*Publ. RIMS, Kyoto Univ.*
**4x** (200x), 000–000

# The irregularity of the direct image of some $\mathcal{D}$-modules

By

Céline Roucairol *

## Abstract

Let $f$ and $g$ be two regular functions on $U$ smooth affine variety. Let $\mathcal{M}$ be a regular holonomic $\mathcal{D}_U$-module. We are interested in the irregularity of the complex $f_+(\mathcal{M}e^g)$. More precisely, we relate the irregularity number at $c$ of the systems $\mathcal{H}^k f_+(\mathcal{M}e^g)$ with the characteristic cycles of the systems $\mathcal{H}^k(f,g)_+(\mathcal{M})$.

## §1. Introduction

• Let $U$ be a smooth affine variety over $\mathbb{C}$ and $g : U \to \mathbb{C}$ be a regular function on $U$. We denote by $\mathcal{O}_U$ the sheaf of regular functions on $U$ and by $\mathcal{D}_U$ the sheaf of algebraic differential operators on $U$.

Let $\mathcal{M}$ be a regular holonomic $\mathcal{D}_U$-module. We denote by $\mathcal{M}e^g$ the $\mathcal{D}_U$-module obtained from $\mathcal{M}$ by twisting by $e^g$. If $\bigtriangledown$ is the connection defined by the $\mathcal{D}_U$-module structure of $\mathcal{M}$, $\bigtriangledown + dg$ is the one associated with $\mathcal{M}e^g$. Although $\mathcal{M}$ is regular, $\mathcal{M}e^g$ is not regular in general. Here, regular means that there exists a smooth compactification $X$ of $U$ and an extension of $\mathcal{M}e^g$ as $\mathcal{D}_X$-module which is regular holonomic on $X$. In [10], C. Sabbah describes a comparison theorem for these $\mathcal{D}$-modules twisted by an exponential. This theorem gives a relation between the irregularity complex of $\mathcal{M}e^g$ (see [6]) and some topological data given by $g$ and $\mathcal{M}$.

In this paper, we consider two regular functions $f, g : U \to \mathbb{C}$. We are interested in the irregularity of the cohomology modules of the direct image by $f$ of a $\mathcal{D}_U$-module, $\mathcal{M}e^g$, where $\mathcal{M}$ is regular and holonomic.

---

Communicated by M. Kashiwara. Received June 28, 2005. Revised September 27, 2005.
2000 Mathematics Subject Classification(s): 32C38, 35B40

*Universität Mannheim, Institut für Mathematik, A5, 6, 68131 Mannheim, Germany
celine.roucairol@uni-mannheim.de

• In section 2, we recall the definitions of a $\mathcal{D}$-module twisted by an exponential of a meromorphic function. We will need the definition in the case of meromorphic function during the proof of the main theorem.

Then, we will consider the case where $\mathcal{M}$ is the sheaf of regular function $\mathcal{O}_U$. According to [4], the system $\mathcal{H}^k(f_+(\mathcal{O}_U e^g))$ extends vector bundle with flat holomorphic connection such that the generic fiber of the sheaf of their horizontal sections is canonically isomorphic to the cohomology group $H^{k+n-1}_{\phi_t}(f^{-1}(t)^{an}, \mathbb{C})$, where $\phi_t$ is the family of closed subsets of $f^{-1}(t)$ on which $e^{-g}$ is rapidly decreasing. Using this result, we will motivate the study of the irregularity of the systems $\mathcal{H}^k f_+(\mathcal{O}_U e^g)$ by observations on some integrals.

• The main theorem of this paper gives us a formula for the irregularity number of the systems $\mathcal{H}^k(f_+(\mathcal{M}e^g))$ at finite distance and at infinity.

In the case where $f$ and $g$ are two polynomials in two variables which are algebraically independents and $\mathcal{M} = \mathcal{O}_{\mathbb{C}^2}$, the complex $f_+(\mathcal{O}_{\mathbb{C}^2} e^g)$ is concentrated in degree 0 except at a finite number of points (see [9]). Then, the irregularity number at a point $c \in \mathbb{C} \cup \{\infty\}$ of the system $\mathcal{H}^0 f_+(\mathcal{O}_{\mathbb{C}^2} e^g)$ can be expressed in terms of some geometric data associated with $f$ and $g$ (see [9]).

In this paper, we calculate the irregularity number at $c \in \mathbb{C} \cup \{\infty\}$ of the systems $\mathcal{H}^k(f_+(\mathcal{M}e^g))$ with the help of the characteristic cycle of the systems $\mathcal{H}^k(f,g)_+(\mathcal{M})$, in the general case where $f$ and $g$ are any regular functions.

