CONNES-MOSCOVICI CHARACTERISTIC MAP IS A LIE ALGEBRA MORPHISM

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Abstract. Let H be a Hopf algebra with a modular pair in involution $(\delta, 1)$. Let A be a (module) algebra over H equipped with a non-degenerated δ-invariant 1-trace τ. We show that Connes-Moscovici characteristic map $\varphi_\tau : HC^*_{(\delta,1)}(H) \to HC^*_\lambda(A)$ is a morphism of graded Lie algebras. We also have a morphism Φ of Batalin-Vilkovisky algebras from the cotorsion product of H, $\mathrm{Cotor}^*_H(\mathbb{k}, \mathbb{k})$, to the Hochschild cohomology of A, $HH^*(A, A)$. Let K be both a Hopf algebra and a symmetric Frobenius algebra. Suppose that the square of its antipode is an inner automorphism by a group-like element. Then this morphism of Batalin-Vilkovisky algebras $\Phi : \mathrm{Cotor}^*_{K^\vee}(\mathbb{F}, \mathbb{F}) \cong \mathrm{Ext}_K(\mathbb{F}, \mathbb{F}) \hookrightarrow HH^*(K, K)$ is injective.

1. Introduction

Let $\mathbb{k}$ be any commutative ring and $\mathbb{F}$ be any field. It is well known that the Hochschild cohomology of an algebra A, $HH^*(A, A)$, is a Gerstenhaber algebra. It is also well known that the homology of a double pointed loop space, $H_*(\Omega^2 X)$, is also a Gerstenhaber algebra [4]. Let H be a bialgebra. It is not well known (See [25] for a recent paper rediscovering it) that the Cotorsion product of H, $\mathrm{Cotor}^*_H(\mathbb{k}, \mathbb{k})$ has a Gerstenhaber algebra structure (this results from [16, p. 65]). But it should. Indeed, by Adams cobar equivalence, there is an isomorphism $\mathrm{Cotor}^*_{S_*(\Omega X)}(\mathbb{k}, \mathbb{k}) \cong H_*(\Omega^2 X)$ between the two Gerstenhaber algebras (See the proof of Corollary 26 for details).

The first goal of this paper is to study (Section 4) this Gerstenhaber algebra $\mathrm{Cotor}^*_H(\mathbb{k}, \mathbb{k})$. In particular, generalizing a result of Farinati and Solotar [9], we show (Theorem 16) that the exterior product $\mathrm{Ext}^*_H(\mathbb{k}, \mathbb{k})$ is a sub Gerstenhaber algebra of the Hochschild cohomology of H, $HH^*(H, H)$.

Key words and phrases. Batalin-Vilkovisky algebra, Hochschild cohomology, cyclic cohomology, Hopf algebra, Frobenius algebra.

In Section 5, we turn our attention to a particular case of Gerstenhaber algebras: the Batalin-Vilkovisky algebras. In [41], we introduced the notion of cyclic operad with multiplication (Definition 35) and we showed (Theorem 36) that every cyclic operad with multiplication $\mathcal{O}$ gives a cocyclic module such that

-the homology of the associated cochain complex $H(\mathcal{C}^*(\mathcal{O}))$ is a Batalin-Vilkovisky algebra and

-the negative cyclic cohomology of $\mathcal{C}^*(\mathcal{O})$, $HC^*_-(\mathcal{O})$, has a Lie bracket of degree -2.

Let M be a simply-connected closed manifold. In [2], Chas and Sullivan showed that $\mathbb{H}_*(LM)$, the free loop space homology of M, is a Batalin-Vilkovisky algebra and that the S^1-equivariant homology $H_*^{S^1}(LM)$ has a Lie bracket. The singular cochains of M, $S^*(M)$ is a (derived) symmetric Frobenius algebra. Motivated by Chas-Sullivan string topology, in [41], as first application of Theorem 36, we obtained that the Hochschild cohomology of a symmetric Frobenius algebra A, $HH^*(A,A)$, is a Batalin-Vilkovisky algebra and that the negative cyclic cohomology of A, $HC^*_-(A)$ has a Lie bracket of degree -2. It is expected that there is an isomorphism of Batalin-Vilkovisky algebras $HH^*(S^*(M),S^*(M))\cong\mathbb{H}_*(LM)$ and an isomorphism of Lie algebras $HC^*_-(S^*(M))\cong H_*^{S^1}(LM)$.

In [18], Getzler showed that the Gerstenhaber algebra $H_*(\Omega^2X)$ is in fact a Batalin-Vilkovisky algebra. Therefore as second application of Theorem 36, in [41], we showed that the Cotorsion product of a Hopf algebra H with an involutive antipode or more generally with a modular pair in involution $(\delta,1)$, $\mathrm{Cotor}^*_H(\mathbb{k},\mathbb{k})$, is a Batalin-Vilkovisky algebra. In this paper, we give the dual result (Theorem 50) which we believe is far more clear: Let K be a Hopf algebra such that the square of its antipode is an inner automorphism by a group-like element. Then $\mathrm{Ext}^*_K(\mathbb{k},\mathbb{k})$ is a Batalin-Vilkovisky algebra.

In [41], we also had that the negative cyclic cohomology of H, $HC^*_{-(\delta,1)}(H)$ has a Lie bracket of degree -2. But Connes and Moscovici never use negative cyclic cohomology: they use the (ordinary) cyclic cohomology. Therefore, in this paper, we show (Corollary 45) that Connes-Moscovici (ordinary) cyclic cohomology of H, $HC^*_{(\delta,1)}(H)$, has also a Lie bracket (of degree -1 this time) and we show (Theorem 54 and its variant Theorem 53) that Connes-Moscovici characteristic map $\chi_\tau : HC^*_{(\delta,1)}(H)\to HC^*_\lambda(A)$ is compatible with the Lie brackets of degree -1. Here A is a symmetric Frobenius algebra equipped an action of the Hopf algebra H compatible with the product and the trace.

In fact, we show that Connes-Moscovici characteristic map is induced by a morphism of cyclic operads with multiplication from the cobar construction of H, ΩH, to the Hochschild cochain complex of A, $\mathcal{C}^*(A,A)$. And we show that the (ordinary) cyclic cohomology of every cyclic operad with multiplication has naturally a Lie bracket of degree -1 (Theorem 37). As a consequence of Theorem 36, we also obtain a morphism of Batalin-Vilkovisky algebras $H^*(\Phi) : \mathrm{Cotor}^*_H(\Bbbk,\Bbbk) \to HH^*(A,A)$(Theorem 54).

Note that this morphism $H^*(\Phi)$ should be the algebraic counterpart of our very recent morphism of Batalin-Vilkovisky algebras [42, Theorem 24]

$$\mathrm{Cotor}_{S_*(G)}(\Bbbk,\Bbbk) \cong H_*(\Omega^2 BG) \to \mathbb{H}_*(LM) \cong HH^*(S^*(M), S^*(M))$$

between the Batalin-Vilkovisky algebra on the homology of double loop space given by by Getzler [18], and the Batalin-Vilkovisky algebra on the free loop space homology of a manifold given by Chas and Sullivan. Here G is a topological group acting on M.

In Section 8, we specialize to the case where the symmetric Frobenius algebra A is the Hopf algebra H itself. And we show that the inclusion of Gerstenhaber algebras $\mathrm{Ext}^*_H(\mathbb{F},\mathbb{F}) \hookrightarrow HH^*(H,H)$, given by Theorem 16, is often an inclusion of Batalin-Vilkovisky algebras (Theorem 63).

In this last section, we compute the Batalin-Vilkovisky algebra structure on $\mathrm{Cotor}_H(\Bbbk,\Bbbk)$ introduced in [41, Theorem 1.1] and recalled in Corollary 44 when H is the universal enveloping algebra of a Lie algebra over a field of characteristic 0.

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2. Hochschild complex and (Co)bar construction

We work over an arbitrary commutative ring $\Bbbk$, except for Conjectures 23 -25 in Section 4, for Proposition 46 to Corollary 49 (almost all Section 6) and for all Section 8, where we use an arbitrary field $\mathbb{F}$ as coefficient.

