then $\gamma^*(\mathcal{H}^0 p_{1+}(\mathcal{M}e^{p_2})) = \mathcal{H}^0 p_{1+}(\widetilde{\gamma}^*(\mathcal{M})e^{p_2})$. We deduce the theorem in the ramified case from the one in the non ramified case using the following lemma:

LEMMA 5.3. *(1) For $\ell \in \Lambda$, $\widetilde{\gamma}^{-1}(Z_\ell) = \bigcup_{i=1}^{p_\ell} \widetilde{Z^i_\ell}$, where $\widetilde{Z^i_\ell}$ are germs at $(0, \infty)$ of irreducible curves.*

The characteristic cycle of $\widetilde{\gamma}^*(\mathcal{M})$ in some neighbourhood $\widetilde{B} = \widetilde{B'_1} \times B_2 \subset \widetilde{B_1} \times \mathbb{P}^1$ of $(0, \infty)$ takes the form $rT^*_{\widetilde{B}}\widetilde{B} + \widetilde{m_1}T^*_{\widetilde{B'_1}\times\{\infty\}}\widetilde{B} + \widetilde{m_2}T^*_{\{0\}\times B_2}\widetilde{B} + \widetilde{m}T^*_{(0,\infty)}\widetilde{B} + \sum_{\ell\in\Lambda}\sum_{i=1}^{p_\ell} m_\ell T^*_{\widetilde{Z^i_\ell}}B$.

Moreover, $s \to (s^{p_\ell}, \alpha_\ell(1/s) + \delta(s))$ is a parametrization of Z_ℓ if and only if $s \to (s, \alpha_\ell\left(1/\xi_i s^{k_\ell}\right) + \delta(\xi_i s^{k_\ell}))$ is a parametrization of $\widetilde{Z^i_\ell}$, where $\xi_i^{p_\ell} = 1$.

In particular, the intersection multiplicity at (c, ∞) of $\widetilde{Z^i_\ell}$ with $\{0\} \times \mathbb{P}^1$ is equal to 1 (assumption of non ramified case).

(2) The monodromy on $R\Psi_{p_1}(\mathrm{DR}\ \ R\Gamma_{Z_\ell}\mathcal{M}[+1])_{(0,\infty)}$ and the one on $R\Psi_{p_1}(\mathrm{DR}\ \ R\Gamma_{\widetilde{Z^i_\ell}}\widetilde{\gamma}^*(\mathcal{M})[+1])_{(0,\infty)}$ have same zeta function.

Proof. This comes easily from the fact that γ is a finite morphism. □

• **Non ramified case:** We suppose that all the p_ℓ's are equal to 1. Let τ be a coordinate on B_1.

- Theorem 5.2 is proved if $\oplus_{\beta(1/\tau)\in\Gamma} M_\beta e^{\beta(1/\tau)}$ belongs to the formal irregular part of $\mathcal{H}^0 p_{1+}(\mathcal{M}e^{p_2})_0$. Indeed, the irregularity number of $\mathcal{H}^0 p_{1+}(\mathcal{M}e^{p_2})$ at 0 is equal to $\sum_{\ell\in\Lambda} m_\ell q_\ell$, where q_ℓ is the intersection multiplicity of Z_ℓ and $B_1 \times \{\infty\}$ (cf. theorem 1.1 of [12]). Then we remark that the irregularity number of $M_\beta e^{\beta(1/\tau)}$ at 0 is equal to $rk(M_\beta)q_\ell$.
- Let $\beta(1/\tau) \in 1/\tau.\mathbb{C}[1/\tau]$. As announced in Section 3.1, we begin by twisting the $\mathbb{C}\{\tau\}\langle\partial_\tau\rangle$-module by the exponential of $-\beta(1/\tau)$. Then we have to consider $\Psi_0((\mathcal{H}^0 p_{1+}(\mathcal{M}e^{p_2}))[\frac{1}{\tau}]e^{-\beta(1/\tau)})_0$. First we want to prove that there exists an isomorphism of $\mathbb{C}$-vector spaces which respects the monodromies between it and:

$$\mathrm{DR}\ \Psi_{\{0\}\times\mathbb{P}^1}(\mathcal{M}[*(\{0\}\times\mathbb{P}^1)]e^{p_2-\beta(1/p_1)})_{(0,\infty)}[+1].$$