In the following, we identify $\mathbb{C} \cup \{\infty\}$ with $\mathbb{P}^1$. Let $i$ be the inclusion of $\mathbb{C}^2$ in $\mathbb{P}^1 \times \mathbb{P}^1$. Let $c \in \mathbb{P}^1$ and $V = V_1 \times V_2 \subset \mathbb{P}^1 \times \mathbb{P}^1$ a neighbourhood of $(c, \infty)$.

Let $Cch(c,k)$ be the characteristic cycle of $\mathcal{H}^k i_+(f,g)_+(\mathcal{M})$ in the neighbourhood $V$:
$Cch(c,k) = mT^*_V V + m' T^*_{(c,\infty)} V + m'' T^*_{\{c\}\times V_2} V + m''' T^*_{V_1 \times \{\infty\}} V + \sum m_l T^*_{Z_l} V$,
where $Z_l$ are some germs of irreducible curves in a neighbourhood of $(c, \infty)$ distinct from $V_1 \times \{\infty\}$ and $\{c\} \times V_2$.

**Theorem 1.1.** *The irregularity number of $\mathcal{H}^k f_+(\mathcal{M}e^g)$ at $c$ is equal to $\sum_l m_l I_{(c,\infty)}(Z_l, \mathbb{P}^1 \times \{\infty\})$, where $I_{(c,\infty)}(Z_l, \mathbb{P}^1 \times \{\infty\})$ is the intersection multiplicity of $Z_l$ and $\mathbb{P}^1 \times \{\infty\}$ at $(c, \infty)$.*

• The theorem of commutation between the irregularity functor and the direct image functor ([6]) allows us to rephrasing theorem 1.1 in terms of an irregularity complex of a regular holonomic $\mathcal{D}$-module twisted by an exponential (cf. lemma 3.2).

Then, using the comparison theorem of [10], we are led to calculate the Euler characteristic of a germ of a complex of nearby cycles.

## §2. The complex $f_+(\mathcal{M}e^g)$

### §2.1. Regular holonomic $\mathcal{D}$-modules twisted by an exponential

Let $X$ be a smooth algebraic variety over $\mathbb{C}$.
We identify $\mathbb{P}^1$ with $\mathbb{C}\cup\{\infty\}$. Let $h : X \to \mathbb{P}^1$ be a meromorphic function.

**Definition 2.1.** We define the $\mathcal{D}_X$-module $\mathcal{O}_X[*h^{-1}(\infty)]e^h$ as a $\mathcal{D}_X$-module which is isomorphic to $\mathcal{O}_X[*h^{-1}(\infty)]$ as $\mathcal{O}_X$-module. The original connection $\bigtriangledown$ on $\mathcal{O}_X[*h^{-1}(\infty)]$ is replaced with the connection $\bigtriangledown + dh$ on $\mathcal{O}_X[*h^{-1}(\infty)]e^h$.

Let $\mathcal{M}$ be a holonomic $\mathcal{D}_X$-module.

**Definition 2.2.** We define the $\mathcal{D}_X$-module $\mathcal{M}[*h^{-1}(\infty)]e^h$ as the $\mathcal{D}_X$-module $\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{O}_X[*h^{-1}(\infty)]e^h$.

*Remark.* $\mathcal{O}_X[*h^{-1}(\infty)]e^h$ is the direct image by an open immersion of a vector bundle with integrable connection. Then, it is a holonomic $\mathcal{D}_X$-module as algebraic direct image of a holonomic $\mathcal{D}$-module.

$\mathcal{M}[*h^{-1}(\infty)]e^h$ is a holonomic left $\mathcal{D}_X$-module as tensor product of two holonomic left $\mathcal{D}_X$-modules (cf. theorem 4.6 of [2]).

We have analogous definitions in the analytic case. We just have to transpose in the analytic setting.

### §2.2. On the solutions of the systems $\mathcal{H}^k(f_+(\mathcal{O}_U e^g))$

The generic fiber of the sheaf of horizonal sections of $\mathcal{H}^k(f_+(\mathcal{O}_U e^g))$ can be describe as follows:

**Theorem 2.1** ([4]). *There exists a finite subset $\Sigma \subset \mathbb{C}$ such that*

- $\mathcal{H}^k(f_+(\mathcal{O}_U e^g))_{|\mathbb{C}\setminus\Sigma}$ *is a vector bundle with flat holomorphic connection.*
- *For all $t \in \mathbb{C} \setminus \Sigma$, $i_t^+\mathcal{H}^{k-n+1}(f_+(\mathcal{O}_U e^g)) \simeq H^k_{\phi_t}((f^{-1}(t))^{an}, \mathbb{C})$, where $i_t$ is the inclusion of $\{t\}$ in $\mathbb{C}$ and $\phi_t$ is the family of closed subsets of $f^{-1}(t)$ on which $e^{-g}$ is rapidly decreasing.*

