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p-th powers in mod p cohomology of fibers

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Abstract. Let k be a non-negative integer. Let $F \hookrightarrow E \twoheadrightarrow B$ be a fibration whose base space B is a finite simply-connected CW-complex of dimension $\leqslant p^k$ and whose total space E is a path-connected CW-complex of dimension $\leqslant p^k - 1$. If $\alpha \in H^+(F;\mathbb{F}_p)$ then $\alpha^{p^k} = 0$. © 2001 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

Les puissances p-èmes dans la cohomologie modulo p de fibres

Résumé. *Soit $k \in \mathbb{N}^*$. Considérons une fibration $F \hookrightarrow E \twoheadrightarrow B$ dont la base B est un CW-complexe fini simplement connexe de dimension $\leqslant p^k$, et dont l'espace total E est un CW-complexe fini connexe par arcs de dimension $\leqslant p^k - 1$. Si $\alpha \in H^+(F;\mathbb{F}_p)$ alors $\alpha^{p^k} = 0$.* © 2001 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

Version française abrégée

Soit p un nombre premier quelconque. Nous notons $H^*(X) = \mathbb{F}_p \oplus H^+(X)$ la cohomologie de l'espace connexe par arcs X à coefficients dans le corps premier $\mathbb{F}_p$. Nous étudions les éléments $(\alpha^{p^k}, \alpha \in H^+(X), k \geqslant 1)$, appelés puissances p^k-èmes, de l'algèbre de cohomologie d'espaces X obtenues comme produit fibré (homotopique) de CW-complexes finis simplement connexes. Nous démontrons :

THÉORÈME A. – *Soient $r, k \in \mathbb{N}^*$. Considérons un produit fibré d'espaces* :

$$\begin{array}{ccc} E \times_B X & \longrightarrow & E \\ \downarrow & & \downarrow{\scriptstyle \pi} \\ X & \longrightarrow & B \end{array}$$

où

- *π est une fibration de Serre* ;
- *B est un CW-complexe fini r-connexe de dimension inférieure ou égale à rp^k* ;
- *E et X sont deux CW-complexes finis $(r-1)$-connexes tels que $E \times X$ soit de dimension inférieure ou égale à $rp^k - 1$.*

Alors les puissances p^k-èmes sont nulles dans $H^+(E \times_B X)$.

Note présentée par Henri CARTAN.

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Dans le cas particulier de l'espace B^{S^1} des lacets libres sur un CW-complexe B (*voir* version anglaise), nous pouvons minimiser les hypothèses du théorème A :

THÉORÈME B. – *Soient $r, k \in \mathbb{N}^*$. Si B est un CW-complexe fini r-connexe de dimension inférieure ou égale à rp^k alors les puissances p^k-èmes s'annulent dans $H^+(B^{S^1})$.*

THÉORÈME C (À comparer avec [8, 10.8]). – *Soient $r, k \in \mathbb{N}^*$. Soit $F \overset{j}{\hookrightarrow} E \overset{\pi}{\twoheadrightarrow} B$ une fibration de Serre d'espace total E connexe par arcs. Si la base B est un CW-complexe fini r-connexe de dimension inférieure ou égale à rp^k alors pour tout $\alpha \in H^*(F)$, $\alpha^{p^k} \in \operatorname{Im} H^*(j)$.*

En Corollaire du théorème A ou du théorème C, nous obtenons

COROLLARY. – *Soit B un CW-complexe fini simplement connexe. Pour tout $\alpha \in H^+(\Omega B)$, il existe $k \in \mathbb{N}^*$ telle que $\alpha^{p^k} = 0$.*

Signalons que ce corollaire résulte aussi du théorème suivant démontré par Lannes et Schwartz en utilisant les opérations de Steenrod dans la suite spectrale d'Eilenberg–Moore.

THÉORÈME [5, proposition 0.6]. – *Soit B un CW-complexe simplement connexe ayant un nombre fini de cellules en chaque dimension. Si l'algèbre de Steenrod agit sur $H^*(B)$ avec des orbites finies, alors elle agit aussi sur $H^*(\Omega B)$ avec des orbites finies.*

We work over the prime field $\mathbb{F}_p$ with p an odd or even prime. The homology and cohomology of spaces are considered with coefficients in $\mathbb{F}_p$.