- Denote by $\mathcal{P}$ the module $\mathcal{M}[*(\{0\}\times\mathbb{P}^1)]e^{p_2-\beta(1/p_1)}$. According to Lemma 3.4, $(\mathcal{H}^0 p_{1+}(\mathcal{M}e^{p_2}))[\frac{1}{\tau}]e^{-\beta(1/\tau)}$ is isomorphic to $\mathcal{H}^0 p_{1+}(\mathcal{P})$. Moreover, we have isomorphisms of $\mathbb{C}$-vector spaces which respect the monodromies:

$$\begin{aligned}
\Psi_0(\mathcal{H}^0 p_{1+}(\mathcal{P}))_0 &= \mathcal{H}^0\widetilde{p_1}_+(\Psi_{\{0\}\times\mathbb{P}^1}(\mathcal{P}))_0,\ \widetilde{p_1}:\{0\}\times\mathbb{P}^1\to\{0\},\\
&\quad \text{(cf. Remark 3.9)},\\
&= R^0\widetilde{p_1}_*(\mathrm{DR}\ \Psi_{\{0\}\times\mathbb{P}^1}(\mathcal{P})[+1])_0,\\
&= R^0\Gamma(\{0\}\times\mathbb{P}^1, \mathrm{DR}\ \Psi_{\{0\}\times\mathbb{P}^1}(\mathcal{P})[+1]),\\
&= \mathcal{H}^0\,\mathrm{DR}\ \Psi_{\{0\}\times\mathbb{P}^1}(\mathcal{P})_{(0,\infty)}[+1],\\
&= \mathrm{DR}\ \Psi_{\{0\}\times\mathbb{P}^1}(\mathcal{P})_{(0,\infty)}[+1].
\end{aligned}$$

The two last isomorphisms come from the fact that $\Psi_{\{0\}\times\mathbb{P}^1}(\mathcal{P})$ has support $(0,\infty)$ (cf. Corollary 7.2).

Then we do a local change of coordinates. Let (x, y) be local coordinates on $B_1\times\mathbb{P}^1$ in the neighbourhood of $(0,\infty)$ such that $(0,\infty)$ has coordinates $(0,0)$ and $x = 0$ (resp. $y = 0$) is the equation of $\{0\}\times\mathbb{P}^1$ (resp. $B_1 \times \{\infty\}$). After this change of coordinates, we have to compute the Euler characteristics of the complex $\mathrm{DR}\ \Psi_{x=0}(\mathcal{M}_{(0,\infty)}[\frac{1}{xy}]e^{1/y-\alpha(1/x)})[+1]$ and the zeta function of its monodromy. Then the corollary is proved using Theorem 4.2. □

We deduce easyly the Newton polygon of $\mathcal{H}^0 p_{1+}(\mathcal{M}e^{p_2})_0$ using Theorem 5.2.

Proof of Theorem 1.1. Let $\beta(1/\tau) \in \Gamma$. For all $\ell \in \Lambda_\beta$, the product $k_\ell q_\ell$ is equal to the degree ρ_β of β in $1/\tau$. Then the Newton polygon of $M_\beta e^{\beta(1/\tau)}$ has slope ρ_β and height $\rho_\beta * rk(M_\beta) = \sum_{\ell\in\Lambda_\beta} k_\ell q_\ell m_\ell$.

Now let $\Gamma_\rho = \{\beta(1/\tau)\in\Gamma \mid \rho_\beta = \rho\}$ and $\Lambda_\rho = \{\ell\in\Lambda \mid k_\ell q_\ell = \rho\}$.

The Newton polygon of $\oplus_{\beta(1/\tau)\in\Gamma_\rho} M_\beta e^{\beta(1/\tau)}$ has slope ρ_β and its height is $\sum_{\beta(1/\tau)\in\Gamma_\rho}\sum_{\ell\in\Lambda_\beta} k_\ell q_\ell m_\ell = \sum_{\ell\in\Lambda_\rho} N q_\ell m_\ell$.

Let $\widetilde{N_\ell}$ be the convex hull of $Q\cup((p_\ell m_\ell, Nq_\ell m_\ell)+Q)$. Then the Newton polygon of $\oplus_{\beta(1/\tau)\in\Gamma_\rho} M_\beta e^{\beta(1/\tau)}$ is the convex hull of $\sum_{\ell\in\Lambda_\rho}\widetilde{N_\ell}$.