More precisely, the family $\phi_t$ is defined as follow. Let $\pi : \widetilde{\mathbb{P}^1} \to \mathbb{P}^1$ be the oriented real blow-up of $\mathbb{P}^1$ at infinity. $\widetilde{\mathbb{P}^1}$ is diffeomorphic to $\mathbb{C} \cup S^1$, where $S^1$ is the circle of directions at infinity. $A$ is in $\phi_t$ if $A$ is a closed subset of $f^{-1}(t)$ and the closure of $g(A)$ in $\mathbb{C} \cup S^1$ intersects $S^1$ in $]-\frac{\pi}{2}, \frac{\pi}{2}[$.

Let us motivate the study of this complex by observations on some integrals. Concerning Gauss-Manin systems, we can express their solutions as period integrals of the type $\int_{\gamma(t)} w_{|f^{-1}(t)}$, where $\gamma(t)$ is an horizontal family of cycles in the fibres $f^{-1}(t)$ and $w$ is a relative algebraic differential form. As the Gauss-Manin connection is regular, these integrals have moderate growth in the neighbourhood of their singularities. In our case, some solutions can also be expressed as integrals.

Let $\Psi_t$ be the family of closed subsets $A$ of $f^{-1}(t)$ such that for all $R$ big enough, $A \setminus g^{-1}(\{t \in \mathbb{C} \mid Re(-t) > R\})$ is compact. We consider the complex of semi-algebraic chains with support in $\Psi_t$ (see [8]). We denote by $H_{k,\Psi_t}(f^{-1}(t)^{an}, \mathbb{C})$ the $k$-th homology group associated with this complex. We can now integrate forms in $H^k_{\Phi_t}(f^{-1}(t)^{an}, \mathbb{C})$ on cycles in $H_{k,\Psi_t}(f^{-1}(t)^{an}, \mathbb{C})$.

According to theorem 1.4 of [1], since $f$ is a submersion outside $\Sigma$, we have an isomorphism

$$\mathcal{H}^{k-n+1}(f_+(\mathcal{O}_U e^g))_{|\mathbb{C}\setminus\Sigma} \simeq R^k f_*(DR_{\mathbb{C}^n/\mathbb{C}}(\mathcal{O}_U)e^g)_{|\mathbb{C}\setminus\Sigma}.$$

Thus, we can extend the integration defined before to a form $we^g_{|f^{-1}(t)}$, where $w$ is a relative algebraic differential form. Indeed, by the definition of $\Psi_t$, $e^g$ is rapidly decreasing on the cycles and semi-algebraic chains with support in $\Psi_t$ behave well at infinity.

In this way, to $\gamma(t)$, horizontal family of cycles in $H_{k,\Psi_t}(f^{-1}(t)^{an}, \mathbb{C})$, we can associate a solution of the $\mathcal{D}_{\mathbb{C}\setminus\Sigma}$-module $\mathcal{H}^{k-n+1}(f_+(\mathcal{O}_U e^g))_{|\mathbb{C}\setminus\Sigma}$. It is a morphism $\alpha$ of $\mathcal{D}_{\mathbb{C}\setminus\Sigma}$-modules defined by $\alpha([we^g]) = \int_{\gamma(t)} we^g_{|f^{-1}(t)}$.

The study of the irregularity of the systems $\mathcal{H}^{k-n+1}(f_+(\mathcal{O}_U e^g))$ gives us informations about the growth of these integrals in the neighbourhood of their singularities.

## §3. On the irregularity of the complex $f_+(\mathcal{M}e^g)$

In the following, we will identify $\mathbb{C} \cup \{\infty\}$ with $\mathbb{P}^1$ and we consider the canonical immersion $j : \mathbb{C} \to \mathbb{P}^1$. Let us fix $k \in \mathbb{Z}$ and $c \in \mathbb{P}^1$.

We are interested in the number $IR_{c,k}$, it being the irregularity number at $c \in \mathbb{P}^1$ of the system $\mathcal{H}^k j_+ f_+(\mathcal{M}e^g)$.

The first step of the proof of theorem 1.1 consists in rephrasing it using an irregularity complex of a $\mathcal{D}_{\mathbb{C}^2}$-module twisted by an exponential.

For the definition of irregularity complex along an hypersurface, we refer the reader to [6] and [7]. We adopt the following notations. If $\mathfrak{M}$ is a complex of $\mathcal{D}_X$-modules and $Z$ is an hypersurface of $X$, we denote by $IR_Z(\mathfrak{M})$ the irregularity complex of $\mathfrak{M}$ along $Z$. For simplicity of notations, we write $IR^k_Z(\mathfrak{M})$ instead of $\mathcal{H}^k(IR_Z(\mathfrak{M}))$.

