Rational homotopy – Sullivan models

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Abstract. This chapter is a short introduction to Sullivan models. In particular, we find the Sullivan model of a free loop space and use it to prove the Vigué-Poirrier-Sullivan theorem on the Betti numbers of a free loop space.

In the previous chapter, we have seen the following theorem due to Gromoll and Meyer.

Theorem 0.1. Let M be a compact simply connected manifold. If the sequence of Betti numbers of the free loop space on M, M^{S^1} , is unbounded then any Riemannian metric on M carries infinitely many non trivial and geometrically distinct closed geodesics.

In this chapter, using Rational homotopy, we will see exactly when the sequence of Betti numbers of M^{S^1} over a field of caracteristic 0 is bounded (See Theorem 6.1 and its converse Proposition 5.5). This was one of the first major applications of rational homotopy.

Rational homotopy associates to any rational simply connected space, a commutative differential graded algebra. If we restrict to almost free commutative differential graded algebras, that is "Sullivan models", this association is unique.

1 Graded differential algebra

1.1 Definition and elementary properties

All the vector spaces are over \mathbb{Q} (or more generally over a field \mathbf{k} of characteric 0). We will denote by \mathbb{N} the set of non-negative integers.

Definition 1.1. A (non-negatively upper) graded vector space V is a family $\{V^n\}_{n\in\mathbb{N}}$ of vector spaces. An element $v \in V_i$ is an element of V of degree i. The degree of v is denoted |v|. A differential d in V is a sequence of linear maps

 $d^n: V^n \to V^{n+1}$ such that $d^{n+1} \circ d^n = 0$, for all $n \in \mathbb{N}$. A differential graded vector space or *complex* is a graded vector space equipped with a differential. A morphism of complexes $f: V \xrightarrow{\simeq} W$ is a *quasi-isomorphism* if the induced map in homology $H(f): H(V) \xrightarrow{\simeq} H(W)$ is an isomorphism in all degrees.

Definition 1.2. A graded algebra is a graded vector space $A = \{A^n\}_{n \in \mathbb{N}}$, equipped with a multiplication $\mu : A^p \otimes A^q \to A^{p+q}$. The algebra is commutative if $ab = (-1)^{|a||b|} ba$ for all a and $b \in A$.

Definition 1.3. A differential graded algebra or dga is a graded algebra equipped with a differential $d: A^n \to A^{n+1}$ which is also a *derivation*: this means that for a and $b \in A$

$$d(ab) = (da)b + (-1)^{|a|}a(db).$$

A *cdga* is a commutative dga.

Example 1.4. 1) Let (B, d_B) and (C, d_C) be two cdgas. Then the tensor product $B \otimes C$ equipped with the multiplication

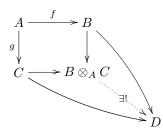
$$(b \otimes c)(b' \otimes c') := (-1)^{|c||b'|} bb' \otimes cc'$$

and the differential

$$d(b \otimes c) = (db) \otimes c + (-1)^{|b|} b \otimes dc.$$

is a cdga. The *tensor product of cdgas* is the sum (or coproduct) in the category of cdgas.

2) More generally, let $f : A \to B$ and $g : A \to C$ be two morphisms of cdgas. Let $B \otimes_A C$ be the quotient of $B \otimes C$ by the sub graded vector spanned by elements of the form $bf(a) \otimes c - b \otimes g(a)c$, $a \in A$, $b \in B$ and $c \in C$. Then $B \otimes_A C$ is a cgda such that the quotient map $B \otimes C \twoheadrightarrow B \otimes_A C$ is a morphism of cdgas. The cdga $B \otimes_A C$ is the pushout of f and g in the category of cdgas:



3) Let V and W be two graded vector spaces. We denote by ΛV the free graded commutative algebra on V.

If $V = \mathbb{Q}v$, i. e. is of dimension 1 and generated by a single element v, then

 $-\Lambda V$ is $E(v) = \mathbb{Q} \oplus \mathbb{Q}v$, the exterior algebra on v if the degree of v is odd and $-\Lambda V$ is $\mathbb{Q}[v] = \bigoplus_{n \in \mathbb{N}} \mathbb{Q}v^n$, the polynomial or symmetric algebra on v if the degree of v is even.

Since Λ is left adjoint to the forgetful functor from the category of commutative graded algebras to the category of graded vector spaces, Λ preserves sums: there

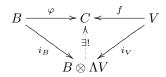
is a natural isomorphism of commutative graded algebras $\Lambda(V \oplus W) \cong \Lambda V \otimes \Lambda W$.

Therefore ΛV is the tensor product $E(V^{odd}) \otimes S(V^{even})$ of the exterior algebra on the generators of odd degree and of the polynomial algebra on the generators of even degree.

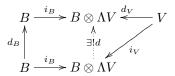
Definition 1.5. Let $f : A \to B$ be a morphism of commutative graded algebras. Let $d : A \to B$ be a linear map of degree k. By definition, d is a (f, f)-derivation if for a and $b \in A$

$$d(ab) = (da)f(b) + (-1)^{k|a|}f(a)(db).$$

Property 1.6 (Universal properties). 1) Let $i_B : B \hookrightarrow B \otimes \Lambda V$, $b \mapsto b \otimes 1$ and $i_V : V \hookrightarrow B \otimes \Lambda V$, $v \mapsto 1 \otimes v$ be the inclusion maps. Let $\varphi : B \to C$ be a morphism of commutative graded algebras. Let $f : V \to C$ be a morphism of graded vector spaces. Then φ and f extend uniquely to a morphism $B \otimes \Lambda V \to C$ of commutative graded algebras such that the following diagram commutes



2) Let $d_B : B \to B$ be a derivation of degree k. Let $d_V : V \to B \otimes \Lambda V$ be a linear map of degree k. Then there is a unique derivation d such that the following diagram commutes.



3) Let $f : \Lambda V \to B$ be a morphism of commutative graded algebras. Let $d_V : V \to B$ be a linear map of degree k. Then there exists a unique (f, f)-derivation d extending d_V :

$$V \xrightarrow{d_V} B$$

$$i_V \downarrow \qquad \exists !d$$

$$\Lambda V$$

Proof. 1) Since ΛV is the free commutative graded algebra on V, f can be extended to a morphism of graded algebras $\Lambda V \to C$. Since the tensor product of commutative graded algebras is the sum in the category of commutative graded algebras, we obtain a morphism of commutative graded algebras from $B \otimes \Lambda V$ to C.

2) Since $b \otimes v_1 \dots v_n$ is the product $(b \otimes 1)(1 \otimes v_1) \dots (1 \otimes v_n), d(b \otimes v_1 \dots v_n)$

is given by

$$d_B(b) \otimes v_1 \dots v_n + \sum_{i=1}^n (-1)^{k(|b|+|v_1|+\dots+|v_{i-1}|)} (b \otimes v_1 \dots v_{i-1}) (d_V v_i) (1 \otimes v_{i+1} \dots v_n)$$

3) Similarly, $d(v_1 \dots v_n)$ is given by

$$\sum_{i=1}^{n} (-1)^{k(|v_1|+\dots+|v_{i-1}|)} f(v_1) \dots f(v_{i-1}) d_V(v_i) f(v_{i+1}) \dots f(v_n)$$

1.2 Sullivan models of spheres

Sullivan models of odd spheres S^{2n+1} , $n \ge 0$.

Consider a cdga $A(S^{2n+1})$ whose cohomology is isomorphic as graded algebras to the cohomology of S^{2n+1} with coefficients in **k**:

$$H^*(A(S^{2n+1})) \cong H^*(S^{2n+1}).$$

When **k** is \mathbb{R} , you can think of A as the De Rham algebra of forms on S^{2n+1} . There exists a cycle v of degree 2n + 1 in $A(S^{2n+1})$ such that

$$H^*(A(S^{2n+1})) = \Lambda[v].$$

The inclusion of complexes $(\mathbf{k}v, 0) \hookrightarrow A(S^{2n+1})$ extends to a unique morphism of cdgas $m : (\Lambda v, 0) \to A(S^{2n+1})$ (Property 1.6):

The induced morphism in homology H(m) is an isomorphism. We say that $m : (\Lambda v, 0) \xrightarrow{\simeq} A(S^{2n+1})$ is a Sullivan model of S^{2n+1}

Sullivan models of even spheres S^{2n} , $n \ge 1$.

Exactly as above, we construct a morphism of cdga $m_1 : (\Lambda v, 0) \to A(S^{2n})$. But now, $H(m_1)$ is not an isomorphism:

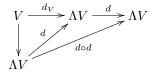
 $H(m_1)(v) = [v]$. Therefore $H(m_1)(v^2) = [v^2] = [v]^2 = 0$. Since $[v^2] = 0$ in $H^*(A(S^{2n}))$, there exists an element $\psi \in A(S^{2n})$ of degree 4n - 1 such that $d\psi = v^2$.