Let A be an algebra and M be an A-bimodule. The *Hochschild chain complex* $\mathcal{C}_*(A,M)$ is the chain complex $\mathcal{C}_n(A,M) = M \otimes A^n$ with differential $d : \mathcal{C}_n(A,M) \to \mathcal{C}_{n-1}(A,M)$ given by

$$d(m \otimes a_1 \otimes \cdots \otimes a_n) = ma_1 \otimes a_2 \otimes \cdots \otimes a_n$$
$$+\sum_{i=1}^{n-1}(-1)^i m\otimes a_1\otimes\cdots\otimes a_ia_{i+1}\otimes\cdots\otimes a_n + (-1)^n a_n m\otimes a_1\otimes\cdots\otimes a_{n-1}.$$

By definition, the *Hochschild homology of A with coefficients in M*, $HH_*(A,M)$ is the homology of $\mathcal{C}_*(A,M)$. The *Hochschild cochain complex* $\mathcal{C}^*(A,M)$ is the cochain complex $\mathcal{C}^n(A,M)=\operatorname{Hom}(A^n,M)$ with differential $d:\mathcal{C}^n(A,M)\to\mathcal{C}^{n+1}(A,M)$ given by

$$d(f)(a_0\otimes\cdots\otimes a_n)=a_0f(a_1\otimes\cdots\otimes a_n)$$
$$+\sum_{i=1}^n(-1)^if(a_0\otimes\cdots\otimes a_{i-1}a_i\otimes\cdots\otimes a_n)+(-1)^{n+1}f(a_0\otimes\cdots\otimes a_{n-1})a_n.$$

By definition, the *Hochschild cohomology of A with coefficients in M*, $HH^*(A,M)$ is the homology of $\mathcal{C}^*(A,M)$. Suppose that A has an augmentation $\varepsilon:A\to\Bbbk$. Then $\Bbbk$ is an A-bimodule. The *(reduced) Bar construction* $B(A)$ is just then the Hochschild chain complex $\mathcal{C}_*(A,\Bbbk)$ and $\operatorname{Ext}^*_A(\Bbbk,\Bbbk)=HH^*(A,\Bbbk)$.

Dually, let C be a coalgebra with diagonal $\Delta_C:C\to C\otimes C$. Let N be a C-bicomodule with left C-coaction $\Delta^l_N:N\to C\otimes N$. and right C-coaction $\Delta^r_N:N\to N\otimes C$. The Hochschild cochain complex $\mathcal{C}^*_{coalg}(C,N)$ ([16, p. 57] or [1, 30.3]) is the cochain complex $\mathcal{C}^n(C,N)=\operatorname{Hom}(N,C^n)$ with differential $d:\mathcal{C}^n(C,N)\to\mathcal{C}^{n+1}(C,N)$ given by

$$d(\varphi)=(C\otimes\varphi)\circ\Delta^l_N+\sum_{i=1}^n(-1)^i(C^{\otimes i-1}\otimes\Delta_C\otimes C^{\otimes n-i})\circ\varphi+(-1)^{n+1}(\varphi\otimes C)\circ\Delta^r_N.$$

The *Hochschild coalgebra cohomology* $HH^*_{coalg}(C,N)$ is the homology of $\mathcal{C}^*_{coalg}(C,N)$. Suppose that C has a coaugmentation $\eta:\Bbbk\to C$. Then $\Bbbk$ is a C-bicomodule. The *(reduced) cobar construction* $\Omega(C)$ [27, p. 432] is just $\mathcal{C}^*_{coalg}(C,\Bbbk)$ and $\operatorname{Cotor}^*_C(\Bbbk,\Bbbk)=HH^*_{coalg}(C,\Bbbk)$.

3. Operads with multiplication

A *Gerstenhaber algebra* is a commutative graded algebra $A=\{A^i\}_{i\in\mathbb{Z}}$ equipped with a bracket of degree -1

$$\{-,-\}:A^i\otimes A^j\to A^{i+j-1}\,,\quad x\otimes y\mapsto\{x,y\}$$

such that the product and the Lie bracket satisfy the Poisson rule: for any $c\in A^k$ the adjunction map $\{-,c\}:A^i\to A^{i+k-1}\,,\quad a\mapsto\{a,c\}$ is a $(k-1)$-derivation: ie. for a, b, $c\in A$, $\{ab,c\}=\{a,c\}b+(-1)^{|a|(|c|-1)}a\{b,c\}$.

In this paper, every Gerstenhaber algebra comes from a (linear) operad with multiplication using the following general theorem:

Theorem 1. [16, 17, 39] *a) Each operad with multiplication O is a cosimplicial module (See 5). Denote by $\mathcal{C}^*(O)$ the associated cochain complex.*

b) Its homology $H(\mathcal{C}^*(O))$ is a Gerstenhaber algebra.

Let us first recall what is a (linear) operad and a (linear) operad with multiplication.

In this paper, *operad* means non-Σ-operad in the category of $\Bbbk$-modules. That is: a sequence of modules $\{O(n)\}_{n\geq 0}$, an identity element $id \in O(1)$ and structure maps
$\gamma : O(n) \otimes O(i_1) \otimes \cdots \otimes O(i_n) \to O(i_1 + \cdots + i_n)$
$f \otimes g_1 \otimes \cdots \otimes g_n \mapsto \gamma(f; g_1, \ldots, g_n)$
satisfying associativity and unit [37].

Hereafter we use mainly the composition operations $\circ_i : O(m) \otimes O(n) \to O(m+n-1)$, $f \otimes g \mapsto f \circ_i g$ defined for $m \in \mathbb{N}^*$, $n \in \mathbb{N}$ and $1 \leq i \leq m$ by $f \circ_i g := \gamma(f; id, \ldots, g, id, \ldots, id)$ where g is the i-th element after the semicolon.

Example 2. Let $(\mathcal{C}, \otimes, \Bbbk)$ be a monoidal category. Suppose that $\mathcal{C}$ is enriched over the category of $\Bbbk$-modules [36, I.8] and that

$$\otimes : \mathrm{Hom}_{\mathcal{C}}(V_1, W_1) \times \mathrm{Hom}_{\mathcal{C}}(V_2, W_2) \to \mathrm{Hom}_{\mathcal{C}}(V_1 \otimes V_2, W_1 \otimes W_2),$$

mapping (g_1, g_2) to $g_1 \otimes g_2$, is $\Bbbk$-bilinear (we say that $\mathcal{C}$ is a $\Bbbk$-*linear monoidal category*). Let V be a object of $\mathcal{C}$. The *endomorphism operad* of V in $\mathcal{C}$ [37, p. 43] is the operad $\mathcal{E}nd_{\mathcal{C}}(V)$ defined by

$$\mathcal{E}nd_{\mathcal{C}}(V)(n) := \mathrm{Hom}_{\mathcal{C}}(V^{\otimes n}, V).$$

The structure maps γ

$$\mathrm{Hom}_{\mathcal{C}}(V^{\otimes n}, V) \otimes \mathrm{Hom}_{\mathcal{C}}(V^{\otimes i_1}, V) \otimes \cdots \otimes \mathrm{Hom}_{\mathcal{C}}(V^{\otimes i_n}, V) \to \mathrm{Hom}_{\mathcal{C}}(V^{\otimes i_1 + \cdots + i_n}, V)$$

are given by $\gamma(f; g_1, \ldots, g_n) = f \circ (g_1 \otimes \cdots \otimes g_n)$. The identity element of $\mathcal{E}nd_{\mathcal{C}}(V)$ is the identity map $id_V : V \to V$.

Example 3. The *coendomorphism operad* of V in $\mathcal{C}$, denoted $\mathcal{C}o\mathcal{E}nd_{\mathcal{C}}(V)$, is by definition the endomorphism operad of V in the opposite category $\mathcal{C}^{op}$, $\mathcal{E}nd_{\mathcal{C}^{op}}(V)$. Explicitly [37, p. 43-4] $\mathcal{C}o\mathcal{E}nd_{\mathcal{C}}(V)$ is the operad given by

$$\mathcal{C}o\mathcal{E}nd_{\mathcal{C}}(V)(n) := \mathrm{Hom}_{\mathcal{C}}(V, V^{\otimes n}).$$

The structure maps γ

$$\mathrm{Hom}_{\mathcal{C}}(V, V^{\otimes n}) \otimes \mathrm{Hom}_{\mathcal{C}}(V, V^{\otimes i_1}) \otimes \cdots \otimes \mathrm{Hom}_{\mathcal{C}}(V, V^{\otimes i_n}) \to \mathrm{Hom}_{\mathcal{C}}(V, V^{\otimes i_1 + \cdots + i_n})$$

are given by $\gamma(f; g_1, \ldots, g_n) = (g_1 \otimes \cdots \otimes g_n) \circ f$. The identity element of $\mathcal{E}nd_{\mathcal{C}}(V)$ is again id.

Definition 4. An *operad with multiplication* is an operad equipped with an element $\mu \in O(2)$ called the multiplication and an element $e \in O(0)$ such that $\mu \circ_1 \mu = \mu \circ_2 \mu$ and $\mu \circ_1 e = id = \mu \circ_2 e$.

Let $\underline{Ass}$ be the (non-Σ) associative operad [37]: $\underline{Ass}(n) := \Bbbk$. An operad O is an operad with multiplication if and only if O is equipped with a morphism of operads $\underline{Ass} \to O$.