Then after a convenient translation, the Newton polygon of the $\mathbb{C}\{\tau\}\langle\partial_\tau\rangle$-module $\gamma^*(\mathcal{H}^0 p_{1+}(\mathcal{M}e^{p_2}))_0$ is the convex hull of $\sum_{\ell\in\Lambda}\widetilde{N_\ell}$.

We deduce the Newton polygon of $\mathbb{C}\{t\}\langle\partial_t\rangle$-module $\mathcal{H}^0 p_{1+}(\mathcal{M}e^{p_2})_0$ from the one of $\gamma^*(\mathcal{H}^0 p_{1+}(\mathcal{M}e^{p_2}))_0$ by a dilatation of the vertical axis in a ratio $1/N$ (see lemma 5.4.3 of [13]).

□

6. On the formal structure of the direct image of some algebraic $\mathcal{D}$-modules

The goal of this section is to state some results about the formal irregular part of the direct image by a regular function of some $\mathcal{D}^{\mathrm{alg}}$-modules of exponential type.

Let U be a smooth affine variety, $f, g : U \to \mathbb{C}$ be two regular functions on U and $\mathcal{M}$ be a regular holonomic $\mathcal{D}_U^{\mathrm{alg}}$-module.

Let $i : \mathbb{C}^2 \hookrightarrow \mathbb{P}^1 \times \mathbb{P}^1$ and $j : \mathbb{C} \hookrightarrow \mathbb{P}^1$ be the inclusions such that we have the following diagrams:

$$\begin{array}{ccc} \mathbb{C}^2 & \overset{i}{\hookrightarrow} & \mathbb{P}^1 \times \mathbb{P}^1 \\ \downarrow{\scriptstyle P_1} & & \downarrow{\scriptstyle \pi_1} \\ \mathbb{C} & \overset{j}{\hookrightarrow} & \mathbb{P}^1 \end{array} \quad , \quad \begin{array}{ccc} \mathbb{C}^2 & \overset{i}{\hookrightarrow} & \mathbb{P}^1 \times \mathbb{P}^1 \\ \downarrow{\scriptstyle P_2} & & \downarrow{\scriptstyle \pi_2} \\ \mathbb{C} & \overset{j}{\hookrightarrow} & \mathbb{P}^1, \end{array}$$

where P_1, P_2 (resp. π_1, π_2) are the two canonical projections.

We consider the $\mathcal{D}_{\mathbb{P}^1}^{\mathrm{alg}}$-module $\mathfrak{M}^k = \mathcal{H}^k(j_+ f_+(\mathcal{M}e^g))$. We want to compute the formal irregular part of $\mathfrak{M}^k$ in the neighbourhood of a point $c \in \mathbb{P}^1$ using the regular holonomic $\mathcal{D}_{\mathbb{P}^1\times\mathbb{P}^1}$-module $\mathcal{H}^k i_+(f,g)_+(\mathcal{M}^{\mathrm{an}})$.

First, we consider the analytization of $\mathfrak{M}^k$. Indeed, the formal irregular part of $\mathfrak{M}^k$ in the neighbourhood of c is the same as the one of $(\mathfrak{M}^k)^{\mathrm{an}}$. Then we give another presentation of the complex of $\mathcal{D}_{\mathbb{P}^1}$-modules $(j_+ f_+(\mathcal{M}e^g))^{\mathrm{an}}$.

LEMMA 6.1. *$(j_+ f_+(\mathcal{M}e^g))^{\mathrm{an}} = \pi_{1+}(\mathcal{N}^\bullet e^{\pi_2})$, where $\mathcal{N}^\bullet = i_+(f,g)_+(\mathcal{M}^{\mathrm{an}})$.*

Proof. As $f = P_1 \circ (f,g)$, $f_+(\mathcal{M}e^g) = P_{1+}(f,g)_+(\mathcal{M}e^g)$.