Let w denote another element of degree 4n - 1. The morphism of graded vector spaces $\mathbf{k}v \oplus \mathbf{k}w \hookrightarrow A(S^{2n})$, mapping v to v and w to ψ extends to a unique morphism of commutative graded algebras $m : \Lambda(v, w) \to A(S^{2n})$ (1) of Property 1.6):

The linear map of degree +1, $d_V : V := \mathbf{k}v \oplus \mathbf{k}w \to \Lambda(v, w)$ mapping v to 0 and w to v^2 extends to a unique derivation $d : \Lambda(v, w) \to \Lambda(v, w)$ (2) of Property 1.6).

$$\begin{aligned} \mathbf{k}v \oplus \mathbf{k}w \xrightarrow{a_V} \Lambda(v,w) \\ \downarrow & \stackrel{\pi}{\exists !d} \\ \Lambda(v,w) \end{aligned}$$

Since d is a derivation of odd degree, $d \circ d$ (which is equal to 1/2[d, d]) is again a derivation. The following diagram commutes



Since the composite $d \circ d_V$ is null, by unicity (2) of Property 1.6), the derivation $d \circ d$ is also null. Therefore $(\Lambda V, d)$ is a cdga. This is the general method to check that $d \circ d = 0$.

Denote by d_A the differential on $A(S^{2n})$. Let's check now that $d_A \circ m = m \circ d$. Since d_A and d are both (id, id)-derivations, $d_A \circ m$ and $m \circ d$ are both (m, m)-derivations.

Since $d_A(m(v)) = d_A(v) = 0 = m(0) = m(d(v))$ and $d_A(m(w)) = d_A(\psi) = v^2 = m(v^2) = m(d(w))$, $d_A \circ m$ and $m \circ d$ coincide on V. Therefore by unicity (3) of Property 1.6), $d_A \circ m = m \circ d$. Again, this method is general. So finally, we have proved that m is a morphism of cdgas. Now we prove that H(m) is an isomorphism, by checking that H(m) sends a basis to a basis.

2 Sullivan models

2.1 Definitions

Let V be a graded vector space. Denote by $V^+ = V^{\geq 1}$ the sub graded vector space of V formed by the elements of V of positive degrees: $V = V^0 \oplus V^+$.

Definition 2.1. A *relative Sullivan model* (or *cofibration* in the category of cdgas) is a morphism of cdgas of the form

$$(B, d_B) \hookrightarrow (B \otimes \Lambda V, d), b \mapsto b \otimes 1$$

where

• $H^0(B) \cong \mathbf{k}$,

• $V = V^{\geq 1}$,

• and V is the direct sum of graded vector spaces V(k):

$$\forall n, V^n = \bigoplus_{k \in \mathbb{N}} V(k)^n$$

such that $d: V(0) \to B \otimes \mathbf{k}$ and $d: V(k) \to B \otimes \Lambda(V(< k))$. Here V(< k) denotes the direct sum $V(0) \oplus \cdots \oplus V(k-1)$.

Let $k \in \mathbb{N}$. Denote by $\Lambda^k V$ the sub graded vector space of ΛV generated by elements of the form $v_1 \wedge \cdots \wedge v_k$, $v_i \in V$. Elements of $\Lambda^k V$ have by definition wordlength k. For example $\Lambda V = \mathbf{k} \oplus V \oplus \Lambda^{\geq 2} V$.

Definition 2.2. A relative Sullivan model $(B, d_B) \hookrightarrow (B \otimes \Lambda V, d)$ is minimal if $d: V \to B^+ \otimes \Lambda V + B \otimes \Lambda^{\geq 2} V$. A (minimal) Sullivan model is a (minimal) relative Sullivan model of the form $(B, d_B) = (\mathbf{k}, 0) \hookrightarrow (\Lambda V, d)$.

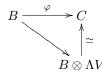
Example 2.3. [5, end of the proof of Lemma 23.1] Let $(\Lambda V, d)$ be cdga such that $V = V^{\geq 2}$. Then $(\Lambda V, d)$ is a Sullivan model.

proof assuming the minimality condition. [5, p. 144] Suppose that $d: V \to \Lambda^{\geq 2}V$. In this case, the V(k) are easy to define: let $V(k) := V^k$ for $k \in N$. Let $v \in V^k$. By the minimality condition, dv is equal to a sum $\sum_i x_i y_i$ where the non trivial elements x_i and y_i are both of positive length and therefore both of degre ≥ 2 . Since $|x_i| + |y_i| = |dv| = k + 1$, both x_i and y_i are of degree less than k. Therefore dv belongs to $\Lambda(V^{\leq k}) = \Lambda(V(\leq k))$.

Property 2.4. The composite of relative Sullivan models is again a Sullivan relative model.

Definition 2.5. Let C be a cdga. A (minimal) Sullivan model of C is a (minimal) Sullivan model $(\Lambda V, d)$ such that there exists a quasi-isomorphism of cdgas $(\Lambda V, d) \xrightarrow{\simeq} C$.

Let $\varphi : B \to C$ be a morphism of cdgas. A (minimal) relative Sullivan model of φ is a (minimal) relative Sullivan model $(B, d_B) \hookrightarrow (B \otimes \Lambda V, d)$ such that φ can be decomposed as the composite of the relative Sullivan model and of a quasi-isomorphism of cdgas:



Theorem 2.6. Any morphism $\varphi : B \to C$ of cdgas admits a minimal relative Sullivan model if $H^0(B) \cong \mathbf{k}$, $H^0(\varphi)$ is an isomorphism and $H^1(\varphi)$ is injective.

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This theorem is proved in general by Proposition 14.3 and Theorem 14.9 of [5]. But in practice, if $H^1(\varphi)$ is an isomorphism, we construct a minimal relative Sullivan model, by induction on degrees as in Proposition 12.2. of [5].

2.2 An example of relative Sullivan model

Consider the minimal Sullivan model of an odd sphere found in section 1.2

$$(\Lambda v, 0) \xrightarrow{\simeq} A(S^{2n+1}).$$

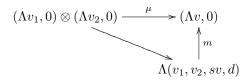
Assume that $n \geq 1$. Consider the multiplication of Λv : the morphism of cdgas

$$\mu: (\Lambda v_1, 0) \otimes (\Lambda v_2, 0) \to (\Lambda v, 0), v_1 \mapsto v, v_2 \mapsto v.$$

Recall that v, v_1 and v_2 are of degree 2n + 1.

Denote by sv an element of degree |sv| = |s| + |v| = -1 + |v|. The operator s of degre -1 is called the *suspension*.

We construct now a minimal relative Sullivan model of μ . Define $d(sv) = v_2 - v_1$. Let $m : \Lambda(v_1, v_2, sv), d \to (\Lambda v, 0)$ be the unique morphism of cdgas extending μ such that m(sv) = 0.



Definition 2.7. Let A be a differential graded algebra such that $A^0 = \mathbf{k}$. The complex of indecomposables of A, denoted Q(A), is the quotient $A^+/\mu(A^+ \otimes A^+)$.

The complex of indecomposables of $(\Lambda v, 0)$, $Q((\Lambda v, 0))$, is $(\mathbf{k}v, 0)$ while

$$Q(\Lambda(v_1, v_2, sv, d)) = (\mathbf{k}v_1 \oplus \mathbf{k}v_2 \oplus \mathbf{k}sv, d(sv) = v_2 - v_1).$$

The morphism of complexes $Q(m) : (\mathbf{k}v, 0) \to (\mathbf{k}v_1 \oplus \mathbf{k}v_2 \oplus \mathbf{k}sv, d(sv) = v_2 - v_1)$ map v_1 to v, v_2 to v and sv to 0. It is easy to check that Q(m) is a quasiisomorphism of complexes.

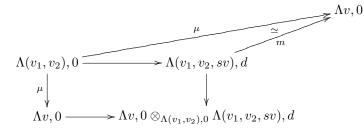
By Proposition 14.13 of [5], since m is a morphism of cdgas between Sullivan model, Q(m) is a quasi-isomorphim of if and only if m is a quasi-isomorphism.

So we have proved that m is a quasi-isomorphism and therefore

$$(\Lambda v_1, 0) \otimes (\Lambda v_2, 0) \hookrightarrow \Lambda(v_1, v_2, sv, d)$$

is a minimal relative Sullivan model of μ . Consider the following commutative

diagram of cdgas where the square is a pushout



It is easy to check that the cdga $\Lambda v, 0 \otimes_{\Lambda(v_1, v_2), 0} \Lambda(v_1, v_2, sv), d$ is isomorphic to $\Lambda(v, sv), 0$. As we will explain later, we have computed in fact, the minimal Sullivan model $\Lambda(v, sv), 0$ of the free loop space $(S^{2n+1})^{S^1}$. In particular, the cohomology algebra $H^*((S^{2n+1})^{S^1}; \mathbf{k})$ is isomorphic to $\Lambda(v, sv)$. We can deduce easily that for $p \in \mathbb{N}$, dim $H^p((S^{2n+1})^{S^1}) \leq 1$. So we have shown that the sequence of Betti numbers of the free loop space on odd dimensional spheres is bounded.

2.3 The relative Sullivan model of the multiplication

Proposition 2.8. [6, Example 2.48] Let $(\Lambda V, d)$ be a relative minimal Sullivan model with $V = V^{\geq 2}$ (concentrated in degrees ≥ 2). Then the multiplication $\mu : (\Lambda V, d) \otimes (\Lambda V, d) \twoheadrightarrow (\Lambda V, d)$ admits a minimal relative Sullivan model of the form $(\Lambda V \otimes \Lambda V \otimes \Lambda sV, D)$.