Sketch of proof of 1. a) The coface maps $\delta_i : O(n) \to O(n+1)$ and codegeneracy maps $\sigma_i : O(n) \to O(n-1)$ are defined [39] by
(5)
$\delta_0 f = \mu \circ_2 f,\ \delta_i f = f \circ_i \mu,\ \delta_{n+1} f = \mu \circ_1 f,\ \sigma_{i-1} f = f \circ_i e$ for $1 \leq i \leq n$.

b) The associated cochain complex $\mathcal{C}^*(O)$ is the cochain complex whose differential d is given by

$$d := \sum_{i=0}^{n+1} (-1)^i \delta_i : O(n) \to O(n+1).$$

The linear maps $\cup : O(m) \otimes O(n) \to O(m+n)$ defined by

$$f \cup g := (\mu \circ_1 f) \circ_{m+1} g = (\mu \circ_2 g) \circ_1 f \tag{6}$$

gives $\mathcal{C}^*(O)$ a structure of differential graded algebra. The linear maps of degree -1

$$\overline{\circ}, \{-,-\} : O(m) \otimes O(n) \to O(m+n-1)$$

are defined by

$$f \overline{\circ} g := (-1)^{(m-1)(n-1)} \sum_{i=1}^{m} (-1)^{(n-1)(i-1)} f \circ_i g \tag{7}$$

and

$$\{f, g\} := f \overline{\circ} g - (-1)^{(m-1)(n-1)} g \overline{\circ} f.$$

The bracket $\{-,-\}$ defines a structure of differential graded Lie algebra of degree -1 on $\mathcal{C}^*(O)$. After passing to cohomology, the cup product $\cup$ and the bracket $\{-,-\}$ satisfy the Poisson rule. □

Remark 8. As pointed by Turchin in [55], the Gerstenhaber algebra $H(\mathcal{C}^*(\mathcal{O}))$ has Dyer-Lashof operations. In particular [16, p. 63], if n is even or if $2 = 0$ in $\Bbbk$, a Steenrod or Dyer-Lashof (non-additive) operation $Sq^{n-1} : H^n(\mathcal{C}^*(\mathcal{O})) \to H^{2n-1}(\mathcal{C}^*(\mathcal{O}))$ is defined by $Sq^{n-1}(f) = f \overline{\circ} f$ for $f \in \mathcal{O}(n)$.

Remark 9. Let $\mathcal{O}$ be an operad. Then $\mathcal{O}(1)$ equipped with $\circ_1 : \mathcal{O}(1) \otimes \mathcal{O}(1) \to \mathcal{O}(1)$ and $id : \Bbbk \to \mathcal{O}(1)$ is an algebra. By (7), the Lie algebra $\mathcal{C}^1(O)$ is just $\mathcal{O}(1)$ equipped with the Lie bracket given by $\{f, g\} := f \circ_1 g - g \circ_1 f$.

Example 10. Let A be a monoid in $\mathcal{C}$, i.e. an object of $\mathcal{C}$ equipped with an associative multiplication $\mu : A \otimes A \to A$ and a unit $e : \Bbbk \to A$. Then the endomorphism operad $\mathcal{E}nd_{\mathcal{C}}(A)$ of A equipped with $\mu \in \mathrm{Hom}_{\mathcal{C}}(A^{\otimes 2}, A) = \mathcal{E}nd_{\mathcal{C}}(A)(2)$ and $e \in \mathrm{Hom}_{\mathcal{C}}(A^{\otimes 0}, A) = \mathcal{E}nd_{\mathcal{C}}(A)(0)$ is an operad with multiplication. The associated cosimplicial module is the cosimplicial module $\{\mathrm{Hom}_{\mathcal{C}}(A^{\otimes n}, A)\}_{n\in\mathbb{N}}$. The coface maps $\delta_i : \mathrm{Hom}_{\mathcal{C}}(A^{\otimes n}, A) \to \mathrm{Hom}_{\mathcal{C}}(A^{\otimes n+1}, A)$ and the codegeneracy map $\sigma_i : \mathrm{Hom}_{\mathcal{C}}(A^{\otimes n}, A) \to \mathrm{Hom}_{\mathcal{C}}(A^{\otimes n-1}, A)$ are given by [41, (2.5)]

(11)
$$\delta_0 f = \mu\circ(id\otimes f),\ \delta_i f = f\circ(id^{\otimes i-1}\otimes\mu\otimes id^{\otimes n-i}),\ \delta_{n+1}f = \mu\circ(f\otimes id),$$

and $\sigma_{i-1}f = f \circ (id^{\otimes i-1} \otimes e \otimes id^{\otimes n-i})$ for $1 \le i \le n$.

If $\mathcal{C}$ is the category of $\Bbbk$-modules, A is an algebra and the cochain complex $\mathcal{C}^*(\mathcal{E}nd_{\mathcal{C}}(A))$ associated to this cosimplicial module is the Hochschild cochain complex of A, denoted $\mathcal{C}^*(A, A)$. This is why Turchin in his work on knots [53, 54] always call the cochain complex associated to a linear operad with multiplication, the Hochschild cochain complex of the operad with multiplication.

Property 12. Let $\mathcal{C}$ and $\mathcal{D}$ be two $\Bbbk$-linear monoidal categories. Let $F : \mathcal{C} \to \mathcal{D}$ be a monoidal functor (in the sense of [36, p. 255]). Let $\psi : F(V) \otimes F(W) \to F(V \otimes W)$ be the associated associative unital natural transformation. Suppose that $F : \mathrm{Hom}_{\mathcal{C}}(V, W) \to \mathrm{Hom}_{\mathcal{D}}(F(V), F(W))$ is $\Bbbk$-linear. Let A be a monoid in $\mathcal{C}$. Then $F(A)$ is a monoid in $\mathcal{D}$ and the map Γ from $\mathcal{E}nd_{\mathcal{C}}(A)$ to $\mathcal{E}nd_{\mathcal{D}}(F(A))$, mapping $f : A^{\otimes n} \to A$ to the composite $F(f) \circ \psi : F(A)^{\otimes n} \to F(A)$, is a morphism of operads with multiplication.

Example 13. Dually, let C be a comonoid in $\mathcal{C}$, i.e. an object of $\mathcal{C}$ equipped with a coassociative diagonal $\Delta : C \to C \otimes C$ and a counit $\varepsilon : C \twoheadrightarrow \Bbbk$. Since C is a monoid in $\mathcal{C}^{op}$, the coendomorphism operad of C, $\mathcal{C}o\mathcal{E}nd_{\mathcal{C}}(C)$ equipped with $\Delta \in \mathrm{Hom}_{\mathcal{C}}(C, C^{\otimes 2}) = \mathcal{C}o\mathcal{E}nd_{\mathcal{C}}(C)(2)$ and $\varepsilon \in \mathrm{Hom}_{\mathcal{C}}(C, C^{\otimes 0}) = \mathcal{C}o\mathcal{E}nd_{\mathcal{C}}(C)(0)$ is also an operad with multiplication. The associated cosimplicial module is the cosimplicial module $\{\mathrm{Hom}_{\mathcal{C}}(C, C^{\otimes n})\}_{n\in\mathbb{N}}$. The coface maps $\delta_i : \mathrm{Hom}_{\mathcal{C}}(C, C^{\otimes n}) \to \mathrm{Hom}_{\mathcal{C}}(C, C^{\otimes n+1})$ and the codegeneracy map $\sigma_i : \mathrm{Hom}_{\mathcal{C}}(C, C^{\otimes n}) \to \mathrm{Hom}_{\mathcal{C}}(C, C^{\otimes n-1})$ are given by

(14)
$$\delta_0 f = (id\otimes f)\circ\Delta,\ \delta_i f = (id^{\otimes i-1}\otimes\Delta\otimes id^{\otimes n-i})\circ f,\ \delta_{n+1}f = (f\otimes id)\circ\Delta,$$

and $\sigma_{i-1}f = (id^{\otimes i-1} \otimes \varepsilon \otimes id^{\otimes n-i}) \circ f$ for $1 \le i \le n$.

If $\mathcal{C}$ is the category of $\Bbbk$-modules, C is a coalgebra and the cochain complex $\mathcal{C}^*(\mathcal{C}o\mathcal{E}nd_{\mathcal{C}}(C))$ associated to this cosimplicial module is the Hochschild cochain complex of the coalgebra C, denoted $\mathcal{C}^*_{coalg}(C, C)$.

More generally, let A be $\mathbb{k}$-algebra. Let $\mathcal{C}$ be the category of A-bimodules. Let C be an A-coring, i.e. a comonoid in $\mathcal{C}$ ([26, 4.2] or [1, 17.1]). The cochain complex $\mathcal{C}^*(\mathcal{C}o\mathcal{E}nd_{\mathcal{C}}(C))$ associated to this cosimplicial module is the Cartier cochain complex of C with coefficients in C, denoted $C_{Ca}(C,C)$. Therefore, without any calculations, we have obtained that $C_{Ca}(C,C)$ is an operad with multiplication [1, 30.8]. This is again an example of our leitmotiv in this paper:

"Every operad with multiplication should be the endomorphism operad of a monoid in a appropriate monoidal category $\mathcal{C}$".