Moreover, $(f,g)_+(\mathcal{M}e^g) = (f,g)_+(\mathcal{M})e^{P_2}$. Then

$$\begin{aligned} (j_+ f_+(\mathcal{M}e^g))^{\mathrm{an}} &= (j_+ P_{1+}((f,g)_+(\mathcal{M})e^{P_2}))^{\mathrm{an}}, \\ &= (\pi_{1+} i_+((f,g)_+(\mathcal{M})e^{P_2}))^{\mathrm{an}}, \\ &= \pi_{1+}((i_+((f,g)_+(\mathcal{M}))e^{\pi_2})^{\mathrm{an}}), \\ &= \pi_{1+}(\mathcal{N}^\bullet e^{\pi_2}). \end{aligned}$$

□

Now the study of the formal irregular part of the module $\mathfrak{M}^k$ is a particular case of the study of the formal irregular part of the cohomology modules $\mathcal{H}^k \pi_{1+}(\mathcal{N}^\bullet e^{\pi_2})$, where $\mathcal{N}^\bullet$ is a complex of $\mathcal{D}_{\mathbb{P}^1\times\mathbb{P}^1}$-modules with regular holonomic cohomology such that $\mathcal{N}^\bullet[*(\mathbb{P}^1 \times \{\infty\})] = \mathcal{N}^\bullet$. We will still denote this cohomology module by $\mathfrak{M}^k$.

Let us begin by describing the different objects that we need to state the theorem.

Notation 6.2.
- *Let $\mathrm{Cch}(c,k)$ be the characteristic cycle of $\mathcal{H}^k(\mathcal{N}^\bullet)$ in some neighbourhood $B = B_1 \times B_2 \subset \mathbb{P}^1 \times \mathbb{P}^1$ of (c,∞): $\mathrm{Cch}(c,k) = rT_B^*B + m_1 T^*_{B_1\times\{\infty\}}B + m_2 T^*_{\{c\}\times B_2}B + mT^*_{(c,\infty)}B + \sum_{\ell\in\Lambda} m_\ell T^*_{Z_\ell}B$, where Z_ℓ are germs at (c,∞) of irreducible curves distincts from $B_1 \times \{\infty\}$ and $\{c\} \times B_2$.*

- *Let p_ℓ be the intersection multiplicity at (c,∞) of Z_ℓ with $\{c\}\times B_2$. Let $x_{c,\ell}(s)=c+s^{p_\ell}$, if $c\neq\infty$ and $x_{c,\ell}(s)=1/s^{p_\ell}$, if $c=\infty$.*
- *Let t be local coordinate on B_1 at c.*
- *Denote by N the least common multiple of p_ℓ, $\ell\in\Lambda$ and let $\gamma:\widetilde{B_1}\to B_1$ defined by $\gamma(\tau)=t=\tau^N$.*
- *For $\ell\in\Lambda$, let $s\to(x_{c,\ell}(s),\alpha_\ell(1/s)+\delta(s))$ be a parametrization of Z_ℓ, where $\alpha_\ell(1/s)\in 1/s.\mathbb{C}[1/s]$ and $\delta(s)\in\mathbb{C}\{s\}$. Let $k_\ell=N/p_\ell$.*
- *Let Γ be the set of $\alpha_\ell(1/\xi\tau^{k_\ell})\in 1/\tau.\mathbb{C}[1/\tau]$, where $\xi^{p_\ell}=1$.*
- *For $\beta(1/\tau)\in\Gamma$, we denote by Λ_β the set of $\ell\in\Lambda$ such that there exists $\xi\in\mathbb{C}$, $\xi^{p_\ell}=1$, which satisfies $\beta(1/\tau)=\alpha_\ell(1/\xi\tau^{k_\ell})$.*
- *For $\ell\in\Lambda$, we consider the complex $R\Gamma_{Z_\ell}(\mathcal{H}^k\mathcal{N}^\bullet)$. The cohomology modules of this complex have support in Z_ℓ and are non trivial only in degree 0 and 1. Moreover, we have the exact sequence : $0\to R^0\Gamma_{Z_\ell}(\mathcal{H}^k\mathcal{N}^\bullet)\to\mathcal{H}^k\mathcal{N}^\bullet\to\mathcal{H}^k\mathcal{N}^\bullet[*Z_\ell]\to R^1\Gamma_{Z_\ell}(\mathcal{H}^k\mathcal{N}^\bullet)\to 0$. Let $\chi^i_\ell(\lambda)$, $i=0,1$, be the zeta function of the monodromy on $R\Psi_{\pi_1}(\mathrm{DR}\ \ R^i\Gamma_{Z_l}(\mathcal{H}^k\mathcal{N}^\bullet))_{(c,\infty)}$. Let us set $\chi_\ell(\lambda)=\chi^1_\ell(\lambda)/\chi^0_\ell(\lambda)$. In other words, $\chi_\ell(\lambda)$ is the zeta function of the monodromy on $R\Psi_{\pi_1}(\mathrm{DR}\ \ R\Gamma_{Z_\ell}(\mathcal{H}^k\mathcal{N}^\bullet)[+1])_{(c,\infty)}$.*