Constructive proof. We proceed by induction on $n \in \mathbb{N}^*$ to construct quasi-isomorphisms of cdgas $\varphi_n : (\Lambda V^{\leq n} \otimes \Lambda V^{\leq n} \otimes \Lambda s V^{\leq n}, D) \xrightarrow{\simeq} (\Lambda V^{\leq n}, d)$ extending the multiplication on $\Lambda V^{\leq n}$.

Suppose that φ_n is constructed. We now define φ_{n+1} extending φ_n and μ , the multiplication on ΛV . Let $v \in V^{n+1}$. Then $d(v) \in \Lambda^{\geq 2}(V^{\leq n})$ and $\varphi_n(dv \otimes 1 \otimes 1 - 1 \otimes dv \otimes 1) = 0$. Since φ_n is a surjective quasi-isomorphism, by the long exact sequence associated to a short exact sequence of complexes, Ker φ_n is acyclic. Therefore since $dv \otimes 1 \otimes 1 - 1 \otimes dv \otimes 1$ is a cycle, there exists an element γ of degree n + 1 of $\Lambda V^{\leq n} \otimes \Lambda V^{\leq n} \otimes \Lambda s V^{\leq n}$ such that $D(\gamma) = dv \otimes 1 \otimes 1 - 1 \otimes dv \otimes 1$ and $\varphi_n(\gamma) = 0$. For degree reasons, γ is decomposable, i. e. has wordlength ≥ 2 . We define $D(1 \otimes 1 \otimes sv) = v \otimes 1 \otimes 1 - 1 \otimes v \otimes 1 - \gamma$ and $\varphi_{n+1}(1 \otimes 1 \otimes sv) = 0$. Since $D \circ D(1 \otimes 1 \otimes sv) = 0$ and $d \circ \varphi_{n+1}(1 \otimes 1 \otimes sv) = \varphi_{n+1} \circ d(1 \otimes 1 \otimes sv)$, by Property 1.6, the derivation D is a differential on $\Lambda V^{\leq n+1} \otimes \Lambda V^{\leq n+1} \otimes \Lambda s V^{\leq n+1}$ and the morphism of graded algebras φ_{n+1} is a morphism of complexes.

The complex of indecomposables of $(\Lambda V^{\leq n+1} \otimes \Lambda V^{\leq n+1} \otimes \Lambda s V^{\leq n+1}, D)$,

$$Q((\Lambda V^{\leq n+1} \otimes \Lambda V^{\leq n+1} \otimes \Lambda s V^{\leq n+1}, D)$$

is $(V^{\leq n+1} \oplus V^{\leq n+1} \oplus sV^{\leq n+1}, d)$ with differential d given by $d(v' \oplus v^* \oplus sv) = v \oplus -v \oplus 0$ for v', v^* and $v \in V^{\leq n+1}$. Therefore it is easy to check that $Q(\varphi_{n+1})$ is a quasi-isomorphism. So by Proposition 14.13 of [5], φ_{n+1} is a quasi-isomorphism. Since γ is of degree n+1 and $sV^{\leq n}$ is of degree < n, this relative Sullivan model

is minimal. We now define $\varphi : (\Lambda V \otimes \Lambda V \otimes \Lambda sV, D) \twoheadrightarrow (\Lambda V, d)$ as

$$\lim_{\longrightarrow} \varphi_n = \bigcup_{n \in \mathbb{N}} \varphi_n : \bigcup_{n \in \mathbb{N}} \left(\Lambda V^{\leq n} \otimes \Lambda V^{\leq n} \otimes \Lambda s V^{\leq n} \right) \to \bigcup_{n \in \mathbb{N}} \Lambda V^{\leq n}.$$

Since homology commutes with direct limits in the category of complexes [14, Chap 4, Sect 2, Theorem 7], $H(\varphi) = \lim H(\varphi_n)$ is an isomorphism.

3 Rational homotopy theory

Let X be a topological space. Denote by $S^*(X)$ the singular cochains of X with coefficients in **k**. The dga $S^*(X)$ is almost never commutative. Nevertheless, Sullivan, inspired by Quillen proved the following theorem.

Theorem 3.1. [5, Corollary 10.10] For any topological space X, there exists two natural quasi-isomorphisms of dgas

$$S^*(X) \xrightarrow{\simeq} D(X) \xleftarrow{\sim} A_{PL}(X)$$

where $A_{PL}(X)$ is commutative.

Remark 3.2. This cdga $A_{PL}(X)$ is called the algebra of *polynomial differential* forms. If $\mathbf{k} = \mathbb{R}$ and X is a smooth manifold M, you can think that $A_{PL}(M)$ is the De Rham algebra of differential forms on M, $A_{DR}(M)$ [5, Theorem 11.4].

Definition 3.3. [6, Definition 2.34] Two topological spaces X and Y have the same *rational homotopy type* if there exists a finite sequence of continuous applications

$$X \xrightarrow{f_0} Y_1 \xleftarrow{f_1} Y_2 \dots Y_{n-1} \xleftarrow{f_{n-1}} Y_n \xrightarrow{f_n} Y$$

such that the induced maps in rational cohomology

$$H^{*}(X;\mathbb{Q}) \stackrel{H^{*}(f_{0})}{\leftarrow} H^{*}(Y_{1};\mathbb{Q}) \stackrel{H^{*}(f_{1})}{\rightarrow} H^{*}(Y_{2};\mathbb{Q}) \dots H^{*}(Y_{n-1}1;\mathbb{Q})$$

$$\stackrel{H^{*}(f_{n-1})}{\rightarrow} H^{*}(Y_{n};\mathbb{Q}) \stackrel{H^{*}(f_{n})}{\leftarrow} H^{*}(Y;\mathbb{Q})$$

are all isomorphisms.

Theorem 3.4. Let X be a path connected topological space.

1) (Unicity of minimal Sullivan models [5, Corollary p. 191]) Two minimal Sullivan models of $A_{PL}(X)$ are isomorphic.

2) Suppose that X is simply connected and $\forall n \in \mathbb{N}$, $H_n(X; \mathbf{k})$ is finite dimensional. Let $(\Lambda V, d)$ be a minimal Sullivan model of X. Then [5, Theorem 15.11] for all $n \in \mathbb{N}$, V^n is isomorphic to $Hom_{\mathbf{k}}(\pi_n(X) \otimes_{\mathbb{Z}} \mathbf{k}, \mathbf{k}) \cong Hom_{\mathbb{Z}}(\pi_n(X), \mathbf{k})$. In particular [5, Remark 1 p.208], Dimension $V^n = Dimension \pi_n(X) \otimes_{\mathbb{Z}} \mathbf{k} < \infty$.

Remark 3.5. The isomorphim of graded vector spaces between V and Hom_k $(\pi_*(X) \otimes_{\mathbb{Z}} \mathbf{k}, \mathbf{k})$ is natural in some sense [6, p. 75-6] with respect to maps $f : X \to Y$. The

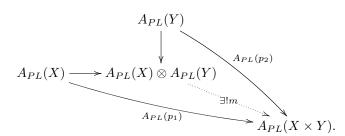
isomorphism behaves well also with respect to the long exact sequence associated to a (Serre) fibration ([5, Proposition 15.13] or [6, Proposition 2.65]).

Theorem 3.6. [6, Proposition 2.35][5, p. 139] Let X and Y be two simply connected topological spaces such that $H^n(X; \mathbb{Q})$ and $H^n(Y; \mathbb{Q})$ are finite dimensional for all $n \in \mathbb{N}$. Let $(\Lambda V, d)$ be a minimal Sullivan model of X and let $(\Lambda W, d)$ be a minimal Sullivan model of Y. Then X and Y have the same rational homotopy type if and only if $(\Lambda V, d)$ is isomorphic to $(\Lambda W, d)$ as cdgas.

4 Sullivan model of a pullback

4.1 Sullivan model of a product

Let X and Y be two topological spaces. Let $p_1 : X \times Y \rightarrow Y$ and $p_2 : X \times Y \rightarrow X$ be the projection maps. Let m be the unique morphism of cdgas given by the universal property of the tensor product (Example 1.4 1))



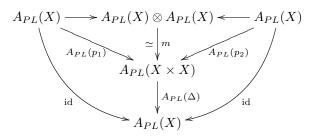
Assume that $H^*(X; \mathbf{k})$ or $H^*(Y; \mathbf{k})$ is finite dimensional in all degrees. Then [5, Example 2, p. 142-3] m is a quasi-isomorphism. Let $m_X : \Lambda V \xrightarrow{\simeq} A_{PL}(X)$ be a Sullivan model of X. Let $m_Y : \Lambda W \xrightarrow{\simeq} A_{PL}(Y)$ be a Sullivan model of Y. Then by Künneth theorem, the composite

$$\Lambda V \otimes \Lambda W \stackrel{m_X \otimes m_Y}{\to} A_{PL}(X) \otimes A_{PL}(Y) \stackrel{m}{\to} A_{PL}(X \times Y)$$

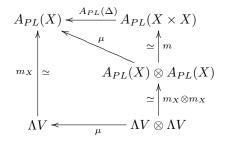
is a quasi-isomorphism of cdgas. Therefore we have proved that "the Sullivan model of a product is the tensor product of the Sullivan models".