4. Gerstenhaber algebra structure on $\mathrm{Ext}^*_A(\mathbb{k},\mathbb{k})$

Let C be a bialgebra. The Cobar construction of C is the cosimplicial module associated to a specific linear operad with multiplication [16, p. 65]. Therefore its cohomology $\mathrm{Cotor}^*_C(\mathbb{k},\mathbb{k})$ has a Gerstenhaber algebra structure. In the following, we show that this operad with multiplication is just the endomorphism operad of a monoid in an appropriate monoidal category and we show:

Theorem 15. *Let C be a bialgebra. Then $Cotor^*_C(\mathbb{k},\mathbb{k})$ is a sub Gerstenhaber algebra of the Hochschild cohomology of the coalgebra C, $HH^*_{coalg}(C,C)$.*

By Property 17, this Lie bracket of degree -1 on the cotorsion product of a bialgebra is an extension of the well-known Lie bracket on the primitive elements of a bialgebra. Dually, we prove

Theorem 16. *Let A be a bialgebra. Then $Ext^*_A(\mathbb{k},\mathbb{k})$ is a sub Gerstenhaber algebra of the Hochschild cohomology of the algebra A, $HH^*(A,A)$.*

When A is a Hopf algebra, this theorem was proved by Farinati and Solotar [9]. But as we would like to emphasize, antipodes are not needed for the Gerstenhaber algebra structure. As we explain in Theorem 50, antipodes are needed only to have a Batalin-Vilkovisky algebra structure.

By Property 20, this inclusion of Gerstenhaber algebras is in degree 1 the inclusion of the Lie algebra of "differentiations" into the Lie algebra of derivations, well known in algebraic groups.

In Proposition 22, we prove that when the bialgebra C is $\mathbb{k}$-free of finite type, Theorem 16 is the dual of Theorem 15. This duality will be later extended in Corollary 49.

In Conjectures 23 and 25, we explain that if the bialgebra A is braided, the Lie bracket of degree -1 given by Theorem 16 on $\mathrm{Ext}^*_A(\mathbb{F},\mathbb{F})$ should vanish and be replaced by a Lie bracket of degree -2. This is related to a conjecture of Kontsevich.

In Corollary 26, we explain that the homology of a double loop space $H_*(\Omega^2 X)$ is always a sub Gerstenhaber algebra of Hochschild cohomology if X is 2-connected.

In Corollary 28, we show that the cohomology algebra of any path-connected topological space is also a sub Gerstenhaber algebra of Hochschild cohomology.

Proof of Theorem 15. The category of left C-modules, C-mod, is a monoidal category. Let M be a comonoid in this monoidal category, i.e. M is a C-module coalgebra [27, Definition IX.2.1]. The coendomorphism operad associated to M is the operad $\{\mathrm{Hom}_{C-mod}(M, M^{\otimes n})\}_{n\in\mathbb{N}}$ with multiplication $\Delta : M \to M \otimes M \in \mathrm{Hom}_{C-mod}(M, M^{\otimes 2})$ and $\varepsilon : M \to \Bbbk \in \mathrm{Hom}_{C-mod}(M, M^{\otimes 0})$. The inclusion maps $i_C : \mathrm{Hom}_{C-mod}(M, M^{\otimes n}) \hookrightarrow \mathrm{Hom}_{\Bbbk-mod}(M, M^{\otimes n})$ defines obviously a morphism of linear operads with multiplication.

The underlying coalgebra C is an example of C-module coalgebra. Therefore we can take in particular $M = C$. The linear morphism $ev : \mathrm{Hom}_{C-mod}(C, C^{\otimes n}) \overset{\cong}{\to} C^{\otimes n}$, mapping $f : C \to C^{\otimes n}$ to $f(1)$ is an isomorphism. The inverse is the linear map $ext : C^{\otimes n} \overset{\cong}{\to} \mathrm{Hom}_{C-mod}(C, C^{\otimes n})$, mapping $c_1 \otimes \cdots \otimes c_n$ to $f : C \to C^{\otimes n}$ defined by $f(c) = c^{(1)}c_1 \otimes \cdots \otimes c^{(n)}c_n$. Here we have denoted by $c^{(1)}\otimes\cdots\otimes c^{(n)}$ the iterated diagonal of c, $\Delta^{n-1}(c)$. Consider the associated cosimplicial set $\{\mathrm{Hom}_{C-mod}(C, C^{\otimes n})\}_{n\in\mathbb{N}}$. The coface maps δ_i and codegeneracy maps σ_i are given by equations (14). Therefore for $1 \leq i \leq n$,

$ev \circ \delta_0 \circ ext(c_1 \otimes \cdots \otimes c_n) = 1 \otimes c_1 \otimes \cdots \otimes c_n$,
$ev \circ \delta_i \circ ext(c_1 \otimes \cdots \otimes c_n) = c_1 \otimes \cdots \otimes \Delta(c_i) \otimes \cdots \otimes c_n$,
$ev \circ \delta_{n+1} \circ ext(c_1 \otimes \cdots \otimes c_n) = c_1 \otimes \cdots \otimes c_n \otimes 1$ and
$ev \circ \sigma_{i-1} \circ ext(c_1 \otimes \cdots \otimes c_n) = c_1 \otimes \cdots \otimes \varepsilon(c_i) \otimes \cdots \otimes c_n$.

So $ext : C^{\otimes n} \overset{\cong}{\to} \mathrm{Hom}_{C-mod}(C, C^{\otimes n})$ is an isomorphism of cosimplicial modules between the Cobar construction of C, ΩC, and the cosimplicial module associated to the operad with multiplication $\mathcal{C}o\mathcal{E}nd_{C-mod}(C)$. Therefore $\mathrm{Cotor}^*_C(\Bbbk, \Bbbk) := H^*(\Omega C)$ is a Gerstenhaber algebra. The composite

$$C^{\otimes n} \overset{ext}{\to} \mathrm{Hom}_{C-mod}(C, C^{\otimes n}) \subset \mathrm{Hom}_{\Bbbk-mod}(C, C^{\otimes n})$$

admits the morphism of differential graded algebras

$$\mathcal{C}^*(C, \eta) : \mathcal{C}^*_{coalg}(C, C) :\to \mathcal{C}^*_{coalg}(C, \Bbbk) = \Omega C$$

mapping $f : C \to C^{\otimes n}$ to $f(1)$ as retract. Passing to cohomology, we obtain an injective morphism of Gerstenhaber algebras

$$\mathrm{Cotor}^*_C(\Bbbk, \Bbbk) \hookrightarrow HH^*_{coalg}(C, C)$$

which admits the morphism of graded algebras

$$HH^*(C,\eta) : HH^*_{coalg}(C,C) \twoheadrightarrow \mathrm{Cotor}^*_C(\mathbb{k},\mathbb{k})$$

as retract. □

Property 17. The Lie algebra structure on $\mathrm{Cotor}^1_C(\mathbb{k},\mathbb{k})$ given by Theorem 15 coincides with the Lie algebra of primitive elements $P(C)$ of the bialgebra C.

Proof. Consider the isomorphisms ext and ev given in the proof of Theorem 15. We have:

$$(18)\quad ev\circ\circ_i\circ(ext\otimes ext)(a_1\otimes\cdots\otimes a_m\otimes b_1\otimes\cdots\otimes b_n) = \\ a_1\otimes\cdots\otimes a_{i-1}\otimes a_i^{(1)}b_1\otimes\cdots\otimes a_i^{(n)}b_n\otimes a_{i+1}\otimes\cdots\otimes a_m.$$

$ev(id_C)=1_C\in C$, $ev(\varepsilon)=1_{\mathbb{k}}\in\mathbb{k}$ and $ev(\Delta)=1_C\otimes 1_C\in C\otimes C$. Therefore $ev:\mathrm{Hom}_{C-mod}(C,C^{\otimes n})\overset{\cong}{\to} C^{\otimes n}$ is an isomorphism of linear operads with multiplication between $\mathcal{C}o\mathcal{E}nd_{C-mod}(C)$ and the operad with multiplication $\mathcal{O}$ of [41, Proof of Corollary 2.9], first considered by Gerstenhaber and Schack [16, p. 65] (See also [39, Example 3.5]). In particular $\circ_1:\mathcal{O}(1)\otimes\mathcal{O}(1)\to\mathcal{O}(1)$ is the multiplication of C, $\mu: C\otimes C\to C$. Therefore, by (9), the Lie algebra $\mathrm{Cotor}^1_C(\mathbb{k},\mathbb{k})$ coincides with the Lie algebra of primitive elements of C, denoted $P(C)$. □

In order to check that the Gerstenhaber algebra structure given by Theorem 16 coincides with the Gerstenhaber algebra structure on $\mathrm{Ext}^*_A(\mathbb{k},\mathbb{k})$ given by Farinati and Solotar [9], we give the proof of Theorem 16.