According to [5], the irregularity number $IR_{c,k}$ is equal to the dimension of the $\mathbb{C}$-vector space $IR^0(\mathcal{H}^k j_+ f_+(\mathcal{M}e^g))_c$.

Denote by $\mathcal{N}^\bullet$ the complex of $\mathcal{D}_{\mathbb{P}^1\times\mathbb{P}^1}$-modules $i_+(f,g)_+(\mathcal{M})$. In the way of rephrasing theorem 1.1, we need the following lemma:

**Lemma 3.1.** *Let $\pi_2 : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$ be the second projection and $D = (\mathbb{P}^1 \times \mathbb{P}^1) \setminus i(\mathbb{C}^2)$.*

$$IR_c(\mathcal{H}^k j_+ f_+(\mathcal{M}e^g))_c = IR_{\{c\}\times\mathbb{P}^1}(\mathcal{H}^k(\mathcal{N}^\bullet)[*D]e^{\pi_2})_{(c,\infty)}[+1].$$

*Proof.* • Reduction to the case of the two projections.

Let $p_1 : \mathbb{C}^2 \to \mathbb{C}$ and $p_2 : \mathbb{C}^2 \to \mathbb{C}$ be the two canonical projections.

As $f = p_1 \circ (f,g)$, we have $f_+(\mathcal{M}e^g) = p_{1+}(f,g)_+(\mathcal{M}e^g)$. Moreover, $(f,g)_+(\mathcal{M}e^g) = (f,g)_+(\mathcal{M})e^{p_2}$.

Finally, we obtain that $f_+(\mathcal{M}e^g) \simeq p_{1+}((f,g)_+(\mathcal{M})e^{p_2})$.

• In this paragraph, we denote by ${}^p\mathcal{H}$ the perverse cohomology. According to corollary 2-1-8 of [6], we have:

$$IR_c(\mathcal{H}^k j_+ f_+(\mathcal{M}e^g)) = {}^p\mathcal{H}^k IR_c(j_+ f_+(\mathcal{M}e^g))$$

Consider the following diagrams:

$$\begin{array}{ccc} \mathbb{C}^2 & \xrightarrow{i} & \mathbb{P}^1\times\mathbb{P}^1 \\ \downarrow{\scriptstyle p_1} & & \downarrow{\scriptstyle \pi_1} \\ \mathbb{C} & \xrightarrow{j} & \mathbb{P}^1 \end{array} \qquad\qquad \begin{array}{ccc} \mathbb{C}^2 & \xrightarrow{i} & \mathbb{P}^1\times\mathbb{P}^1 \\ \downarrow{\scriptstyle p_2} & & \downarrow{\scriptstyle \pi_2} \\ \mathbb{C} & \xrightarrow{j} & \mathbb{P}^1 \end{array}.$$

Then,

$$\begin{aligned} IR_c(j_+f_+(\mathcal{M}e^g)) &= IR_c(j_+p_{1+}((f,g)_+(\mathcal{M})e^{p_2})) \\ &= IR_c(\pi_{1+}i_+((f,g)_+(\mathcal{M})e^{p_2})) \\ &= IR_c(\pi_{1+}i_+(\mathcal{N}^\bullet[*D]e^{\pi_2})). \end{aligned}$$

Then, $IR_c(\mathcal{H}^k j_+ f_+(\mathcal{M}e^g)) = {}^p\mathcal{H}^k IR_c(\pi_{1+}i_+(\mathcal{N}^\bullet[*D]e^{\pi_2}))$.

• According to proposition 3-6-4 of [7], the irregularity functor commutes with the direct image functor. Thus:

$$\begin{aligned} IR_c(\pi_{1+}i_+(\mathcal{N}^\bullet[*D]e^{\pi_2}))_c &= \mathbb{R}\pi_{1*}IR_{\{c\}\times\mathbb{P}^1}(\mathcal{N}^\bullet[*D]e^{\pi_2})_c[+1] \\ &= \mathbb{R}\Gamma(\{c\}\times\mathbb{P}^1, IR_{\{c\}\times\mathbb{P}^1}(\mathcal{N}^\bullet[*D]e^{\pi_2}))[+1]. \end{aligned}$$

• Then, we remark that $\pi_2$ is holomorphic out of $(c,\infty)$ and $\mathcal{N}^\bullet$ is regular holonomic (direct image complex of an algebraic regular holonomic $\mathcal{D}$-module).