4.2 the model of the diagonal

Let X be a topological space such that $H^*(X)$ is finite dimensional in all degrees. Denote by $\Delta : X \to X \times X$, $x \mapsto (x, x)$ the diagonal map of X. Using the previous paragraph, since $A_{PL}(p_1 \circ \Delta) = A_{PL}(p_2 \circ \Delta) = A_{PL}(id) = id$, we have the commutative diagram of cdgas.



Therefore the composite $A_{PL}(X) \otimes A_{PL}(X) \xrightarrow{m} A_{PL}(X \times X) \xrightarrow{A_{PL}(\Delta)} A_{PL}(X)$ coincides with the multiplication $\mu : A_{PL}(X) \otimes A_{PL}(X) \to A_{PL}(X)$. Therefore the following diagram of cdgas commutes



Here $m_X : \Lambda V \xrightarrow{\simeq} A_{PL}(X)$ denotes a Sullivan model of X. Therefore we have proved that "the morphism modelling the diagonal map is the multiplication of the Sullivan model".

4.3 Sullivan model of a fibre product

Consider a pullback square in the category of topological spaces

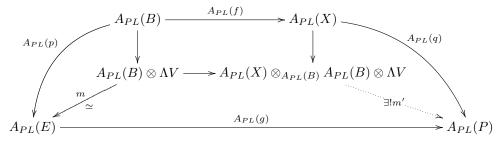
$$\begin{array}{c} P \xrightarrow{g} E \\ q \\ \downarrow \\ X \xrightarrow{f} B \end{array}$$

where

- $p: E \to B$ is a (Serre) fibration between two topological spaces,
- for every $i \in \mathbb{N}$, $H^i(X)$ and $H^i(B)$ are finite dimensional,

• the topological spaces X and E are path-connected and B is simply-connected. Since p is a (Serre) fibration, the pullback map q is also a (Serre) fibration. Let $A_{PL}(B) \otimes \Lambda V$ be a relative Sullivan model of A(p). Consider the corresponding commutative diagram of cdgas

.



where the rectangle is a pushout and m' is given by the universal property. Explicitly, for $x \in A_{PL}(X)$ and $e \in A_{PL}(B) \otimes \Lambda V$, $m'(x \otimes e)$ is the product of $A_{PL}(q)(x)$ and $A_{PL}(g) \circ m(e)$.

Since $A_{PL}(B) \hookrightarrow A_{PL}(B) \otimes \Lambda V$ is a relative Sullivan model, the inclusion obtained via pullback $A_{PL}(X) \hookrightarrow A_{PL}(X) \otimes_{A_{PL}(B)} (A_{PL}(B) \otimes \Lambda V, d) \cong (A_{PL}(X) \otimes$ $\Lambda V, d$) is also a relative Sullivan model (minimal if $A_{PL}(B) \hookrightarrow A_{PL}(B) \otimes \Lambda V$ is minimal).

By [5, Proposition 15.8] (or for weaker hypothesis [6, Theorem 2.70]),

Theorem 4.1. The morphism of cdqas m' is a quasi-isomorphism.

We can summarize this theorem by saying that: "The push-out of a (minimal) relative Sullivan model of a fibration is a (minimal) relative Sullivan model of the pullback of the fibration."

Idea of the proof. Since by [5, Lemma 14.1], $A_{PL}(B) \otimes \Lambda V$ is a "semi-free" resolution of $A_{PL}(E)$ as left $A_{PL}(B)$ -modules, by definition of the differential torsion product,

$$\operatorname{Tor}^{A_{PL}(B)}(A_{PL}(X), A_{PL}(E)) := H(A_{PL}(X) \otimes_{A_{PL}(B)} (A_{PL}(B) \otimes \Lambda V).$$

By Theorem 3.1 and naturality, we have an isomorphim of graded vector spaces

$$\operatorname{Tor}^{A_{PL}(B)}(A_{PL}(X), A_{PL}(E)) \cong \operatorname{Tor}^{S^{*}(B)}(S^{*}(X), S^{*}(E)).$$

The Eilenberg-Moore formula gives an isomorphism of graded vector spaces

$$\operatorname{Tor}^{S^*(B)}(S^*(X), S^*(E)) \cong H^*(P).$$

We claimed that the resulting isomorphism between the homology of $A_{PL}(X) \otimes_{A_{PL}(B)}$ $(A_{PL}(B) \otimes \Lambda V)$ and $H^*(P)$ can be identified with H(m). Therefore m is a quasiisomorphism.

Instead of working with A_{PL} , we prefer usually to work at the level of Sullivan models. Let $m_B : \Lambda B \xrightarrow{\simeq} A_{PL}(B)$ be a Sullivan model of B. Let $m_X : \Lambda X \xrightarrow{\simeq}$ $A_{PL}(X)$ be a Sullivan model of X. Let φ be a morphism of cdgas such the following diagram commutes exactly

$$\begin{array}{c|c} A_{PL}(B) \xrightarrow{A_{PL}(f)} A_{PL}(X) \\ & \xrightarrow{m_B} & \swarrow & m_X \\ & \xrightarrow{m_B} & \swarrow & m_X \\ & & & & & & & \\ & & & & & & & \\ & & & & & & & \\ & & & & & & & \\ & & & & & & & \\ & & & & & & & \\ & & & & & & & \\ & & & & & & & \\ & & & & & & & \\ & & & & & & & \\ & & & & & & & \\ & & & & & & & \\ & & & & & & & \\ & & & & & & & \\ & & & & & & \\ & & & & & & \\ & & & & & & \\ & & & & & & \\ & & & & & & \\ & & & & & & \\ & & & & & & \\ & & & & & & \\ & & & & \\ & & & & \\ & & & & \\ & & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & &$$

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Let $\Lambda B \hookrightarrow \Lambda B \otimes \Lambda V$ be a relative Sullivan model of $A_{PL}(p) \circ m_B$. Consider the corresponding commutative diagram of cdgas

where the rectangle is a pushout and m' is given by the universal property. Then again, $\Lambda X \hookrightarrow \Lambda X \otimes_{\Lambda B} (\Lambda B \otimes \Lambda V)$ is a relative Sullivan model and the morphism of cdgas m' is a quasi-isomorphism.

The reader should skip the following remark on his first reading.

Remark 4.2. 1) In the previous proof, if the composites $m_X \circ \varphi$ and $A_{PL}(f) \circ$ m_B are not strictly equal then the map m' is not well defined. In general, the composites $m_X \circ \varphi$ and $A_{PL}(f) \circ m_B$ are only homotopic and the situation is more complicated: see part 2) of this remark.

2) Let $m_B : \Lambda B \xrightarrow{\simeq} A_{PL}(B)$ be a Sullivan model of B. Let $m'_X : \Lambda X' \xrightarrow{\simeq} A_{PL}(X)$ be a Sullivan model of X. By the lifting Lemma of Sullivan models [5, Proposition 14.6], there exists a morphism of cdgas $\varphi' : \Lambda B \to \Lambda X'$ such that the following diagram commutes only up to homotopy (in the sense of [6, Section 2.2])

$$\begin{array}{c} A_{PL}(B) \xrightarrow{A_{PL}(f)} A_{PL}(X) \\ \xrightarrow{m_B} \simeq & m'_X \simeq \\ & \Lambda B \xrightarrow{\varphi'} & \Lambda X'. \end{array}$$

In general, this square is not strictly commutative. Let $\Lambda B \hookrightarrow \Lambda B \otimes \Lambda V$ be a relative Sullivan model of $A_{PL}(p) \circ m_B$. Then there exists a commutative diagram of cdgas

$$A_{PL}(X) \xrightarrow{A_{PL}(q)} A_{PL}(P)$$

$$\simeq \uparrow \qquad \simeq \uparrow$$

$$\Lambda X \longrightarrow \Lambda X \otimes_{\Lambda B} (\Lambda B \otimes \Lambda V)$$

$$\downarrow \simeq \qquad \qquad \downarrow \simeq$$

$$\Lambda X' \longrightarrow \Lambda X' \otimes_{\Lambda B} (\Lambda B \otimes \Lambda V)$$

Proof of part 2) of Remark 4.2. Let $\Lambda B \xrightarrow{\varphi} \Lambda X \xrightarrow{\theta} \Lambda X'$ be a relative Sullivan model of φ' . Since the composites $m'_X \circ \theta \circ \varphi$ and $A_{PL}(f) \circ m_B$ are homotopic, by the homotopy extension property [6, Proposition 2.22] of the relative Sullivan model $\varphi : \Lambda B \to \Lambda X$, there exists a morphism of cdgas $m_X : \Lambda X \to A_{PL}(X)$ homotopic to $m'_X \circ \theta$ such that $m_X \circ \varphi = A_{PL}(f) \circ m_B$. Therefore using diagram (1), we obtain the following commutative diagram of cdgas:

$$\begin{array}{c|c} A_{PL}(X) & \xrightarrow{A_{PL}(q)} & A_{PL}(P) \lessdot \xrightarrow{A_{PL}(g)} & A_{PL}(E) \\ \simeq & \uparrow^{m_X} & \simeq & \uparrow^{m'} & \simeq & \uparrow^{m} \\ & \Lambda X \longrightarrow & \Lambda X \otimes_{\Lambda B} (\Lambda B \otimes \Lambda V) \twoheadleftarrow & \Lambda B \otimes \Lambda V \\ \simeq & \downarrow^{\theta} & \simeq & \downarrow^{\theta \otimes_{\Lambda B} (\Lambda B \otimes \Lambda V)} \\ & \Lambda X' \longrightarrow & \Lambda X' \otimes_{\Lambda B} (\Lambda B \otimes \Lambda V). \end{array}$$

Here, since θ is a quasi-isomorphism, the pushout morphism $\theta \otimes_{\Lambda B} (\Lambda B \otimes \Lambda V)$ along the relative Sullivan model $\Lambda X \hookrightarrow \Lambda X \otimes_{\Lambda B} (\Lambda B \otimes \Lambda V)$ is also a quasi-isomorphism [5, Lemma 14.2].