THEOREM 6.3. *The formal irregular part of $\gamma^*(\mathfrak{M}^k)_c$ takes the form $\oplus_{\beta(1/\tau)\in\Gamma}M_\beta e^{\beta(1/\tau)}$, where M_β are regular holonomic $\mathbb{C}[[\tau]]\langle\partial_\tau\rangle$-modules, such that $M_\beta=M_\beta[\frac{1}{\tau}]$.*

Moreover, the rank of M_β is equal to $\sum_{\ell\in\Lambda_\beta}m_\ell$ and the characteristic polynomial of the monodromy of M_β is $\prod_{\ell\in\Lambda_\beta}\chi_\ell(\lambda)$.

REMARK 6.4. *In the case where $f,g:\mathbb{C}^2\to\mathbb{C}$ are two polynomials which are algebraically independent and $\mathcal{N}^\bullet=i_+(f,g)_+(\mathcal{O}^{\mathrm{alg}}_{\mathbb{C}^2})$, we can prove that the Z_ℓ's of the characteristic cycle are those which are described in the introduction.*

Proof. Let $p_1:B_1\times\mathbb{P}^1\to B_1$ and $p_2:B_1\times\mathbb{P}^1\to\mathbb{P}^1$ be the two canonical projections. It is obvious that $(\mathfrak{M}^k)_c=\mathcal{H}^kp_{1+}(\mathcal{N}^\bullet_{|B_1\times\mathbb{P}^1}e^{p_2})_c$. After a change of coordinates, we can assume that $c=0$. Then, Theorem 6.3 comes from Theorem 5.2 and the following lemma. For simplicity of notation, we write $\mathcal{N}^\bullet$ instead of $\mathcal{N}^\bullet_{|B_1\times\mathbb{P}^1}$.

LEMMA 6.5. *$\mathcal{H}^kp_{1+}(\mathcal{N}^\bullet e^{p_2})$ and $\mathcal{H}^0p_{1+}(\mathcal{H}^k\mathcal{N}^\bullet e^{p_2})$ have the same formal irregular part at 0.*

Proof. It is sufficient to prove that, after a ramification γ, the formal irregular part of $\gamma^*\mathcal{H}^kp_{1+}(\mathcal{N}^\bullet e^{p_2})$ is equal to the one of $\gamma^*\mathcal{H}^0p_{1+}(\mathcal{H}^k\mathcal{N}^\bullet e^{p_2})$.

For $\alpha(1/\tau)\in 1/\tau.\mathbb{C}[1/\tau]$, we have to prove that:
$\mathrm{sp}(\gamma^*\mathcal{H}^kp_{1+}(\mathcal{N}^\bullet e^{p_2})[\frac{1}{\tau}]e^{-\alpha(1/\tau)})_0=\mathrm{sp}(\gamma^*\mathcal{H}^0p_{1+}(\mathcal{H}^k\mathcal{N}^\bullet e^{p_2})[\frac{1}{\tau}]e^{-\alpha(1/\tau)})_0$.

Denotes by $\mathcal{P}^\bullet$ the complex $\widetilde{\gamma}^*(\mathcal{N}[*(\{0\}\times\mathbb{P}^1)])$, where $\widetilde{\gamma}=(\gamma,id):\widetilde{B_1}\times\mathbb{P}^1\to B_1\times\mathbb{P}^1$.