In [1], Anick proved using algebraic models:

THEOREM [1, 9.1]. – *Let r be a non-negative integer. Let B be a simply-connected space with a finite type homology concentrated in degrees $i \in [r+1, rp]$. Then all p-th powers vanish in $H^+(\Omega B)$.*

This result was suggested by McGibbon and Wilkerson [7, p. 699]. The aim of this Note is to give two different generalisations of Anick theorem: Theorem A and Theorem C below.

The first one, whose proof is inspired by the proof of a result of Lannes and Schwartz [5, Proposition 0.6], uses the (vertical) Steenrod operations in the Eilenberg–Moore spectral sequence:

THEOREM A. – *Let r and k be two non-negative integers. Consider a fiber product of spaces*:

$$\begin{tikzcd} E \times_B X \arrow[r] \arrow[d] & E \arrow[d, "\pi"] \\ X \arrow[r] & B \end{tikzcd}$$

where

- *π is a Serre fibration and*
- *$H^*(E)$, $H^*(X)$ and $H^*(B)$ are of finite type.*

If B is simply-connected with homology concentrated in degrees $i \in [r+1, rp^k]$, and the product space $E \times X$ is path connected with homology $H_*(E \times X)$ concentrated in degrees $i \in [r, rp^k - 1]$, then all p^k-th powers vanish in $H^+(E \times_B X)$.

Proof. – We suppose that p is an odd prime. The case $p = 2$ is similar. Let $\mathcal{A}$ denote the mod p Steenrod algebra. The degree of an element α is denoted $|\alpha|$. Recall from [9,11,12], that the Eilenberg–Moore spectral sequence is a strongly convergent second quadrant cohomological spectral sequence of $\mathcal{A}$-modules:

$$E_2^{-s,*} \cong \operatorname{Tor}^{-s,*}_{H^*(B)}\big(H^*(E), H^*(X)\big) \Rightarrow H^*(E \times_B X).$$

More precisely, there exists a convergent filtration of $\mathcal{A}$-modules on $H^*(E\times_B X)$:

$$H^*(E\times_B X)\supset\dots F_s\supset F_{s-1}\dots F_1\supset F_0\supset F_{-1}=\{0\},$$

such that $\Sigma^{-s}F_s/F_{s-1}\cong E_\infty^{-s,*}$, $s\geqslant 0$. Here Σ^{-s} denotes the s-th desuspension of an $\mathcal{A}$-module.

Let $\alpha\in F_s$ such that the class $[\alpha]\in F_s/F_{s-1}$ is non-zero. We want to prove that $\alpha^{p^k}=0$. As an $\mathcal{A}$-module, $\mathrm{Tor}^{-s,*}_{H^*(B)}(H^*(E),H^*(X))$ is the s-th homology group of a complex of $\mathcal{A}$-modules, namely the Bar construction, whose s-th term is $H^*(E)\otimes H^+(B)^{\otimes s}\otimes H^*(X)$.

The element $\Sigma^{-s}[\alpha]\in E_\infty^{-s,*}$ is represented by a cycle of the form $e[b_1|\dots|b_s]x$, where $e\in H^*(E)$, $(b_i)_{1\leqslant i\leqslant s}\in H^+(B)$ and $x\in H^*(X)$. So $rs\leqslant|\alpha|\leqslant(rp^k-1)(s+1)$.

Case 1. – When $|e|+|x|\geqslant r$. Then $|\alpha|\geqslant r(s+1)$. Therefore, by a degree argument, the element α^{p^k} of F_s is zero.

Case 2. – When $e=x=1$. Since the Cartan formula applies, $\Sigma^{-s}[\alpha^p]=\mathrm{P}^{|\alpha|/2}\Sigma^{-s}[\alpha]$ is represented by the element of the Bar construction,

$$\sum_{i_1+\dots+i_s=|\alpha|/2}\big[\mathrm{P}^{i_1}b_1|\dots|\mathrm{P}^{i_s}b_s\big]\in H^+(B)^{\otimes s}.$$

So $\Sigma^{-s}[\alpha^{p^k}]$ is zero for degree reasons. Therefore α^{p^k} belongs to F_{s-1} which is concentrated in degrees $\leqslant(rp^k-1)s$, thus $\alpha^{p^k}=0$. □

Let B be a space. The *free loop space* on B, denoted B^{S^1}, is the set of continuous (unpointed) maps from the circle S^1 to B. It can be defined as a fibre product:

$$\begin{tikzcd} B^{S^1} \arrow[r, hook] \arrow[d, "ev"'] & B^{[0,1]} \arrow[d, "\pi"] \\ B \arrow[r, "\Delta"'] & B\times B \end{tikzcd}$$

In this particular case, we can improve Theorem A.