4.4 Sullivan model of a fibration

Let $p: E \to B$ be a (Serre) fibration with fibre $F := p^{-1}(b_0)$.

$$F \xrightarrow{j} E \\ \downarrow \qquad \qquad \downarrow^{p} \\ b_{0} \longrightarrow B$$

Taking X to be the point b_0 , we can apply the results of the previous section. Let $m_B : (\Lambda V, d) \xrightarrow{\sim} A_{PL}(B)$ be a Sullivan model of B. Let $(\Lambda V, d) \hookrightarrow (\Lambda V \otimes \Lambda W, d)$ be a relative Sullivan model of $A_{PL}(p) \circ m_B$.

Since $A_{PL}(\{b_0\})$ is equal to $(\mathbf{k}, 0)$, there is a unique morphism of cdgas m' such that the following diagram commutes

$$\begin{array}{ccc} A_{PL}(B) & \xrightarrow{A_{PL}(p)} & A_{PL}(E) & \xrightarrow{A_{PL}(j)} & A_{PL}(F) \\ \simeq & & & \uparrow & & \uparrow & \\ \simeq & & & \uparrow & & \uparrow & \\ (\Lambda V, d) & \longrightarrow & (\Lambda V \otimes \Lambda W, d) & \longrightarrow & (k, 0) \otimes_{(\Lambda V, d)} (\Lambda V \otimes \Lambda W, d) \end{array}$$

Suppose that the base B is a simply connected space and that the total space E is path-connected. Then by the previous section, the morphism of cdga's

$$m': (k,0) \otimes_{(\Lambda V,d)} (\Lambda V \otimes \Lambda W, d) \cong (\Lambda W, \bar{d}) \xrightarrow{\simeq} A_{PL}(F)$$

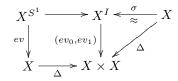
is a quasi-isomorphism:

" The cofiber of a relative Sullivan model of a fibration is a Sullivan model of the fiber of the fibration."

Note that the cofiber of a relative Sullivan model is minimal if and only if the relative Sullivan model is minimal.

4.5 Sullivan model of free loop spaces

Let X be a simply-connected space. Consider the commutative diagram of spaces



where the square is a pullback. Here I denotes the closed interval [0, 1], ev, ev_0 , ev_1 are the evaluation maps and the homotopy equivalence $\sigma : X \xrightarrow{\approx} X^I$ is the inclusion of constant paths. Let $m_X : \Lambda V \xrightarrow{\approx} A_{PL}(X)$ be a minimal Sullivan model of X. By Proposition 2.8, the multiplication $\mu : \Lambda V \otimes \Lambda V \to \Lambda V$ admits a minimal relative Sullivan model of the form

$$\Lambda V \otimes \Lambda V \hookrightarrow \Lambda V \otimes \Lambda V \otimes \Lambda s V.$$

Since μ is a model of the diagonal (Section 4.2) and since $\Delta = (ev_0, ev_1) \circ \sigma$, we have the commutative rectangle of cdgas

$$\begin{array}{c} A_{PL}(X \times X) \xrightarrow{A_{PL}((ev_0, ev_1))} & A_{PL}(X^I) \xrightarrow{A_{PL}(\sigma)} & A_{PL}(X) \\ \\ m_{X \times X} & \uparrow \simeq & \simeq \uparrow m_X \\ & \Lambda V \otimes \Lambda V \xrightarrow{} & \Lambda V \otimes \Lambda V \otimes \Lambda sV \xrightarrow{} \simeq & \Lambda V \end{array}$$

Since σ is a homotopy equivalence, $S^*(\sigma)$ is a homotopy equivalence of complexes and in particular a quasi-isomorphim. So by Theorem 3.1 and naturality, $A_{PL}(\sigma)$ is also a quasi-isomorphism. Therefore, by the lifting property of relative Sullivan models [5, Proposition 14.6], there exists a morphism of cdgas $\varphi : \Lambda V \otimes \Lambda V \otimes$ $\Lambda sV \to A_{PL}(X^I)$ such that, in the diagram of cdgas

$$\begin{array}{c} A_{PL}(X \times X) \xrightarrow{A_{PL}((ev_0, ev_1))} A_{PL}(X^I) \xrightarrow{A_{PL}(\sigma)} A_{PL}(X) \\ \xrightarrow{m_{X \times X}} & \stackrel{\wedge}{\simeq} & \stackrel{\wedge}{\sim} & \stackrel{\vee}{\sim} & \stackrel{\wedge}{\sim} & \stackrel{\vee}{\sim} &$$

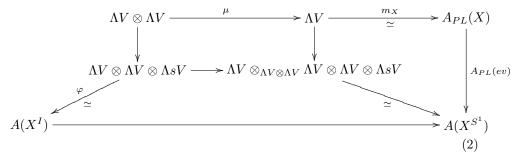
the left square commutes exactly and the right square commutes in homology. Therefore φ is also a quasi-isomorphism. This means that

$$\Lambda V \otimes \Lambda V \hookrightarrow \Lambda V \otimes \Lambda V \otimes \Lambda sV.$$

is a relative Sullivan model of the composite

$$\Lambda V \otimes \Lambda V \stackrel{m_{X \times X}}{\to} A_{PL}(X \times X) \stackrel{A_{PL}((ev_0, ev_1))}{\to} A_{PL}(X^I).$$

Here diagram (1) specializes to the following commutative diagram of cdgas



where the rectangle is a pushout. Therefore

$$\Lambda V \hookrightarrow \Lambda V \otimes_{\Lambda V \otimes \Lambda V} (\Lambda V \otimes \Lambda V \otimes \Lambda s V) \cong (\Lambda V \otimes \Lambda s V, \delta)$$

is a minimal relative Sullivan model of $A_{PL}(ev) \circ m_X$.

Corollary 4.3. Let X be a simply-connected space. Then the free loop space cohomology of $H^*(X^{S^1}; \mathbf{k})$ with coefficients in a field \mathbf{k} of characteristic 0 is isomorphic to the Hochschild homology of $A_{PL}(X)$, $HH_*(A_{PL}(X), A_{PL}(X))$.

Replacing $A_{PL}(X)$ by $A_{DR}(M)$ (Remark 3.2), this Corollary is a theorem of Chen [3, 3.2.3 Theorem] when X is a smooth manifold M.

Proof. The quasi-isomorphism of cdgas $m_X : \Lambda V \xrightarrow{\simeq} A_{PL}(X)$ induces an isomorphism between Hochschild homologies

$$HH_*(m_X, m_X): HH_*(\Lambda V, \Lambda V) \xrightarrow{\cong} HH_*(A_{PL}(X), A_{PL}(X))$$

By [5, Lemma 14.1], $\Lambda V \otimes \Lambda V \otimes \Lambda sV$ is a semi-free resolution of ΛV as a $\Lambda V \otimes \Lambda V^{op}$ module. Therefore the Hochschild homology $HH_*(\Lambda V, \Lambda V)$ can be defined as the homology of the cdga ($\Lambda V \otimes \Lambda sV, \delta$). We have just seen above that $H(\Lambda V \otimes \Lambda sV, \delta)$ is isomorphic to the free loop space cohomology $H^*(X^{S^1}; \mathbf{k})$.

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We have shown that a Sullivan model of X^{S^1} is of the form $(\Lambda V \otimes \Lambda sV, \delta)$. The following theorem of Vigué-Poirrier and Sullivan gives a precise description of the differential δ .

Theorem 4.4. ([17, Theorem p. 637] or [6, Theorem 5.11]) Let X be a simply connected topological space. Let $(\Lambda V, d)$ be a minimal Sullivan model of X. For all $v \in V$, denote by sv an element of degree |v| - 1. Let $s : \Lambda V \otimes \Lambda sV \to \Lambda V \otimes \Lambda sV$ be the unique derivation of (upper) degree -1 such that on the generators v, sv, $v \in V$, s(v) = sv and s(sv) = 0. We have $s \circ s = 0$. Then there exists a unique Sullivan model of X^{S^1} of the form $(\Lambda V \otimes \Lambda sV, \delta)$ such that $\delta \circ s + s \circ \delta = 0$ on $\Lambda V \otimes \Lambda sV$.