$$
\begin{aligned}
&\mathrm{sp}_0(\gamma^*\mathcal{H}^k p_{1+}(\mathcal{N}^\bullet e^{p_2})[\frac{1}{\tau}]e^{-\alpha(1/\tau)})_0 =\\
&\quad = \mathrm{sp}_0(\mathcal{H}^k p_{1+}(\mathcal{P}^\bullet e^{p_2-\alpha(1/p_1)})_0,\ \text{(Lemma 3.4 and } \gamma \text{ finite morphism)},\\
&\quad = \mathcal{H}^k\overline{p_1}_+\,\mathrm{sp}_{\{0\}\times\mathbb{P}^1}(\mathcal{P}^\bullet e^{p_2-\alpha(1/p_1)})_0,\ \text{(Theorem 3.8)},\\
&\quad = R^k p_{1*}\,\mathrm{DR}\ \ \mathrm{sp}_{\{0\}\times\mathbb{P}^1}(\mathcal{P}^\bullet e^{p_2-\alpha(1/p_1)})_0,\\
&\quad (\mathrm{DR}\ \mathrm{gr}^V\mathcal{D}_{B_1\times\mathbb{P}^1} \text{ is a resolution of } \mathcal{D}^{\mathrm{alg}}_{B_1\times\mathbb{P}^1\to\mathbb{P}^1}),\\
&\quad = \mathcal{H}^0 p_{1*}\,\mathrm{DR}\ \ \mathrm{sp}_{\{0\}\times\mathbb{P}^1}(\mathcal{H}^k\mathcal{P}^\bullet e^{p_2-\alpha(1/p_1)})_0,\\
&\quad (\mathrm{sp}_{\{0\}\times\mathbb{P}^1}(\mathcal{P}^\bullet e^{p_2-\alpha(1/p_1)}) \text{ has support } (0,\infty)\ \text{(Corollary 7.2))}\\
&\quad = \mathcal{H}^0\overline{p_1}_+\,\mathrm{sp}_{\{0\}\times\mathbb{P}^1}(\mathcal{H}^k\mathcal{P}^\bullet e^{p_2-\alpha(1/p_1)})_0\\
&\quad = \mathrm{sp}_0(\mathcal{H}^0 p_{1+}(\mathcal{H}^k\mathcal{P}^\bullet e^{p_2-\alpha(1/p_1)}))_0,\ \text{(Theorem 3.8)},\\
&\quad = \mathrm{sp}_0(\gamma^*\mathcal{H}^0 p_{1+}(\mathcal{H}^k\mathcal{N}^\bullet e^{p_2})[\frac{1}{\tau}]e^{-\alpha(1/\tau)})_0,\ \text{(Lemma 3.4 and } \gamma \text{ finite)}.
\end{aligned}
$$

□

□

7. Appendix: Local computations

In all this section, $\mathcal{M}$ is a regular holonomic $\mathcal{D}_{\mathbb{C}^2}$-module, $k,l\geq 1$ and $m,n\geq 1$.

• *Nearby cycles module along $x=0$ of $\mathcal{D}_{\mathbb{C}^2}$-modules of exponential type:*

In this paragraph, we suppose that the singular support of $\mathcal{M}$ is included in $\{xy=0\}$.

LEMMA 7.1. $\Psi_{x^m=0}(\mathcal{M}[\frac{1}{xy}]e^{1/x^ky^l})=0$ *and* $\Psi_{x^m=0}(\mathcal{M}[\frac{1}{x}]e^{1/x^k})=0$.

Proof. • In lemma 4.5.9 and lemma 4.5.10 of [14], C. Sabbah proved that $\Psi_{x^m=0}(\mathcal{M}[\frac{1}{xy}]e^{1/x^ky^l})=0$ and $\Psi_{x^m=0}(\mathcal{M}[\frac{1}{xy}]e^{1/x^k})=0$.

• Now, we have to prove that $\Psi_{x^m=0}(\mathcal{M}[\frac{1}{x}]e^{1/x^k})=0$, when $\mathcal{M}$ has support in $\{y=0\}$. We will prove that $\mathrm{sp}_{x^m=0}(\mathcal{M}[\frac{1}{x}]e^{1/x^k})=0$. Let i be the inclusion of $\mathbb{C}\times\{0\}$ in $\mathbb{C}^2$. Then $\mathcal{M}[\frac{1}{x}]=i_+Li^*(\mathcal{M}[\frac{1}{x}])[+1]$ and $Li^*(\mathcal{M}[\frac{1}{x}])[+1]$ is just a module. Then according to Theorem 3.8, we have $\mathrm{sp}_{x^m=0}(\mathcal{M}[\frac{1}{x}]e^{1/x^k})=\overline{i}_+(\mathrm{sp}_{x^m=0}(Li^*(\mathcal{M}[\frac{1}{x}])[+1]e^{1/x^k}))$. Moreover, as the module $Li^*(\mathcal{M}[\frac{1}{x}])[+1]e^{1/x^k}$ is purely irregular along $x=0$, we can prove that $\mathrm{sp}_{x^m=0}(Li^*(\mathcal{M}[\frac{1}{x}])[+1]e^{1/x^k})=0$.