Then, $IR_{\{c\}\times\mathbb{P}^1}(\mathcal{N}^\bullet[*D]e^{\pi_2})$ has its support in $(c,\infty)$ and we have an isomorphism of complexes of vector spaces

$$IR_c(\pi_{1+}i_+(\mathcal{N}^\bullet[*D]e^{\pi_2}))_c = IR_{\{c\}\times\mathbb{P}^1}(\mathcal{N}^\bullet[*D]e^{\pi_2})_{(c,\infty)}[+1].$$

- We conclude that

$$\begin{aligned} IR_c(\mathcal{H}^k j_+ f_+(\mathcal{M}e^g))_c &= {}^p\mathcal{H}^k IR_c(\pi_{1+}i_+(\mathcal{N}^\bullet[*D]e^{\pi_2}))_c \\ &= {}^p\mathcal{H}^k IR_{\{c\}\times\mathbb{P}^1}(\mathcal{N}^\bullet[*D]e^{\pi_2})_{(c,\infty)}[+1] \\ &= IR_{\{c\}\times\mathbb{P}^1}(\mathcal{H}^k(\mathcal{N}^\bullet[*D]e^{\pi_2}))_{(c,\infty)}[+1] \\ &= IR_{\{c\}\times\mathbb{P}^1}(\mathcal{H}^k(\mathcal{N}^\bullet)[*D]e^{\pi_2})_{(c,\infty)}[+1]. \end{aligned}$$

□

Now, we can rephrase theorem 1.1.

Let us choose some local coordinates $(x,z)$ of $\mathbb{P}^1\times\mathbb{P}^1$ in a neighbourhood of $(c,\infty)$ such that:

- $(c,\infty)$ has for coordinates $(0,0)$,
- $\{c\}\times\mathbb{P}^1$ has equation $x=0$ in a neighbourhood of $(c,\infty)$,
- $\mathbb{P}^1\times\{\infty\}$ has equation $z=0$ in a neighbourhood of $(c,\infty)$.

In these coordinates, $\pi_2$ is equal to $\frac{1}{z}$ in a neighbourhood of $(c,\infty)$. Then, according to lemma 3.1, we are led to prove the following lemma:

**Lemma 3.2.** *Let $\mathfrak{M}$ be a holonomic regular $\mathcal{D}_{\mathbb{C}^2}$-module. We denote the characteristic cycle of $\mathfrak{M}$ in a neighbourhood of $(0,0)$ by:*

$$Cch(\mathfrak{M}) = mT^*_{\mathbb{C}^2}\mathbb{C}^2 + m^{'}T^*_{(c,\infty)}\mathbb{C}^2 + m^{''}T^*_{x=0}\mathbb{C}^2 + m^{'''}T^*_{z=0}\mathbb{C}^2 + \sum m_l T^*_{Z_l}\mathbb{C}^2,$$

*where $Z_l$ are some germs of irreducible curves in a neighbourhood of $(0,0)$ distinct from $x=0$ and $z=0$.*

*Then,*

$$\chi(IR_{x=0}(\mathfrak{M}[\frac{1}{z}]e^{\frac{1}{z}})_{(0,0)} = -\sum_l m_l I_{(c,\infty)}(Z_l,\{z=0\}).$$

## §4. Proof of lemma 3.2

We break up the proof of lemma 3.2 in three steps:

**Lemma 4.1.** $\chi(IR_{x=0}(\mathfrak{M}[\frac{1}{z}]e^{\frac{1}{z}})_{(0,0)}) = \chi(IR_{z=0}(\mathfrak{M}[\frac{1}{xz}]e^{\frac{1}{z}})_{(0,0)})$.

We denote by $\Psi_z(\mathfrak{M}[\frac{1}{x}])$ the complex of nearby cycles of $\mathfrak{M}[\frac{1}{x}]$ relative to $z$. It is a complex of constructible sheaves on $\mathbb{C}\times\{0\}$ defined as follows.

Let $\eta$ small enough. We denote by $\widetilde{D^*(0,\eta)}$ the universal covering of $D^*(0,\eta)$. Let $(E,\pi,\tilde{z})$ be the fiber product over $D^*(0,\eta)$ of $\mathbb{C}\times D^*(0,\eta)$ and $\widetilde{D^*(0,\eta)}$. Then, we have the following diagram:

$$\begin{array}{ccccccc} \mathbb{C}\times\{0\} & \xhookrightarrow{\alpha} & \mathbb{C}^2 & \xhookleftarrow{\tilde{i}} & \mathbb{C}\times D^*(0,\eta) & \xleftarrow{\pi} & E \\ & & & & \Big\downarrow{\scriptstyle z} & & \Big\downarrow \\ & & & & D^*(0,\eta) & \longleftarrow & \widetilde{D^*(0,\eta)} \end{array} .$$

$$\Psi_z(\mathfrak{M}[\frac{1}{x}]) = \alpha^{-1}R(\tilde{i}\circ\pi)_*(\tilde{i}\circ\pi)^{-1}(DR(\mathfrak{M}[\frac{1}{x}]))$$

.