Remark 4.5. Consider the free loop fibration $\Omega X \hookrightarrow X^{S^1} \xrightarrow{ev} X$. Since $(\Lambda V, d) \hookrightarrow (\Lambda V \otimes \Lambda s V, \delta)$ is a minimal relative Sullivan model of $A_{PL}(ev) \circ m_X$, by Section 4.4,

$$\Bbbk \otimes_{(\Lambda V,d)} (\Lambda V \otimes \Lambda sV, \delta) \cong (\Lambda sV, \delta)$$

is a minimal Sullivan model of ΩX . Let $v \in V$. By Theorem 4.4, $\delta(sv) = -s\delta v = -sdv$. Since $dv \in \Lambda^{\geq 2}V$, $\delta(sv) \in \Lambda^{\geq 1}V \otimes \Lambda^{1}sV$. Therefore $\overline{\delta} = 0$. Since ΩX is a *H*-space, this follows also from Theorem 5.3 and from the unicity of minimal Sullivan models (part 1) of Theorem 3.4).

5 Examples of Sullivan models

5.1 Sullivan model of spaces with polynomial cohomology

The following proposition is a straightforward generalisation [5, p. 144] of the Sullivan model of odd-dimensional spheres (see section 1.2).

Proposition 5.1. Let X be a path connected topological space such that its cohomology $H^*(X; \mathbf{k})$ is a free graded commutative algebra ΛV (for example, polynomial). Then a Sullivan model of X is $(\Lambda V, 0)$.

Example 5.2. Odd-dimensional spheres S^{2n+1} , complex or quartenionic Stiefel manifolds [6, Example 2.40] $V_k(\mathbb{C}^n)$ or $V_k(\mathbb{H}^n)$, classifying spaces BG of simply connected Lie groups [6, Example 2.42], connected Lie groups G as we will see in the following section.

5.2 Sullivan model of an *H*-space

An *H*-space is a pointed topological space (G, e) equipped with a pointed continuous map $\mu : (G, e) \times (G, e) \to (G, e)$ such that the two pointed maps $g \mapsto \mu(e, g)$ and $g \mapsto \mu(g, e)$ are pointed homotopic to the identity map of (G, e).

Theorem 5.3. [5, Example 3 p. 143] Let G be a path connected H-space such that $\forall n \in \mathbb{N}, H_n(G; \mathbf{k})$ is finite dimensional. Then

- 1) its cohomology $H^*(G; \mathbf{k})$ is a free graded commutative algebra ΛV ,
- 2) G has a Sullivan model of the form $(\Lambda V, 0)$, that is with zero differential.

Proof. 1) Let A be $H^*(G; \mathbf{k})$ the cohomology of G. By hypothesis, A is a connected commutative graded Hopf algebra (not necessarily associative). Now the theorem of Hopf-Borel in caracteristic 0 [4, VII.10.16] says that A is a free graded commutative algebra.

2) By Proposition 5.1, 1) and 2) are equivalent.

Example 5.4. Let G be a path-connected Lie group (or more generally a H-space with finitely generated integral homology). Then G has a Sullivan model of the form $(\Lambda V, 0)$. By Theorem 3.4, V^n and $\pi_n(G) \otimes_{\mathbb{Z}} \mathbf{k}$ have the same dimension for any $n \in \mathbb{N}$. Since $H_*(G; \mathbf{k})$ is of finite (total) dimension, V and therefore $\pi_*(G) \otimes_{\mathbb{Z}} \mathbf{k}$ are concentrated in odd degrees. In fact, more generally [2, Theorem 6.11], $\pi_2(G) = \{0\}$. Note, however that $\pi_4(S^3) = \mathbb{Z}/2\mathbb{Z} \neq \{0\}$.

5.3 Sullivan model of projective spaces

Consider the complex projective space \mathbb{CP}^n , $n \geq 1$. The construction of the Sullivan model of \mathbb{CP}^n is similar to the construction of the Sullivan model of $S^2 = \mathbb{CP}^1$ done in section 1.2:

The cohomology algebra $H^*(A_{PL}(\mathbb{CP}^n)) \cong H^*(\mathbb{CP}^n)$ is the truncated polynomial algebra $\frac{\mathbf{k}[x]}{x^{n+1}=0}$ where x is an element of degree 2. Let v be a cycle of $A_{PL}(\mathbb{CP}^n)$ representing x := [v]. The inclusion of complexes $(\mathbf{k}v, 0) \hookrightarrow A_{PL}(\mathbb{CP}^n)$ extends to a unique morphism of cdgas $m : (\Lambda v, 0) \to A_{PL}(\mathbb{CP}^n)$ (Property 1.6). Since $[v^{n+1}] = x^{n+1} = 0$, there exists an element $\psi \in A_{PL}(\mathbb{CP}^n)$ of degree 2n + 1 such that $d\psi = v^{n+1}$. Let w denote another element of degree 2n + 1. Let d be the unique derivation of $\Lambda(v, w)$ such that d(v) = 0 and $d(w) = v^{n+1}$. The unique morphism of graded algebras $m : (\Lambda(v, w), d) \to A_{PL}(\mathbb{CP}^n)$ such that m(v) = v and $m(w) = \psi$, is a morphism of cdgas. In homology, H(m) sends 1, $[v], \ldots, [v^n]$ to 1, x, \ldots, x^n . Therefore m is a quasi-isomorphism.

More generally, let X be a simply connected space such that $H^*(X)$ is a truncated polynomial algebra $\frac{\mathbf{k}[x]}{x^{n+1}=0}$ where $n \ge 1$ and x is an element of even degree $d \ge 2$. Then the Sullivan model of X is $(\Lambda(v, w), d)$ where v is an element of degree d, w is an element of degree d(n+1) - 1, d(v) = 0 and $d(w) = v^{n+1}$.

5.4 Free loop space cohomology for even-dimensional spheres and projective spaces

In this section, we compute the free loop space cohomology of any simply connected space X whose cohomology is a truncated polynomial algebra $\frac{\mathbf{k}[x]}{x^{n+1}=0}$ where $n \ge 1$ and x is an element of even degree $d \ge 2$.

Mainly, this is the even-dimensional sphere S^d (n = 1), the complex projective space \mathbb{CP}^n (d = 2), the quaternionic projective space \mathbb{HP}^n (d = 4) and the Cayley plane \mathbb{OP}^2 (n = 2 and d = 8).

In the previous section, we have seen that the minimal Sullivan model of X is $(\Lambda(v, w), d(v) = 0, d(w) = v^{n+1})$ where v is an element of degree d and w is an element of degree d(n + 1) - 1. By the constructive proof of Proposition 2.8, the multiplication μ of this minimal Sullivan model $(\Lambda(v, w), d)$ admits the relative Sullivan model $(\Lambda(v, w) \otimes \Lambda(v, w) \otimes \Lambda(sv, sw), D)$ where

$$D(1 \otimes 1 \otimes sv) = v \otimes 1 \otimes 1 - 1 \otimes v \otimes 1 \text{ and}$$
$$D(1 \otimes 1 \otimes sw) = w \otimes 1 \otimes 1 - 1 \otimes w \otimes 1 - \sum_{i=0}^{n} v^{i} \otimes v^{n-i} \otimes sv$$

`

Therefore, by taking the pushout along μ of this relative Sullivan model (diagram (2)), or simply by applying Theorem 4.4, a relative Sullivan model of $A_{PL}(ev) \circ m_X$ is given by the inclusion of cdgas $(\Lambda(v, w), d) \hookrightarrow (\Lambda(v, w, sv, sw), \delta)$ where $\delta(sv) = -sd(v) = 0$ and $\delta(sw) = -s(v^{n+1}) = -(n+1)v^n sv$. Consider the pushout square of cdgas

Here, since θ is a quasi-isomorphism, the pushout morphism $\theta \otimes_{\Lambda(v,w)} \Lambda(sv, sw)$ along the relative Sullivan model $\Lambda(v, w) \hookrightarrow \Lambda(v, w, sv, sw)$ is also a quasi-isomorphism [5, Lemma 14.2]. Therefore, $H^*(X^{S^1}; \mathbf{k})$ is the graded vector space

$$\Bbbk \oplus \bigoplus_{1 \leq p \leq n, \ i \in \mathbb{N}} \Bbbk v^p (sw)^i \oplus \bigoplus_{0 \leq p \leq n-1, \ i \in \mathbb{N}} \Bbbk v^p sv (sw)^i$$

(In [11, Section 8], the author extends these rational computations over any commutative ring.) Since for all $i \in \mathbb{N}$, the degree of $v(sw)^{i+1}$ is strictly greater than the degree of $v^n(sw)^i$, the generators 1, $v^p(sw)^i$, $1 \le p \le n$, $i \in \mathbb{N}$, have all distinct (even) degrees. Since for all $i \in \mathbb{N}$, the degree of $sv(sw)^{i+1}$ is strictly greater than the degree of $v^{n-1}sv(sw)^i$, the generators $v^psv(sw)^i$, $0 \le p \le n-1$, $i \in \mathbb{N}$, have also distinct (odd) degrees. Therefore, for all $p \in \mathbb{N}$, Dim $H^p(X^{S^1}; \mathbf{k}) \le 1$.