□

COROLLARY 7.2. *Let $\alpha(1/x)\in 1/x.\mathbb{C}[1/x]$ and $\mathcal{M}^\bullet$ be a complex of regular holonomic $\mathcal{D}_{\mathbb{C}^2}$-modules. Then $\Psi_{x=0}(\mathcal{M}^\bullet[\frac{1}{x}]e^{-\alpha(1/x)})=0$.*

Proof. As $\mathcal{M}^\bullet$ is a complex of regular holonomic $\mathcal{D}_{\mathbb{C}^2}$-modules, we can apply the nearby cycles functor to each terms of the complex. Then it is sufficient to prove the theorem for a single module. This comes from Lemma 7.1 and the fact that $\mathcal{M}^\bullet[\frac{1}{x}]e^{-\alpha(1/x)}$ is isomorphic to $\mathcal{M}^\bullet[\frac{1}{x}]e^{1/x^k}$, where k is the degree of $\alpha(1/x)$ in $1/x$ (we are in the analytic setting). □

• *Nearby cycles module along $xy = 0$ of $\mathcal{D}_{\mathbb{C}^2}$-modules of exponential type:*

In this paragraph, we suppose that the singular support of $\mathcal{M}$ is included in $\{xy = 0\}$.

LEMMA 7.3. *Let $i : \mathbb{C}^2 \hookrightarrow \mathbb{C}^3$, $i(x, y) = (x, y, x^m y^n)$. Let $\chi_r(\lambda)$ be the zeta function of the monodromy on $R\Psi_x\ \mathrm{DR}(Rj_*j^{-1}\mathcal{M})_{(0,0)}$, where j is the inclusion of $\mathbb{C}^* \times \mathbb{C}^*$ in $\mathbb{C}^2$.*

(1) $\Psi_{z=0}(i_+(\mathcal{M}[\frac{1}{xy}]e^{1/x^k y^l})) = 0.$

(2)
- $\chi(\mathrm{DR}\ \Psi_{z=0}(i_+(\mathcal{M}[\frac{1}{xy}]e^{1/x})))_{(0,0)} = mr.$
- *Moreover, the zeta function of the monodromy on the complex* $\mathrm{DR}\ \Psi_{z=0}(i_+(\mathcal{M}[\frac{1}{xy}]e^{1/x})))_{(0,0)}$ *is equal to* $\chi_r(\lambda)^m$.

Proof. (1) cf. lemma 4.5.10 (2) of [14].

(2) cf. lemma 4.5.10 (3) of [14] and its proof.

□

• *Nearby cycles module along $x = 0$ of regular $\mathcal{D}_{\mathbb{C}^2}$-modules:*

In this paragraph, we assume that the singular support of $\mathcal{M}$ is not necessary a normal crossing. Let $\mathrm{Cch}(\mathcal{M})$ be the characteristic cycle of $\mathcal{M}$ in the neighbourhood of $(0, 0)$: $\mathrm{Cch}(\mathcal{M}) = rT^*_{\mathbb{C}^2}\mathbb{C}^2 + n_x T^*_{x=0}\mathbb{C}^2 + \sum_{\ell\in\widetilde{\Lambda}} n_\ell T^*_{Y_\ell}\mathbb{C}^2 + nT^*_{(0,0)}\mathbb{C}^2$, where Y_ℓ are germs of irreducible curves of $\mathbb{C}^2$ distincts from $\{x = 0\}$. We will assume that **the intersection multiplicity at** $(0, 0)$ **of** Y_ℓ **with** $\{x = 0\}$ **is equal to 1**.

Notation 7.4.
- *Let $\chi_\ell(\lambda)$ be the zeta function of the monodromy on $R\Psi_x\ \mathrm{DR}(R\Gamma_{Y_\ell}\mathcal{M}[+1])_{(0,0)}$.*
- *Let $\chi_r(\lambda)$ be the zeta function of the monodromy on the complex $R\Psi_x\ \mathrm{DR}(Rj_*j^{-1}\mathcal{M})_{(0,0)}$, where j is the inclusion of the complementary of $\{x = 0\} \cup \bigcup_{l\in\widetilde{\Lambda}} Y_\ell$ in $\mathbb{C}^2$.*
- *Let $\chi(\lambda)$ be the zeta function of the monodromy on the complex $R\Psi_x\ \mathrm{DR}(\mathcal{M}[\frac{1}{x}])_{(0,0)}$.*

LEMMA 7.5.
- $\chi(R\Psi_x\ \mathrm{DR}(\mathcal{M}[\frac{1}{x}]))_{(0,0)} = r - \sum_{\ell\in\widetilde{\Lambda}} m_\ell.$
- $\chi(\lambda) = \chi_r(\lambda)/\prod_{\ell\in\widetilde{\Lambda}} \chi_\ell(\lambda).$

Proof. The first point is proved using the index theorem of Kashiwara (cf. [2]). Let us prove the second point.