**Lemma 4.2.** $\chi(IR_{z=0}(\mathfrak{M}[\frac{1}{xz}]e^{\frac{1}{z}})_{(0,0)}) = \chi(\Psi_z(\mathfrak{M}[\frac{1}{x}])_{(0,0)})$.

**Lemma 4.3.** $\chi(\Psi_z(\mathfrak{M}[\frac{1}{x}])_{(0,0)}) = -\sum_l m_l I_{(c,\infty)}(Z_l, \mathbb{P}^1\times\{\infty\})$.

*Proof of lemma 4.1.* Let us first show that

$$IR_{x=0}(\mathfrak{M}[\frac{1}{z}]e^{\frac{1}{z}}) = R\Gamma_{x=0}(IR_{z=0}(\mathfrak{M}[\frac{1}{xz}]e^{\frac{1}{z}})).$$

Let $\eta$ be the inclusion of $\mathbb{C}\times\mathbb{C}^*$ in $\mathbb{C}^2$. By definition,

$$\begin{aligned} IR_{z=0}(\mathfrak{M}[\tfrac{1}{xz}]e^{\frac{1}{z}}) &= cone\left(DR(\mathfrak{M}[\tfrac{1}{xz}]e^{\frac{1}{z}}) \to R\eta_*\eta^{-1}(DR(\mathfrak{M}[\tfrac{1}{xz}]e^{\frac{1}{z}}))\right) \\ &= cone\left(DR(\mathfrak{M}[\tfrac{1}{xz}]e^{\frac{1}{z}}) \to R\eta_*(DR(\mathfrak{M}[\tfrac{1}{x}])_{|\mathbb{C}\times\mathbb{C}^*})\right) \end{aligned}$$

Now, consider the following diagram:

$$\begin{array}{ccc} \mathbb{C}\times\mathbb{C}^* & \xhookrightarrow{\eta} & \mathbb{C}^2 \\ \Big\uparrow{\scriptstyle J'} & & \Big\uparrow{\scriptstyle J} \\ \mathbb{C}^*\times\mathbb{C}^* & \xhookrightarrow{\eta'} & \mathbb{C}^*\times\mathbb{C} \end{array} .$$

As $\mathfrak{M}$ is regular, we have:

$$\begin{aligned} R\eta_*(DR(\mathfrak{M}[\tfrac{1}{x}])_{|\mathbb{C}\times\mathbb{C}^*}) &= R\eta_* RJ'_*(DR(\mathfrak{M})_{|\mathbb{C}^*\times\mathbb{C}^*}) \\ &= RJ_* R\eta'_*(DR(\mathfrak{M})_{|\mathbb{C}^*\times\mathbb{C}^*}). \end{aligned}$$

But $R\Gamma_{x=0}RJ_* = 0$. Then:

$$\begin{aligned} R\Gamma_{x=0}(IR_{z=0}(\mathfrak{M}[\tfrac{1}{xz}]e^{\frac{1}{z}})) &= R\Gamma_{x=0}(DR(\mathfrak{M}[\tfrac{1}{xz}]e^{\frac{1}{z}})) \\ &= IR_{x=0}(\mathfrak{M}(\tfrac{1}{z}]e^{\frac{1}{z}}), \end{aligned}$$

by definition of irregularity complex.

Then, we are led to prove that the complexes $R\Gamma_{x=0}(IR_{z=0}(\mathfrak{M}[\frac{1}{xz}]e^{\frac{1}{z}}))$ and $IR_{z=0}(\mathfrak{M}[\frac{1}{xz}]e^{\frac{1}{z}})$ have the same characteristic function at $(0,0)$.

Using the following distinguished triangle,

$$\begin{array}{ccc} & RJ_*J^{-1}(IR_{z=0}(\mathfrak{M}[\frac{1}{xz}]e^{\frac{1}{z}})) & \\ {\scriptstyle [+1]}\swarrow & & \nwarrow \\ R\Gamma_{x=0}(IR_{z=0}(\mathfrak{M}[\frac{1}{xz}]e^{\frac{1}{z}})) & \longrightarrow & IR_{z=0}(\mathfrak{M}[\frac{1}{xz}]e^{\frac{1}{z}}), \end{array}$$

it is sufficient to show that the characteristic function on $\{x = 0\}$ of the complex $RJ_*J^{-1}(IR_{z=0}(\mathfrak{M}[\frac{1}{xz}]e^{\frac{1}{z}}))$ is zero.