At the end of section 2.2, we have shown the same inequalities when X is an odd-dimensional sphere, or more generally for a simply-connected space X whose cohomology $H^*(X; \mathbf{k})$ is an exterior algebra Λx on an odd degree generator x. Since every finite dimensional graded commutative algebra generated by a single element x is either Λx or $\frac{\mathbf{k}[x]}{x^{n+1}=0}$, we have shown the following proposition:

Proposition 5.5. Let X be a simply connected topological space such that its cohomology $H^*(X; \mathbf{k})$ is generated by a single element and is finite dimensional. Then the sequence of Betti numbers of the free loop space on X, $b_n := \dim H^n(X^{S^1}; \mathbf{k})$

 $is \ bounded.$

The goal of the following section will be to prove the converse of this proposition.

6 Vigué-Poirrier-Sullivan theorem on closed geodesics

The goal of this section is to prove (See section 6.4) the following theorem due to Vigué-Poirrier and Sullivan.

6.1 Statement of Vigué-Poirrier-Sullivan theorem and of its generalisations

Theorem 6.1. ([17, Theorem p. 637] or [6, Proposition 5.14]) Let M be a simply connected topological space such that the rational cohomology of M, $H^*(M; \mathbb{Q})$ is of finite (total) dimension (in particular, vanishes in higher degrees).

If the cohomology algebra $H^*(M; \mathbb{Q})$ requires at least two generators then the sequence of Betti numbers of the free loop space on M, $b_n := \dim H^n(M^{S^1}; \mathbb{Q})$ is unbounded.

Example 6.2. (Betti numbers of $(S^3 \times S^3)^{S^1}$ over \mathbb{Q})

Let V and W be two graded vector spaces such $\forall n \in \mathbb{N}, V^n$ and W^n are finite dimensional. We denote by

$$P_V(z) := \sum_{n=0}^{+\infty} (\text{Dim } V^n) z^n$$

the sum of the *Poincaré serie* of V. If V is the cohomology of a space X, we denote $P_{H^*(X)}(z)$ simply by $P_X(z)$. Note that $P_{V\otimes W}(z)$ is the product $P_V(z)P_W(z)$. We saw at the end of section 2.2 that $H^*((S^3)^{S^1}; \mathbb{Q}) \cong \Lambda v \otimes \Lambda sv$ where v is an element of degree 3. Therefore

$$P_{(S^3)^{S^1}}(z) = (1+z^3) \sum_{n=0}^{+\infty} z^{2n} = \frac{1+z^3}{1-z^2}.$$

Since the free loops on a product is the product of the free loops

$$H^*((S^3 \times S^3)^{S^1}) \cong H^*((S^3)^{S^1}) \otimes H^*((S^3)^{S^1}).$$

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Therefore, since $\frac{1}{1-z^2} = \sum_{n=0}^{+\infty} (n+1)z^{2n}$, $P_{(S^3 \times S^3)^{S^1}}(z) = \left(\frac{1+z^3}{1-z^2}\right)^2 = 1 + 2z^2 + \sum_{n=3}^{+\infty} (n-1)z^n$.

So the Betti numbers over \mathbb{Q} of the free loop space on $S^3 \times S^3$, $b_n := \text{Dim } H^n((S^3 \times S^3)^{S^1}; \mathbb{Q})$ are equal to n-1 if $n \geq 3$. In particular, they are unbounded.

Conjecture 6.3. The theorem of Vigué-Poirrier and Sullivan holds replacing \mathbb{Q} by any field \mathbb{F} .

Example 6.4. (Betti numbers of $(S^3 \times S^3)^{S^1}$ over \mathbb{F})

The calculation of Example 6.2 over \mathbb{Q} can be extended over any field \mathbb{F} as follows: Since S^3 is a topological group, the map $\Omega S^3 \times S^3 \to (S^3)^{S^1}$, sending (w,g) to the free loop $t \mapsto w(t)g$, is a homeomorphism. Using Serre spectral sequence ([13, Proposition 17] or [14, Chap 9. Sect 7. Lemma 3]) or Bott-Samelson theorem ([12, Corollary 7.3.3] or [9, Appendix 2 Theorem 1.4]), the cohomology of the pointed loops on S^3 , $H^*(\Omega S^3)$ is again isomorphic (as graded vector spaces only!) to the polynomial algebra Λsv where sv is of degree 2. Therefore exactly as over \mathbb{Q} , $H^*((S^3)^{S^1}; \mathbb{F}) \cong \Lambda v \otimes \Lambda sv$ where v is an element of degree 3. Now the same proof as in Example 6.2 shows that the Betti numbers over \mathbb{F} of the free loop space on $S^3 \times S^3$, $b_n := \text{Dim } H^n((S^3 \times S^3)^{S^1}; \mathbb{F})$ are again equal to n-1 if $n \geq 3$.

In fact, the theorem of Vigué-Poirrier and Sullivan is completely algebraic:

Theorem 6.5. ([17] when $\mathbb{F} = \mathbb{Q}$, [7, Theorem III p. 315] over any field \mathbb{F}) Let \mathbb{F} be a field. Let A be a cdga such that $H^{<0}(A) = 0$, $H^0(A) = \mathbb{F}$ and $H^*(A)$ is of finite (total) dimension. If the algebra $H^*(A)$ requires at least two generators then the sequence of dimensions of the Hochschild homology of A, $b_n := \dim HH_{-n}(A, A)$ is unbounded.

Generalising Chen's theorem (Corollary 4.3) over any field \mathbb{F} , Jones theorem [10] gives the isomorphisms of vector spaces

$$H^n(X^{S^1}; \mathbb{F}) \cong HH_{-n}(S^*(X; \mathbb{F}), S^*(X; \mathbb{F})), \quad n \in \mathbb{Z}$$

between the free loop space cohomology of X and the Hochschild homology of the algebra of singular cochains on X. But since the algebra of singular cochains $S^*(X; \mathbb{F})$ is not commutative, Conjecture 6.3 does not follow from Theorem 6.5.

6.2 A first result of Sullivan

In this section, we start by a first result of Sullivan whose simple proof illustrates the technics used in the proof of Vigué-Poirrier-Sullivan theorem.

Theorem 6.6. [15] Let X be a simply-connected space such that $H^*(X; \mathbb{Q})$ is not concentrated in degree 0 and $H^n(X; \mathbb{Q})$ is null for n large enough. Then on the contrary, $H^n(X^{S^1}; \mathbb{Q}) \neq 0$ for an infinite set of integers n.

Proof. Let $(\Lambda V, d)$ be a minimal Sullivan model of X. Suppose that V is concentrated in even degree. Then d = 0. Therefore $H^*(\Lambda V, d) = \Lambda V$ is either concentrated in degree 0 or is not null for an infinite sequence of degrees. By hypothesis, we have excluded theses two cases. Therefore dim $V^{odd} \geq 1$.

Let $x_1, x_2, \ldots, x_m, y, x_{m+1}, \ldots$ be a basis of V ordered by degree where y denotes the first generator of odd degree $(m \ge 0)$. For all $1 \le i \le m, dx_i \in \Lambda x_{\le i}$. But dx_i is of odd degree and $\Lambda x_{\le i}$ is concentrated in even degre. So $dx_i = 0$. Since $dy \in \Lambda x_{\le m}, dy$ is equal to a polynomial $P(x_1, \ldots, x_m)$ which belongs to $\Lambda^{\ge 2}(x_1, \ldots, x_m)$.

Consider $(\Lambda V \otimes \Lambda sV, \delta)$, the Sullivan model of X^{S^1} , given by Theorem 4.4. We have $\forall 1 \leq i \leq m, \ \delta(sx_i) = -sdx_i = 0$ and $\delta(sy) = -sdy \in \Lambda^{\geq 1}(x_1, \ldots, x_m) \otimes \Lambda^1(sx_1, \ldots, sx_m)$. Therefore, since sx_1, \ldots, sx_m are all of odd degree, $\forall p \geq 0$,

$$\delta(sx_1\dots sx_m(sy)^p) = \pm sx_1\dots sx_m p\delta(sy)(sy)^{p-1} = 0.$$

For all $p \ge 0$, the cocycle $sx_1 \dots sx_m (sy)^p$ gives a non trivial cohomology class in $H^*(X^{S^1}; \mathbb{Q})$, since by Remark 4.5, the image of this cohomology class in $H^*(\Omega X; \mathbb{Q}) \cong \Lambda V$ is different from 0.

6.3 Dimension of $V^{odd} \geq 2$

In this section, we show the following proposition:

Proposition 6.7. Let X be a simply connected space such that $H^*(X; \mathbb{Q})$ is of finite (total) dimension and requires at least two generators. Let $(\Lambda V, d)$ be the minimal Sullivan model of X. Then dim $V^{odd} \geq 2$.