Let $X_{\epsilon,\eta} = B(0, \epsilon) \cap \{x = \eta\}$. We have:

$$R\Psi_x\ \mathrm{DR}(\mathcal{M}[\frac{1}{x}])_{(0,0)} = \varinjlim_{\epsilon,\eta>0} R\Gamma(X_{\epsilon,\eta}, \mathrm{DR}\,\mathcal{M}).$$

Let η small enough such that all the Z_ℓ intersect $X_{\epsilon,\eta}$ and denote by P_ℓ, $\ell \in \widetilde{\Lambda}$, these intersection points. Denote by $X^c_{\epsilon,\eta} = X_{\epsilon,\eta} \setminus \{P_\ell, \ell \in \widetilde{\Lambda}\}$. As $\mathrm{DR}\,(M)_{|X_{\epsilon,\eta}}$ is a constructible sheaf with respect to the stratification $\{X^c_{\epsilon,\eta}, P_\ell; \ell \in \widetilde{\Lambda}\}$, we have $\chi(\lambda) = \overline{\chi_g}(\lambda) \prod_{\ell\in\widetilde{\Lambda}} \overline{\chi_\ell}(\lambda)$, where $\overline{\chi_\ell}(\lambda)$ is the zeta function of the monodromy on $\mathrm{DR}(\mathcal{M})_{P_\ell}$ and $\overline{\chi_g}(\lambda)$ is the zeta function of the monodromy on $\Gamma(X^c_{\epsilon,\eta}, \mathrm{DR}(\mathcal{M}))$.

- First, we have:

$$\begin{aligned} R\Gamma(X^c_{\epsilon,\eta}, \mathrm{DR}(\mathcal{M})) &= R\Gamma(j^{-1}(X_{\epsilon,\eta}), j^{-1}\,\mathrm{DR}(\mathcal{M})), \\ &= R\Gamma(X_{\epsilon,\eta}, Rj_*j^{-1}\,\mathrm{DR}(\mathcal{M})), \\ &= R\Psi_x\ \mathrm{DR}(Rj_*j^{-1}\mathcal{M})_{(0,0)}. \end{aligned}$$

  Then $\overline{\chi_g}(\lambda) = \chi_r(\lambda)$.

- Consider the triangle $R\Gamma_{Y_\ell}\mathcal{M} \to \mathcal{M} \to Ri_*i^{-1}\mathcal{M} \xrightarrow{+1}$, where i is the inclusion of $\mathbb{C}^2 \setminus Y_\ell$ in $\mathbb{C}^2$. Then we have:

$$\mathrm{DR}(R\Gamma_{Y_\ell}\mathcal{M})_{P_\ell} \to \mathrm{DR}(\mathcal{M})_{P_\ell} \to \mathrm{DR}(Ri_*i^{-1}\mathcal{M})_{P_\ell} \xrightarrow{+1}.$$

  Moreover, the zeta function of $\mathrm{DR}(Ri_*i^{-1}\mathcal{M})_{P_\ell}$ is equal to 1. Indeed, $\mathrm{DR}(Ri_*i^{-1}\mathcal{M})_{P_\ell} = \varinjlim_{\epsilon>0} R\Gamma(B(P_\ell,\epsilon) \setminus Y_\ell, \mathrm{DR}\,\mathcal{M})$ and the Euler characteristics of $B(P_\ell,\epsilon) \setminus Y_\ell$ is 0.

  As $\mathrm{DR}(R\Gamma_{Y_\ell}\mathcal{M})_{P_\ell} = R\Psi_x\ \mathrm{DR}(R\Gamma_{Y_\ell}\mathcal{M}[+1])_{(0,0)}[-1]$, we conclude that $\overline{\chi_\ell}(\lambda) = \chi_\ell(\lambda)^{-1}$.

□

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