Property 19. Let C be a coalgebra. Let $\varepsilon: C\twoheadrightarrow\mathbb{k}$ be its counit. Let N be a left C-comodule. Then the linear morphism

$$proj:\mathrm{Hom}_{C-comod}(N,C)\overset{\cong}{\to} N^\vee,\quad F\mapsto\varepsilon\circ F,$$

is an isomorphism. Its inverse is the linear map $lift: N^\vee\overset{\cong}{\to}\mathrm{Hom}_{C-comod}(N,C)$ mapping $f:N\to\mathbb{k}$ to the composite $N\overset{\Delta_N}{\to} C\otimes N\overset{C\otimes f}{\to} C\otimes\mathbb{k}=C$.

Proof of Theorem 16. The category of left A-comodules, A-comod, is a monoidal category. Let M be a monoid in this monoidal category, i.e. M is an A-comodule algebra [27, Definition III.7.1]. The endomorphism operad associated to M is the operad $\{\mathrm{Hom}_{A-comod}(M^{\otimes n},M)\}_{n\in\mathbb{N}}$ with multiplication $\mu: M\otimes M\to M\in\mathrm{Hom}_{A-comod}(M^{\otimes 2},M)$ and $\eta:\mathbb{k}\to M\in\mathrm{Hom}_{A-comod}(M^{\otimes 0},M)$. The coaction of A on $M^{\otimes n}$, $\Delta_{M^{\otimes n}}$, is the composite $M^{\otimes n}\overset{\Delta_M^{\otimes n}}{\to}(A\otimes M)^{\otimes n}\overset{\tau}{\to}A^{\otimes n}\otimes M^{\otimes n}\overset{\mu_A\otimes M^{\otimes n}}{\to} A\otimes M^{\otimes n}$

where τ is the exchange isomorphism. Explicitly $\Delta_{M^{\otimes n}}(a_1\otimes\cdots\otimes a_n) = a_1^{(1)}\dots a_n^{(1)}\otimes(a_1^{(2)}\otimes\cdots\otimes a_n^{(2)})$ where $\Delta_M a_i = a_i^{(1)}\otimes a_i^{(2)}$.

We now take $M = A$. Using Property 19 with $C = A$ and $N = A^{\otimes n}$, we obtain that

$$proj : \operatorname{Hom}_{A-comod}(A^{\otimes n}, A) \overset{\cong}{\to} (A^{\otimes n})^\vee, \quad F \mapsto \varepsilon\circ F,$$

is an isomorphism. Its inverse is the linear map $lift : (A^{\otimes n})^\vee \overset{\cong}{\to} \operatorname{Hom}_{A-comod}(A^{\otimes n}, A)$ mapping $f : A^{\otimes n}\to \Bbbk$ to $F : A^{\otimes n}\to A$ defined by $F(a_1\otimes\cdots\otimes a_n) = a_1^{(1)}\dots a_n^{(1)} f(a_1^{(2)}\otimes\cdots\otimes a_n^{(2)})$.

Therefore the composite

$$(A^{\otimes n})^\vee \overset{lift}{\to} \operatorname{Hom}_{A-comod}(A^{\otimes n}, A) \subset \operatorname{Hom}_{\Bbbk-mod}(A^{\otimes n}, A)$$

coincides with the section of $\mathcal{C}^*(A,\varepsilon) : \mathcal{C}^*(A,A)\to BA^\vee$ defined by Farinati and Solotar [9, p. 2862].

Consider the associated cosimplicial set $\{\operatorname{Hom}_{A-comod}(A^{\otimes n}, A)\}_{n\in\mathbb{N}}$. The coface maps δ_i and codegeneracy maps σ_i are given by equations (11). Therefore for $1\leq i\leq n$,

$proj\circ\delta_0\circ lift(f)(a_1\otimes\cdots\otimes a_{n+1}) = \varepsilon(a_1)f(a_2\otimes\cdots\otimes a_{n+1})$,

$proj\circ\delta_i\circ lift(f)(a_1\otimes\cdots\otimes a_{n+1}) = f(a_1\otimes\cdots\otimes a_ia_{i+1}\otimes\cdots\otimes a_{n+1})$,

$proj\circ\delta_{n+1}\circ lift(f)(a_1\otimes\cdots\otimes a_{n+1}) = f(a_1\otimes\cdots\otimes a_n)\varepsilon(a_{n+1})$ and

$proj\circ\sigma_{i-1}\circ lift(f)(a_1\otimes\cdots\otimes a_{n-1}) = f(a_1\otimes\cdots\otimes a_{i-1}\otimes 1_A\otimes a_i\otimes\cdots\otimes a_n)$.

So $lift : (A^{\otimes n})^\vee \overset{\cong}{\to} \operatorname{Hom}_{A-comod}(A^{\otimes n}, A)$ is an isomorphism of cosimplicial modules between the dual of the bar construction of A, $BA^\vee$, and the cosimplicial module associated to the operad with multiplication $\mathcal{E}nd_{A-comod}(A)$. Therefore $\operatorname{Ext}^*_A(\Bbbk,\Bbbk) := H^*(BA^\vee)$ is a Gerstenhaber algebra. $\square$

Let A be an algebra and M be an A-bimodule. The cocycles of degree 1 of the Hochschild complex $\mathcal{C}^*(A,M)$ are exactly the module of derivations $\operatorname{Der}(A,M)$. A linear map $f : A\to M$ is a *derivation* if and only if $\forall a,\ b\in A,\ f(ab) = f(a)b + af(b)$. The boundaries of degree 1 of $\mathcal{C}^*(A,M)$ are the inner derivations, i.e. the linear maps $f : A\to M,\ a\mapsto am - ma$, where m is a given element of M. The degree 1 component of Hochschild cohomology, $HH^1(A,M)$, can be identified with the quotient $\operatorname{Der}(A,M)/\{\text{inner derivations}\}$ [34, 1.5.2]. In particular, suppose that A has an augmentation $\varepsilon : A\to\Bbbk$. Then $\operatorname{Ext}^1_A(\Bbbk,\Bbbk) = HH^1(A,\Bbbk) = \operatorname{Der}(A,\Bbbk)$.

Property 20. Let A be a bialgebra. The inclusion of Lie algebra $\operatorname{Ext}^1_A(\Bbbk,\Bbbk)\hookrightarrow HH^1(A,A)$ given by Theorem 16 can be identified with the following composite of Lie algebra morphisms

$$\mathrm{Der}(A,\Bbbk) \overset{i}{\hookrightarrow} \mathrm{Der}(A,A) \overset{q}{\twoheadrightarrow} \mathrm{Der}(A,A)/\{innerderivations\}.$$

Here q is the obvious quotient map and i is the inclusion of the Lie algebra of "differentiations" of A into the Lie algebra of derivations of A given by [23, p. 36].

Let G be an affine algebraic group. Then the *algebra of polynomial functions* on G, $\mathcal{P}(G)$, is a commutative Hopf algebra. By definition [23, p. 36], the Lie algebra of G is $\mathrm{Ext}^1_{\mathcal{P}(G)}(\Bbbk,\Bbbk) = \mathrm{Der}(\mathcal{P}(G),\Bbbk)$.

Let G be a Lie group. The algebra of smooth maps on G, $C^\infty(G)$, is a module algebra over the group ring $\mathbb{R}[G]$, but is not a bialgebra (except when G is finite and discrete). However there is still an analogue of the inclusion i: the composite

$$T_e(G) \underset{lift}{\overset{\cong}{\to}} \mathrm{Hom}_{mod-\mathbb{R}[G]}(C^\infty(G), C^\infty(G)) \cap \mathrm{Der}(C^\infty(G)) \subset \mathrm{Der}(C^\infty(G)).$$

Here $lift$ is the isomorphism between the tangent space and the right invariant vector fields on G.

Proof. Consider the inverse isomorphisms $proj : \mathrm{Hom}_{A-comod}(A^n, A) \overset{\cong}{\to} (A^{\otimes n})^\vee$ and $lift : (A^{\otimes n})^\vee \overset{\cong}{\to} \mathrm{Hom}_{A-comod}(A^n, A)$ given in the proof of Theorem 16. Let $\mathcal{O}$ denote the linear operad with multiplication such that $proj : \mathcal{E}nd_{A-comod}(A) \overset{\cong}{\to} \mathcal{O}$ is an isomorphism of linear operads with multiplication. Explicitly, for $f \in \mathcal{O}(m) = (A^{\otimes m})^\vee$ and $g \in \mathcal{O}(n) = (A^{\otimes n})^\vee$, $f \circ_i g$ is given by

$$f \circ_i g(a_1 \otimes \cdots \otimes a_{m+n-1}) =$$
$$f(a_1 \otimes \cdots \otimes a_{i-1} \otimes a_i^{(1)} \dots a_{i+n-1}^{(1)} g(a_i^{(2)} \otimes \dots a_{i+n-1}^{(2)}) \otimes a_{i+n} \otimes \cdots \otimes a_{m+n-1})$$