Now, if $\mathcal{F}$ is a constructible sheaf on $X$ and $P \in \{x = 0\}$,

$$\chi((RJ_*J^{-1}\mathcal{F})_P) = \chi((\mathbb{D}(J_!J^{-1}\mathbb{D}\mathcal{F}))_P) = \chi((J_!J^{-1}\mathbb{D}\mathcal{F})_P) = 0,$$

where $\mathbb{D}$ is the Verdier duality (see [11]). □

*Proof of lemma 4.2.* This is a particular case of a result of C. Sabbah (cf. corollary 5-2 of [10]). □

*Proof of lemma 4.3.* Denote by $\mathcal{C}^\bullet$ the complex $\Psi_z(\mathfrak{M}[\frac{1}{x}])$. By definition, $\mathcal{C}^\bullet_{(0,0)} = R(\tilde{i}\circ\pi)_*(\tilde{i}\circ\pi)^{-1}(DR(\mathfrak{M}[\frac{1}{x}]))_{(0,0)}$.

• Then, for all $k \in \mathbb{Z}$,

$$\mathcal{H}^k\mathcal{C}^\bullet_{(0,0)} = \underset{(0,0)\in U open}{indlim} \mathbb{R}^k\Gamma(U, R(\tilde{i}\circ\pi)_*(\tilde{i}\circ\pi)^{-1}(DR(\mathfrak{M}[\frac{1}{x}]))).$$

As $\{D(0,\eta_1)\times D(0,\eta_2)\}_{\eta_1,\eta_2}$ is a fundamental system of neighbourhoods of $(0,0)$, we have

$$\mathcal{H}^k\mathcal{C}^\bullet_{(0,0)} = \underset{\eta_1,\eta_2}{indlim}\mathbb{R}^k\Gamma(D(0,\eta_1)\times D(0,\eta_2), R(\tilde{i}\circ\pi)_*(\tilde{i}\circ\pi)^{-1}(DR(\mathfrak{M}[\frac{1}{x}]))).$$

• Let $\Sigma$ be a Whitney stratification associated with the constructible sheaf $DR(\mathfrak{M}[\frac{1}{x}])$. Then, for $\eta_1$ and $\eta_2$ small enough, there exists a homotopy equivalence $p : (\tilde{i}\circ\pi)^{-1}(D(0,\eta_1)\times D(0,\eta_2)) \to D(0,\eta_1)\times\{\tilde{\eta}\}$ compatible with $\Sigma$. Thus, $\mathcal{H}^k\mathcal{C}^\bullet_{(0,0)} = \underset{\eta_1,\eta_2}{indlim}\mathbb{R}\Gamma(D(0,\eta_1)\times\{\tilde{\eta}\}, DR(\mathfrak{M}[\frac{1}{x}]))$.

• Now, as $\mathfrak{M}$ is regular, $DR(\mathfrak{M}[\frac{1}{x}]) = RJ_*J^{-1}(DR(\mathfrak{M}))$. Then,

$$\begin{aligned}\mathcal{H}^k\mathcal{C}^\bullet_{(0,0)} &= \underset{\eta_1,\eta_2}{indlim}\mathbb{R}\Gamma(D(0,\eta_1)\times\{\tilde{\eta}\}, RJ_*J^{-1}(DR(\mathfrak{M}))) \\ &= \underset{\eta_1,\eta_2}{indlim}\mathbb{R}\Gamma(D^*(0,\eta_1)\times\{\tilde{\eta}\}, J^{-1}(DR(\mathfrak{M})))\end{aligned}$$

• Let fix $\eta_1$ and $\tilde{\eta}$ small enough such that the singular support of $\mathfrak{M}$ in $D^*(0,\eta_1)\times\{\tilde{\eta}\}$ is a finite number of points. Denote by $P_1,\ldots,P_s$ these points. They are the intersection points of $D^*(0,\eta_1)\times\{\tilde{\eta}\}$ and $\cup Z_l$. As $DR(\mathfrak{M})_{|D^*(0,\eta_1)\times\{\tilde{\eta}\}}$ is a complex of constructible sheaves with respect to the stratification $\{D^*(0,\eta_1)\times\{\tilde{\eta}\}\setminus\{P_1,\ldots,P_l\}, P_1,\ldots,P_l\}$, the Euler characteristic of $\mathbb{R}\Gamma(D^*(0,\eta_1)\times\{\tilde{\eta}\}, J^{'-1}(DR(\mathfrak{M}))$ is equal to:

$$\chi(\mathbb{R}\Gamma(D^*(0,\eta_1)\times\{\tilde{\eta}\}\setminus\{P_1,\ldots,P_l\}, DR(\mathfrak{M}))) + \sum_{i=1}^{l}\chi(DR(\mathfrak{M})_{P_i}).$$