Property 6.8. (Koszul complexes) Let A be a graded algebra. Let z be a central element of even degree of A which is not a divisor of zero. Then we have a quasi-isomorphism of dgas

$$(A \otimes \Lambda sz, d) \xrightarrow{=} A/z.A \quad a \otimes 1 \mapsto a, a \otimes sz \mapsto 0,$$

where $d(a \otimes 1) = 0$ and $d(a \otimes sz) = (-1)^{|a|}az$ for all $a \in A$.

Proof of Proposition 6.7 (following $(2) \Rightarrow (3)$ of p. 214 of [6]). As we saw in the proof of Theorem 6.6, there is at least one generator y of odd degree, that is dim $V^{odd} \ge 1$. Suppose that there is only one. Let $x_1, x_2, \ldots, x_m, y, x_{m+1}, \ldots$ be a basis of V ordered by degree $(m \ge 0)$.

First case: dy = 0. If $m \ge 1$, $dx_1 = 0$. If m = 0, $dx_1 \in \Lambda^{\ge 2}(y) = \{0\}$ and therefore again $dx_1 = 0$. Suppose that for $n \ge 1$, x_1^n is a coboundary. Then $x_1^n = d(yP(x_1,\ldots)) = yd(P(x_1,\ldots))$ where $P(x_1,\ldots)$ is a polynomial in the x_i 's. But this is impossible since x_1^n does not belong to the ideal generated by y. Therefore for all $n \ge 1$, x_1^n gives a non trivial cohomology class in $H^*(X)$. But $H^*(X)$ is finite dimensional.

Second case: $dy \neq 0$. In particular $m \geq 1$. Since dy is a non zero polynomial, dy is not a zero divisor, so by Property 6.8, we have a quasi-isomorphism of cdgas

$$\Lambda(x_1,\ldots,x_m,y) \xrightarrow{\simeq} \Lambda(x_1,\ldots,x_m)/(dy).$$

Consider the push out in the category of cdgas

$$\begin{array}{c|c} \Lambda(x_1,\ldots,x_m,y) & \longrightarrow \Lambda(x_1,\ldots,x_m,y,x_{m+1},\ldots), d \\ & \simeq & & \downarrow \\ \Lambda(x_1,\ldots,x_m)/(dy) & \longrightarrow \Lambda(x_1,\ldots,x_m)/(dy) \otimes \Lambda(x_{m+1},\ldots), \bar{d} \end{array}$$

Since $\Lambda(x_1,\ldots,x_m)/(dy) \otimes \Lambda(x_{m+1},\ldots)$ is concentrated in even degrees, $\bar{d} = 0$. Since the top arrow is a Sullivan relative model and the left arrow is a quasiisomorphism, the right arrow is also a quasi-isomorphism ([5, Lemma 14.2], or more generally the category of cdgas over \mathbb{Q} is a Quillen model category). Therefore the algebra $H^*(X)$ is isomorphic to $\Lambda(x_1,\ldots,x_m)/(dy) \otimes \Lambda(x_{m+1},\ldots)$. If $m \geq 2$, $\Lambda(x_1,\ldots,x_m)/(dy)$ and so $H^*(X)$ is infinite dimensional. If m = 1, since $\Lambda x_1/(dy)$ is generated by only one generator, we must have another generator x_2 . But $\Lambda(x_1)/(dy) \otimes \Lambda(x_2,\ldots)$ is also infinite dimensional. \Box

6.4 Proof of Vigué-Poirrier-Sullivan theorem

Lemma 6.9. [17, Proposition 4] Let A be a dga over any field such that the multiplication by a cocycle x of any degre $A \to A$, $a \mapsto xa$ is injective (Our example will be $A = (\Lambda V, d)$ and x a non-zero element of V of even degree such that dx = 0). If the Betti numbers $b_n = \dim H^n(A)$ of A are bounded then the Betti numbers $b_n = \dim H^n(A/xA)$ of A/xA are also bounded.

Proof. Since $H^n(xA) \cong H^{n-|x|}(A)$, the short exact sequence of complexes

$$0 \to xA \to A \to A/xA \to 0$$

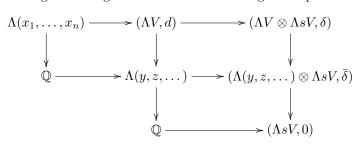
gives the long exact sequence in homology

$$\cdots \to H^n(A) \to H^n(A/xA) \to H^{n+1-|x|}(A) \to \dots$$

Therefore dim $H^n(A/xA) \leq \dim H^n(A) + \dim H^{n+1-|x|}(A)$

Proof of Vigué-Poirrier-Sullivan theorem (Theorem 6.1). Let $(\Lambda V, d)$ be the minimal Sullivan model of X. Let $(\Lambda V \otimes \Lambda sV, \delta)$ be the Sullivan model of X^{S^1} given by Theorem 4.4. From Proposition 6.7, we know that dim $V^{odd} \geq 2$. Let x_1 , $x_2, \ldots, x_m, y, x_{m+1}, \ldots, x_n, z = x_{n+1}, \ldots$ be a basis of V ordered by degrees where x_1, \ldots, x_n are of even degrees and y, z are of odd degrees. Consider the

commutative diagram of cdgas where the three rectangles are push outs



Note that by Remark 4.5, the differential on ΛsV is 0.

For all $1 \leq j \leq n+1$,

$$\delta x_j = dx_j \in \Lambda^{\geq 2}(x_{< j}, y) \subset \Lambda^{\geq 1}(x_{< j}) \otimes \Lambda y.$$

Therefore

$$\delta(sx_j) = -s\delta x_j \in \Lambda x_{< j} \otimes \Lambda^1 sx_{< j} \otimes \Lambda y + \Lambda^{\ge 1}(x_{< j}) \otimes \Lambda^1 sy.$$

Since $(sx_1)^2 = \cdots = (sx_{j-1})^2 = 0$, the product

$$sx_1 \dots sx_{j-1}\delta(sx_j) \in \Lambda^{\geq 1}(x_{< j}) \otimes \Lambda^1 sy$$

So $\forall 1 \leq j \leq n+1$, $sx_1 \dots sx_{j-1}\overline{\delta}(sx_j) = 0$. In particular $sx_1 \dots sx_n\overline{\delta}(sz) = 0$. Similarly, since $dy \in \Lambda^{\geq 2}x_{\leq m}$, $sx_1 \dots sx_m\delta(sy) = 0$ and so $sx_1 \dots sx_n\overline{\delta}(sy) = 0$. By induction, $\forall 1 \leq j \leq n$, $\overline{\delta}(sx_1 \dots sx_j) = 0$. In particular, $\overline{\delta}(sx_1 \dots sx_n) = 0$. So finally, for all $p \geq 0$ and all $q \geq 0$, $\overline{\delta}(sx_1 \dots sx_n(sy)^p(sz)^q) = 0$. The cocycles $sx_1 \dots sx_n(sy)^p(sz)^q$, $p \geq 0$, $q \geq 0$, give linearly independent cohomology classes in $H^*(\Lambda(y, z, \dots) \otimes \Lambda sV, \overline{\delta})$ since their images in $(\Lambda sV, 0)$ are linearly independent.

For all $k \ge 0$, there is at least k + 1 elements of the form $sx_1 \dots sx_n(sy)^p(sz)^q$ in degree $|sx_1| + \dots + |sx_n| + k \cdot \operatorname{lcm}(|sy|, |sz|)$ (just take $p = i \cdot \operatorname{lcm}(|sy|, |sz|)/|sy|$ and $q = (k - i)\operatorname{lcm}(|sy|, |sz|)/|sz|$ for i between 0 and k). Therefore the Betti numbers of $H^*(\Lambda(y, z, \dots) \otimes \Lambda sV, \overline{\delta})$ are unbounded.

Suppose that the Betti numbers of $(\Lambda V \otimes \Lambda sV, \delta)$ are bounded. Then by Lemma 6.9 applied to $A = (\Lambda V \otimes \Lambda sV, \delta)$ and $x = x_1$, the Betti numbers of the quotient cdga $(\Lambda(x_2, \ldots) \otimes \Lambda sV, \bar{\delta})$ are bounded. By continuing to apply Lemma 6.9 to x_2, x_3, \ldots, x_n , we obtain that the Betti numbers of the quotient cdga $(\Lambda(y, z, \ldots) \otimes \Lambda sV, \bar{\delta})$ are bounded. But we saw just above that they are unbounded.

7 Further readings

In this last section, we suggest some further readings that we find appropriate for the student.

In [1, Chapter 19], one can find a very short and gentle introduction to rational homotopy that the reader should compare to our introduction.

In this introduction, we have tried to explain that rational homotopy is a functor which transforms homotopy pullbacks of spaces into homotopy pushouts of cdgas. Therefore after our introduction, we advise the reader to look at [8], a more advanced introduction to rational homotopy, which explains the model category of cdgas.

The canonical reference for rational homotopy [5] is highly readable.

In the recent book [6], you will find many geometric applications of rational homotopy. The proof of Vigué-Poirrier-Sullivan theorem we give here, follows more or less the proof given in [6].

We also like [16] recently reprinted because it is the only book where you can find the Quillen model of a space: a differential graded Lie algebra representing its rational homotopy type (instead of a commutative differential graded algebra as the Sullivan model).

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