THEOREM B. – *Let r and k be two non-negative integers. If B is a simply-connected space with finite type homology concentrated in degrees $i\in[r+1,rp^k]$, then all p^k-th powers vanish in $H^+(B^{S^1})$.*

Proof. – The Eilenberg–Moore spectral sequence for the previous fiber product satisfies:

$$E_2^{-s,*}\cong\mathrm{HH}_s\big(H^*(B)\big)\Rightarrow H^*\big(B^{S^1}\big).$$

Here HH_* denotes the Hochschild homology. As an $\mathcal{A}$-module [1], $\mathrm{HH}_s(H^*(B))$ is the s-th homology group of a complex of $\mathcal{A}$-modules, namely the Hochschild complex, whose s-th term is $H^*(B)\otimes H^+(B)^{\otimes s}$.

The same arguments as in the proof of Theorem A allow us to conclude except in case 2 for $s=1$. If $\alpha\in F_1\subset H^*(B^{S^1})$, we can only affirm that $\alpha^{p^k}\in F_0$. The evaluation map $ev: B^{S^1}\twoheadrightarrow B$ admits a section σ. So $H^*(ev):H^*(B)\hookrightarrow H^*(B^{S^1})$ admits $H^*(\sigma)$ as retract. The edge homomorphism:

$$H^*(B)=E_2^{0,*}\twoheadrightarrow E_3^{0,*}\dots\twoheadrightarrow E_\infty^{0,*}=F_0\subset H^*\big(B^{S^1}\big)$$

correspond to $H^*(ev)$. Since $\alpha^{p^k}\in F_0=H^*(B)$, $\alpha^{p^k}=[H^*(\sigma)(\alpha)]^{p^k}$. For degree reason, all p^k-th powers are zero in $H^+(B)$. So $\alpha^{p^k}=0$. □

In [3], Félix, Halperin and Thomas give a slightly more complicated proof of Anick theorem. Their proof uses the vertical and horizontal Steenrod operations in the Serre spectral sequence:

THEOREM [3, 2.9(i)]. – *Let r and k be two non-negative integers. If B is a simply-connected space with a finite type homology concentrated in degrees $i \in [r+1, rp^k]$ then all p^k-th powers vanish in $H^+(\Omega B)$.*

This result generalizes in:

THEOREM C (Compare with [8, 10.8]). – *Let r and k be two non-negative integers. Let $F \overset{j}{\hookrightarrow} E \overset{\pi}{\twoheadrightarrow} B$ be a Serre fibration with E path connected. If B is a simply-connected space with finite type homology concentrated in degrees $i \in [r+1, rp^k]$ then, for any $\alpha \in H^*(F)$, $\alpha^{p^k} \in \operatorname{Im} H^*(j)$.*

Proof. – The proof follows the lines of [3, 2.9]. Since $H^{\leqslant r}(B) = 0$, $\alpha \in E_2^{0,*}$ survives till $E_{r+1}^{0,*}$. Therefore by a theorem of Araki [2] and Vázquez [13] (*see* also [10], Proposition 2.5, Case 2), $\alpha^{p^k} \in E_2^{0,*}$ survives till $E_{rp^k+1}^{0,*}$. Since $H^{>rp^k}(B) = 0$,

$$E_{rp^k+1}^{0,*} = E_\infty^{0,*} = \operatorname{Im} H^*(j).$$

□

In order to see that the hypothesis in the Félix–Halperin–Thomas theorem (and in Theorem B) cannot be improved, consider $B = \Sigma\mathbb{CP}^{p^k}$, the suspension of the p^k-dimension complex projective space.

Observe also that in Theorem C, α^{p^k} is not zero in general. Indeed, take π to be the fibration associated to the suspension of the Hopf map from S^{2p^k-1} to $\mathbb{CP}^{p^k-1}$ [8, Remark 9.9].

Finally, we remark that the following question of McGibbon and Wilkerson remains unsolved.

Question [7, p. 699] (See also [6], Section 9, Question 3). – Let B be a finite simply-connected CW-complex and p a prime large enough. Do all the Steenrod operations act trivially on $H^*(\Omega B)$?

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[1] To prove it, redo [9] using the cocyclic Cobar construction of Jones ([4], exemple 1.2) instead of the geometric Cobar construction.

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