Then, according to the index theorem of Kashiwara (cf. [3]),

$$\begin{aligned}&\chi(\mathbb{R}\Gamma(D^*(0,\eta_1)\times\{\tilde{\eta}\}, J^{'-1}(DR(\mathfrak{M}))) \\ &= rk(\mathfrak{M})\sum_l I_{(0,0)}(Z_l,\{z=0\}) + \sum_l(rk(\mathfrak{M})-m_l)I_{(0,0)}(Z_l,\{z=0\}) \\ &= -\sum_l m_l I_{(0,0)}(Z_l,\{z=0\})\end{aligned}$$

□

*Remark.* If $f$ and $g$ are two polynomials in two variables, we can compare theorem 1.1 and theorem 1 of [9]. Let us recall this theorem:

Let $\mathbb{X}$ be a smooth projective compactification of $\mathbb{C}^2$ such that there exists $F,G:\mathbb{X}\to\mathbb{P}^1$, two meromorphic maps, which extend $f$ and $g$. Let us denotes by $D$ the divisor $\mathbb{X}\setminus\mathbb{C}^2$. Let $\Gamma$ be the critical locus of $(F,G)$.

Let $c\in\mathbb{P}^1$. We denote by $\Delta_1$ the cycle in $\mathbb{P}^1\times\mathbb{P}^1$ which is the closure in $\mathbb{P}^1\times\mathbb{P}^1$ of $(F,G)(\Gamma)\cap(\mathbb{C}^2\setminus\{c\}\times\mathbb{C})$, where the image is counted with multiplicity and by $\Delta_2$ the cycle in $\mathbb{P}^1\times\mathbb{P}^1$ which is the closure in $\mathbb{P}^1\times\mathbb{P}^1$ of $(F,G)(D)\cap(\mathbb{C}^2\setminus\{c\}\times\mathbb{C})$, where the image is counted with multiplicity.

**Theorem 4.1.** *If $f$ and $g$ are algebraically independent, the irregularity number of $\mathcal{H}^0(f_+(\mathcal{O}_{\mathbb{C}^2}e^g))$ is equal to*

$$I_{(c,\infty)}(\Delta_1,\mathbb{P}^1\times\{\infty\}) + I_{(c,\infty)}(\Delta_2,\mathbb{P}^1\times\{\infty\}).$$

Then, we can prove that the germs $Z_l$ of irreducible curves in theorem 1.1 are the germs at $(c,\infty)$ of the irreducible branches of $\Delta_1\cup\Delta_2$. The multiplicity $m_l$ of $i_+(f,g)_+(\mathcal{O}_{\mathbb{C}^2})$ on $T^*_{Z_l}V$ are the multiplicity of $Z_k$ in $\Delta_1\cup\Delta_2$.

## References


[1] Dimca, A., Maaref, F., Sabbah, C. and Saito, M., Dwork cohomology and algebraic $\mathcal{D}$-modules, *Math. Ann.*, **318**, no. 1, (2000), 107–125.

[2] Kashiwara, M., Systems of microdifferential equation, *Progress in Math.*, **34**, Birkhäuser, Boston, (1983).

[3] Kashiwara, M., Index theorem for constructible sheaves, in: systèmes différentiels et singularités, *Astérisque*, **130**, (1985), 193–209.

[4] Maaref, F., Sur un analogue irrégulier de la connexion de Gauss-Manin, *Ann. Fac. Sc. Toul.*, **VIII(1)**, (1999), 117–124.

[5] Malgrange, B., Sur les points singuliers des équations différentielles, *L'enseignement Mathématique*, **20**, (1974), 147–176.

[6] Mebkhout, Z., Le théorème de positivité de l'irrégularité pour les $\mathcal{D}_X$-modules, *Grothendieck festschrift*, **III**, *Progress in Math.*, **88**, (1990), 84–131.

[7] Mebkhout, Z., Le théorème de positivité, le théorème de comparaison, le théorème d'existence de Riemann, *séminaire et congrès*, **8**, (2004), 165–307.

[8] Pham, F., La descente des cols par les onglets de Lefschetz avec vue sur Gauss-Manin, in: systèmes différentiels et singularités, *Astérisque*, **130**, (1985), 11–47.

[9] Roucairol, C., Irregularity of an analogue of the Gauss-Manin systems, *ArXiv:AG/*0505075, (2005).

[10] Sabbah, C., On the comparaison theorem for elementary irregular $\mathcal{D}$-modules, *Nagoya J. Math.*, **141** (1996), 107–124.

[11] Verdier, J.L., Dualité de Poincaré, *Séminaire Heidelberg-Strasbourg* 1966 − 67*, Publ. I.R.M.A.* **3**, Strasbourg, (1969).