where $\Delta a_j = a_j^{(1)} \otimes a_j^{(2)}$. The identity element of $\mathcal{O}$ is the counit of A, $\varepsilon \in A^\vee = \mathcal{O}(1)$. The multiplication of $\mathcal{O}$ is the composite $\varepsilon \circ \mu \in (A \otimes A)^\vee = \mathcal{O}(2)$ and the unit is $id_\Bbbk \in (A^{\otimes 0})^\vee = \mathcal{O}(0)$. In particular, $\circ_1 : \mathcal{O}(1) \otimes \mathcal{O}(1) \to \mathcal{O}(1)$ is the multiplication of $A^\vee$, $\mu_{A^\vee} : A^\vee \otimes A^\vee \to A^\vee$ obtained by dualizing the diagonal of A. Therefore, by (9), the Lie algebra $\mathcal{C}^1(\mathcal{O})$ is just the Lie algebra associated to the associative algebra $A^\vee$. The composite $\mathcal{O} \underset{\cong}{\overset{lift}{\to}} \mathcal{E}nd_{A-comod}(A) \subset \mathcal{E}nd_{\Bbbk-mod}(A)$ is an injective morphism of linear operads with multiplication. Therefore this composite $\mathcal{C}^*(\mathcal{O}) \underset{\cong}{\overset{lift}{\to}} \mathcal{C}^*(\mathcal{E}nd_{A-comod}(A)) \subset \mathcal{C}^*(\mathcal{E}nd_{\Bbbk-mod}(A))$ is an injective morphism of differential graded Lie algebras. In degree 1, this composite $\mathcal{O}(1) \hookrightarrow \mathcal{E}nd_{\Bbbk-mod}(A)(1) = \mathrm{Hom}_{\Bbbk-mod}(A,A)$ is the injective morphism of (associative) algebras, mapping $f : A \to \Bbbk$ to $(A \otimes f) \circ \Delta_A$, given by [23, I.Proposition 2.1]. Restricted at the cycles in degree 1, this composite gives the injective morphism of Lie algebras $\mathrm{Der}(A,\mathbb{k}) \overset{i}{\hookrightarrow} \mathrm{Der}(A,A)$ considered in [23, p. 36]. □

Let us prove that Theorem 16 is the dual of Theorem 15.

Lemma 21. *Let C be a coalgebra with coaugmentation $\eta : \mathbb{k} \to C$. Let $A = C^\vee$ be the dual algebra with augmentation $\varepsilon : A \to \mathbb{k}$. Then*

i) the linear map $\Gamma : \mathcal{C}o\mathcal{E}nd_{\mathbb{k}-mod}(C) \to \mathcal{E}nd_{\mathbb{k}-mod}(A)$, mapping $f : C \to C^{\otimes n}$ to the composite $A^{\otimes n} \to (C^{\otimes n})^\vee \overset{f^\vee}{\to} A$, is a morphism of linear operads with multiplication,

ii) the linear map $\phi : \Omega C \to (BA)^\vee$, such that $\phi(c_1 \otimes \cdots \otimes c_n)$ is the form on $A^{\otimes n}$, mapping $\varphi_1 \otimes \cdots \otimes \varphi_n$ to the product $\varphi_1(c_1)\ldots\varphi_n(c_n)$, is a morphism of differential graded algebras.

iii) We have the commutative diagram of differential graded algebras

$$\begin{tikzcd}
\mathcal{C}^*_{coalg}(C,C) \arrow[r, "\mathcal{C}^*_{coalg}(C,\eta)"] \arrow[d, "\Gamma"'] & \Omega C \arrow[d, "\phi"] \\
\mathcal{C}^*(A,A) \arrow[r, "\mathcal{C}^*(A,\varepsilon)"'] & (BA)^\vee
\end{tikzcd}$$

If C is $\mathbb{k}$-free of finite type then both Γ and ϕ are isomorphisms.

Proof. Let $\psi : V^\vee \otimes W^\vee \to (V \otimes W)^\vee$ be the linear map, mapping the tensor product $\varphi_1 \otimes \varphi_2$ of a form on V and of a form on W, to the form on $V \otimes W$, also denoted $\varphi_1 \otimes \varphi_2$, mapping $v \otimes w$ to the product $\varphi_1(v)\varphi_2(w)$. The functor $^\vee$ from the opposite category of $\mathbb{k}$-modules to the category of $\mathbb{k}$-modules, mapping a $\mathbb{k}$-module V, to its dual $V^\vee := \mathrm{Hom}(V,\mathbb{k})$ is a monoidal functor. Therefore by applying Property 12, we obtain i). ii) is well known and iii) is easy to check. □

Note that in [14], together with Felix and Thomas, we gave a different proof that $H^*(\Gamma) : HH^*_{coalg}(C,C) \to HH^*(A,A)$ is a morphism of Gerstenhaber algebras.

Proposition 22. *Let C be a bialgebra $\mathbb{k}$-free of finite type. Let A be the dual bialgebra. Then the inclusions of Gerstenhaber algebras given by Theorems 15 and 16 fit into the commutative diagram of Gerstenhaber algebras.*

$$\begin{tikzcd}
Cotor^*_C(\mathbb{k},\mathbb{k}) \arrow[r] \arrow[d, "H^*(\phi)"', "\cong"] & HH^*_{coalg}(C,C) \arrow[d, "\cong"', "H^*(\Gamma)"] \\
Ext^*_A(\mathbb{k},\mathbb{k}) \arrow[r] & HH^*(A,A)
\end{tikzcd}$$

Proof. Since C is an algebra $\Bbbk$-free of finite type, the dualizing functor $^\vee$, defined in the proof of Lemma 21, restrict to a functor F from the opposite category of left C-modules to the category of left A-comodules. If M and N are left C-modules, $\psi : M^\vee \otimes N^\vee \to (M \otimes N)^\vee$ is a morphism of left A-comodules. Therefore by Property 12, we obtain the morphism of linear operads with multiplication $\Gamma_F : \mathcal{C}o\mathcal{E}nd_{C-mod}(C) \to \mathcal{E}nd_{A-comod}(A)$. Consider the two commutatives squares

$$\begin{tikzcd}
\mathcal{C}o\mathcal{E}nd_{C-mod}(C) \arrow[r, "i_C"] \arrow[d, "\Gamma_F"', "\cong"] & \mathcal{C}o\mathcal{E}nd_{\Bbbk-mod}(C) \arrow[r, "\mathcal{C}^*_{coalg}(C,\eta)"] \arrow[d, "\Gamma"', "\cong"] & \Omega C \arrow[d, "\cong"', "\phi"] \\
\mathcal{E}nd_{A-comod}(A) \arrow[r, "i_A"] & \mathcal{E}nd_{\Bbbk-mod}(A) \arrow[r, "\mathcal{C}^*(A,\varepsilon)"'] & (BA)^\vee
\end{tikzcd}$$

The left square commutes by definition of Γ_F since the two horizontal maps i_C and i_A are just the inclusions. Part iii) of Lemma 21 says that the right square commutes. The composite $\mathcal{C}o\mathcal{E}nd_{C-mod}(C) \overset{i_C}{\hookrightarrow} \mathcal{C}o\mathcal{E}nd_{\Bbbk-mod}(C) \overset{\mathcal{C}^*_{coalg}(C,\eta)}{\to} \Omega C$ is the isomorphism ev considered in the proof of Theorem 15. The composite $\mathcal{E}nd_{A-comod}(A) \overset{i_A}{\hookrightarrow} \mathcal{E}nd_{\Bbbk-mod}(A) \overset{\mathcal{C}^*(A,\varepsilon)}{\to} (BA)^\vee$ is the isomorphism $proj$ considered in the proof of Theorem 16. Therefore,we have the commutative square of linear operads with multiplication

$$\begin{tikzcd}
\Omega C \arrow[r, "i_c \circ ev^{-1}"] \arrow[d, "\phi"', "\cong"] & \mathcal{C}^*_{coalg}(C,C) \arrow[d, "\cong"', "\Gamma"] \\
(BA)^\vee \arrow[r, "i_A \circ proj^{-1}"] & \mathcal{C}^*(A,A)
\end{tikzcd}$$

Applying homology, we obtain the Proposition. □

Let H be a finite dimensional Hopf algebra. Let $D(H)$ be the Drinfeld double of H. Then Taillefer [52] proved that the Gerstenhaber-Schack cohomology of H, $H_{GS}(H,H)$ is isomorphic as graded algebras to $\mathrm{Ext}_{D(H)^{op}}(\mathbb{F},\mathbb{F})$. Since $D(H)$ is a Hopf algebra, by Theorem 16, Farinati and Solotar [9] have obtained a Gerstenhaber algebra structure on $\mathrm{Ext}_{D(H)^{op}}(\mathbb{F},\mathbb{F}) = H_{GS}(H,H)$. But Taillefer using a braiding [52, Beginning of Section 5] shows that the Lie bracket in this Gerstenhaber algebra structure is trivial. The Drinfeld double $D(H)$ is a braided Hopf algebra. Therefore, following the proof of Taillefer, it should be easy to prove

Conjecture 23. *Let A be braided bialgebra. Then the Lie algebra of the Gerstenhaber algebra $Ext^*_A(\mathbb{F},\mathbb{F})$ given by Theorem 16 is trivial.*

Proof when A is a cocommutative Hopf algebra. Let A be a cocommutative Hopf algebra. Since A is cocommutative, the antipode S is involutive. Therefore by Theorem 50, $\mathrm{Ext}_A^*(\mathbb{F},\mathbb{F})$ is a Batalin-Vilkovisky algebra. By [29, Theorem 4.1], the operator B of this Batalin-Vilkovisky algebra is trivial. Therefore by (30), the Lie bracket is null. $\square$

In [48], Shoikhet mentions the following conjecture of Kontsevich.

Conjecture 24. *(Kontsevich) Let H be a bialgebra. Then $H_{GS}(H,H)$ is a 3-algebra, i.e.* [37, Theorem p. 26-7] *an algebra over the homology of the little 3-cubes operad, $\mathcal{C}_3$.*

Shoikhet [48, Corollary 0.3] has announced that the proof of this conjecture when H is a Hopf algebra. We formulate the following related conjecture:

Conjecture 25. *Let A be braided bialgebra. Then $Ext_A^*(\mathbb{F},\mathbb{F})$ is a 3-algebra.*

If A is cocommutative, again this Lie algebra bracket (of degree -2 this time) should vanish: modulo p, only the Steenrod or Dyer-Lashof operations on $\mathrm{Ext}_A^*(\mathbb{F},\mathbb{F})$ (Remark 8 and [38, Theorem 11.8]) should be non-trivial.

As an algebraic topologist, we find the following Corollaries of Theorem 15 and Theorem 16, highly interesting.

Corollary 26. *Let X be a 2-connected pointed topological space. Denote by $\Omega_M X$ the pointed Moore loops on X. Then the homology of the double loop spaces on X, $H_*(\Omega^2 X)$, equipped with the Pontryagin product, is a sub Gerstenhaber algebra of $HH^*_{coalg}(S_*(\Omega_M X), S_*(\Omega_M X))$, the Hochschild cohomology of the coalgebra $S_*(\Omega_M X)$.*

Proof. The bialgebra C in Theorem 15 can be differential graded. Since $\Omega_M X$ is a topological monoid, the (reduced normalized) singular chains on $\Omega_M X$ form a differential graded bialgebra $C = S_*(\Omega_M X)$. Therefore, by Theorem 15, $\mathrm{Cotor}_{S_*(\Omega_M X)}(\mathbb{k},\mathbb{k})$ is a sub Gerstenhaber algebra of $HH^*_{coalg}(S_*(\Omega_M X), S_*(\Omega_M X))$. By Adams cobar equivalence, there is an isomorphism of graded algebras $\mathrm{Cotor}_{S_*(\Omega_M X)}(\mathbb{k},\mathbb{k}) \cong H_*(\Omega_M\Omega_M X)$. The inclusion of the (ordinary) pointed loops into the Moore loops $\Omega X \overset{\approx}{\hookrightarrow} \Omega_M X$ is a both a homotopy equivalence [59, p. 112-3] and a morphism of H-spaces. So as graded algebras, $H_*(\Omega_M\Omega_M X)$ is isomorphic to $H_*(\Omega^2 X)$. $\square$

Corollary 26 gives in particular a Gerstenhaber algebra structure on $H_*(\Omega^2 X)$ extending the Pontryagin product. Of course, we believe that this Gerstenhaber algebra structure coincides with the usual one given by Cohen in [4]:

Conjecture 27. *Let X be a 2-connected pointed topological space. There is an isomorphism of Gerstenhaber algebras between the Gerstenhaber algebra* $\mathrm{Cotor}_{S_*(\Omega_M X)}(\Bbbk,\Bbbk)$ *given by Theorem 15 and the Gerstenhaber algebra* $H_*(\Omega^2 X)$ *given by Cohen in* [4].

Recall that the Gerstenhaber algebra on $H_*(\Omega^2 X)$ is usually defined as follows: the little 2-cube operad $\mathcal{C}_2$ acts on the double loop space, $\Omega^2 X$. So its homology $H_*(\Omega^2 X)$ is an algebra over the homology of $\mathcal{C}_2$, i.e. is a Gerstenhaber algebra by Cohen [4].

Corollary 28. *Let X be a path-connected topological space. Denote by $\Omega_M X$ the pointed Moore loops on X. Then the cohomology of X, $H^*(X)$, equipped with the cup product, is a sub Gerstenhaber algebra of $HH^*(S_*(\Omega_M X), S_*(\Omega_M X))$, the Hochschild cohomology of the algebra $S_*(\Omega_M X)$.*

Proof. By Theorem 16 applied to $A = S_*(\Omega_M X)$, $\mathrm{Ext}^*_{S_*(\Omega_M X)}(\Bbbk,\Bbbk)$ is a sub Gerstenhaber algebra of $HH^*(S_*(\Omega_M X), S_*(\Omega_M X))$. Applying homology to [12, Theorem 7.2 ii)], gives the natural isomorphism of graded algebras

$$H^*(X) \cong \mathrm{Ext}^*_{S_*(\Omega_M X)}(\Bbbk,\Bbbk).$$

□

We believe that the Lie bracket on $H^*(X) \cong \mathrm{Ext}^*_{S_*(\Omega_M X)}(\Bbbk,\Bbbk)$ given by Corollary 28 is trivial since $S_*(\Omega_M X)$ is cocommutative up to homotopy in some E_∞-sense.

5. Batalin-Vilkovisky algebras

29. A *Batalin-Vilkovisky algebra* is a Gerstenhaber algebra A equipped with a degree -1 linear map $B : A^i \to A^{i-1}$ such that $B \circ B = 0$ and

$$\{a,b\} = (-1)^{|a|}\left(B(a \cup b) - (Ba) \cup b - (-1)^{|a|} a \cup (Bb)\right) \tag{30}$$

for a and $b \in A$.

Definition 31. A *cyclic operad* is a non-Σ-operad $\mathcal{O}$ equipped with linear maps $\tau_n : O(n) \to O(n)$ for $n \in \mathbb{N}$ such that

$$\forall n \in \mathbb{N}, \quad \tau_n^{n+1} = id_{O(n)}, \tag{32}$$

$$\forall m \geq 1, n \geq 1, \quad \tau_{m+n-1}(f \circ_1 g) = \tau_n g \circ_n \tau_m f, \tag{33}$$

$$\forall m \geq 2, n \geq 0, 2 \leq i \leq m, \quad \tau_{m+n-1}(f \circ_i g) = \tau_m f \circ_{i-1} g, \tag{34}$$

for each $f \in O(m)$ and $g \in O(n)$. In particular, we have $\tau_1 id = id$.

Definition 35. [41] A *cyclic operad with multiplication* is an operad which is both an operad with multiplication and a cyclic operad such that

$$\tau_2\mu = \mu.$$

Theorem 36. [41] *If $\mathcal{O}$ is a cyclic operad with a multiplication then*

a) the structure of cosimplicial module on $\mathcal{O}$ extends to a structure of cocyclic module and

b) the Connes coboundary map B on $\mathcal{C}^*(\mathcal{O})$ induces a natural structure of Batalin-Vilkovisky algebra on the Gerstenhaber algebra $H^*(\mathcal{C}^*(\mathcal{O}))$.

Theorem 37. *Let $\mathcal{O}$ be a linear cyclic operad with multiplication. Consider the associated cocyclic module. Then the cyclic cochains $\mathcal{C}^*_\lambda(\mathcal{O})$ forms a subcomplex of $\mathcal{C}^*(\mathcal{O})$, stable under the Lie bracket of degree -1. In particular, the cyclic cohomology $HC^*_\lambda(\mathcal{C}^*(\mathcal{O}))$ has naturally a graded Lie algebra structure of degree -1.*

Proof. Let $\mathcal{O}$ be a linear cyclic operad. Let $f \in \mathcal{C}^m_\lambda(\mathcal{O})$ and $g \in \mathcal{C}^n_\lambda(\mathcal{O})$. Using (33), (34) and the change of variable $i' = i - 1$ for the first equation and using $\tau_m(f) = (-1)^m f$ and $\tau_m(g) = (-1)^n g$ for the second equation, we have

$$\tau_{m+n-1}(f\bar{\circ}g) = (-1)^{(m-1)(n-1)}\left(\tau_n g \circ_n \tau_m f + \sum_{i'=1}^{m-1}(-1)^{(n-1)i'}\tau_m f \circ_{i'} g\right)$$

$$= (-1)^{(m-1)(n-1)}\left((-1)^{m+n} g \circ_n f + (-1)^{m+n-1}\sum_{i=1}^{m-1}(-1)^{(n-1)(i-1)} f \circ_i g\right).$$

By symmetry

$$(-1)^{(m-1)(n-1)}\tau_{m+n-1}(g\bar{\circ}f) = (-1)^{m+n} f\circ_m g + (-1)^{m+n-1}\sum_{i=1}^{n-1}(-1)^{(m-1)(i-1)} g\circ_i f.$$

Therefore $\tau_{m+n-1}\{f,g\} = (-1)^{m+n-1}\{f,g\}$.

Suppose now that $\mathcal{O}$ has an associative multiplication μ such that $\tau_2\mu = \mu$. Since $\mu \in \mathcal{C}^2_\lambda(\mathcal{O})$, we have just proved above that for any $g \in \mathcal{C}^n_\lambda(\mathcal{O})$, the differential of g, $d(g) = \{\mu, g\} \in \mathcal{C}_\lambda(\mathcal{O})$. □

Remark 38. Let $\mathcal{O}$ be a cyclic operad. Then $\tau_1 : \mathcal{O}(1) \to \mathcal{O}(1)$ is an involutive morphism of anti-algebras. And $\mathcal{C}^1_\lambda(\mathcal{O}) = \mathrm{Ker}(\tau_1 + Id : \mathcal{O}(1) \to \mathcal{O}(1))$ is a sub Lie algebra of the Lie algebra associated to the associative algebra $\mathcal{O}(1)$(Compare with Remark 9).

Remark 39. In [41, Corollary 1.5], motivated by applications to string topology [2], we proved that the negative cyclic cohomology of a cyclic operad $\mathcal{O}$ with multiplication, $HC^*_-(\mathcal{C}^*(\mathcal{O}))$, has a Lie bracket of degree -2.

Remark 40. The (ordinary) cyclic cohomology of $\mathcal{O}$, $HC^*(\mathcal{C}^*(\mathcal{O}))$, has also a Lie bracket of degree -1. This was stated only in the case of the cyclic cohomology of the group ring $\Bbbk[G]$ of a finite group G [3, Theorem 67 a)]. But the proof of [3, Theorem 67 a)] works for any cyclic operad with multiplication.

Remark 41. The proof of Theorem 37 is a lot more simple than the proofs of remarks 39 and 40. Indeed, the proofs of remarks 39 and 40 use that $H^*(\mathcal{C}^*(\mathcal{O}))$ is a Batalin-Vilkovisky algebra (Theorem 36). On the contrary, in the proof of Theorem 37, we don't even use that $H^*(\mathcal{C}^*(\mathcal{O}))$ is a Gerstenhaber algebra: we use only the Lie algebra on $\mathcal{C}^*(\mathcal{O})$.

If our ground ring $\Bbbk$ contains $\mathbb{Q}$, there is a natural isomorphism [34, p. 72]

$$HC^n_\lambda(\mathcal{C}^*(\mathcal{O})) \overset{\cong}{\to} HC^n(\mathcal{C}^*(\mathcal{O})).$$

This isomorphism obviously should be compatible with the brackets.

Recall the following well-known result in string topology.

Corollary 42. *(*[56],[41, Theorem 1.6]*) Let A be a symmetric Frobenius algebra (Definition 55). Then its Hochschild cohomology $HH^*(A, A)$ is a Batalin-Vilkovisky algebra.*

We need to sketch our proof given in [41].

Proof. Let $\Theta : A \overset{\cong}{\to} A^\vee$ be an isomorphism of A-bimodules given by the symmetric Frobenius algebra structure on A. Then $\mathcal{C}^*(A, \Theta) : \mathcal{C}^*(A, A) \overset{\cong}{\to} \mathcal{C}^*(A, A^\vee)$ is an isomorphism of cosimplicial modules. Let $Ad : \mathcal{C}^*(A, A^\vee) \overset{\cong}{\to} \mathcal{C}_*(A, A)^\vee$ be the adjunction map [41, (4.1)] which associates to any $g \in \mathrm{Hom}(A^n, A^\vee)$, the linear map $Ad(g) : A \otimes A^{\otimes n} \to \Bbbk$ given by $Ad(g)(a_0 \otimes \cdots \otimes a_n) = g(a_1, \ldots, a_n)(a_0)$. Then $Ad : \mathcal{C}^*(A, A^\vee) \overset{\cong}{\to} \mathcal{C}_*(A, A)^\vee$ is an isomorphism of cosimplicial modules. By [41, Proof of Theorem 1.6]

$$\mathcal{C}^*(A, A) \underset{\cong}{\overset{\mathcal{C}^*(A,\Theta)}{\to}} \mathcal{C}^*(A, A^\vee) \underset{\cong}{\overset{Ad}{\to}} \mathcal{C}_*(A, A)^\vee$$

equipped with the τ_n [41, (4.2)] is a cyclic operad with multiplication. Using Theorem 36, $HH^*(A, A) \overset{\cong}{\to} HH^*(A, A^\vee)$ is a Batalin-Vilkovisky algebra. □

If instead of using Theorem 36, we apply Theorem 37 in the previous proof, we obtain the following Corollary:

Corollary 43. *Let A be a symmetric Frobenius algebra. Then its cyclic cohomology $HC^*_\lambda(A)$ (in the sense of* [34, 2.4.2]*), is a graded Lie algebra of degree -1.*

We wonder if this Corollary is not a particular simple case of [22, Proposition 2.11]?

In [41], our main objective was the following result:

Corollary 44. [41, Theorem 1.1] *Let H be a Hopf algebra endowed with a modular pair in involution $(\delta, 1)$ (Definition 48). Then the Connes-Moscovici cocyclic on the Cobar construction on H, defines a Batalin-Vilkovisky algebra structure on $Cotor^*_H(\mathbb{k}, \mathbb{k})$.*

Proof. A computation [41, Section 5] shows that the operad with multiplication $\mathcal{C}o\mathcal{E}nd_{H-mod}(H) \cong \Omega H$ considered in the proof of Theorem 15 equipped with the τ_n defined by Connes and Moscovici, is cyclic. Therefore, by Theorem 36, its homology $\mathrm{Cotor}^*_H(\mathbb{k}, \mathbb{k})$ is a Batalin-Vilkovisky algebra. □

If instead of using Theorem 36, we apply Theorem 37 in the previous proof, we obtain the following Corollary:

Corollary 45. *Let H be a Hopf algebra endowed with a modular pair in involution $(\delta, 1)$ (Definition 48). Then its cyclic cohomology, $HC^*_{(\delta,1)}(H)$, is a graded Lie algebra of degree -1.*

6. Batalin-Vilkovisky algebra structure on $\mathrm{Ext}^*_H(\mathbb{k}, \mathbb{k})$

Everybody is more familiar with an algebra A than with a coalgebra C. And therefore, one usually prefers the Exterior product $\mathrm{Ext}^*_A(\mathbb{k}, \mathbb{k})$ instead of the Cotorsion product $\mathrm{Cotor}^*_C(\mathbb{k}, \mathbb{k})$. The goal of this section is to give the duals of Corollaries 44 and 45, Theorem 50 below. Taillefer [51], Khalkhali and Rangipour [29] developed a theory dual to Connes-Moscovici cyclic cohomology of Hopf algebras. First, we are going to explain this duality.

Proposition 46. *Let K be a finite dimensional Hopf algebra with a modular pair in involution (δ, σ) in the sense of Khalkhali-Rangipour* [29, (1)]. *Then i) its dual $K^\vee$ is a Hopf algebra equipped with a modular pair in involution (ev_σ, δ) in the sense of Connes-Moscovici where $ev_\sigma : K^\vee \to \mathbb{F}$ is defined by $ev_\sigma(\varphi) = \varphi(\sigma)$.*

Let $\psi_n : (K^\vee)^{\otimes n} \overset{\cong}{\to} (K^{\otimes n})^\vee$ be the linear map mapping the tensor product $\varphi_1 \otimes \cdots \otimes \varphi_n$ of n forms on K to the form on $K^{\otimes n}$, also denoted $\varphi_1 \otimes \cdots \otimes \varphi_n$, mapping $k_1 \otimes \cdots \otimes k_n$ to the product

$\varphi_1(k_1)\dots\varphi_n(k_n)$. Then ii) ψ_* is an isomorphism of cocyclic modules between the cocyclic modules $\Omega(K^\vee)_{(ev_\sigma,\delta)}$ introduced by Connes-Moscovici and the dual of the cyclic module $B(K)^{(\delta,\sigma)}$ introduced by Khalkhali-Rangipour [29, Theorem 2.1] *and Taillefer* [51].

iii) In particular, ψ_* induces an isomorphism between Connes-Moscovici cyclic cohomology of $K^\vee$, $HC^*_{(ev_\sigma,\delta)}(K^\vee)$ and the dual of Khalkhali-Rangipour-Taillefer cyclic homology of K, $\widetilde{HC}_*^{(\delta,\sigma)}(K)^\vee$.

The cocyclic module $\Omega(H)_{(\delta,\sigma)}$ is denoted $H^\natural_{(\delta,\sigma)}$ in [6, Theorem 1]. The cyclic module $B(K)^{(\delta,\sigma)}$ is denoted $\widetilde{K}^{(\delta,\sigma)}$ in [29, Theorem 2.1] and $C_*^{(\sigma,\varepsilon,\delta)}(K)$ in [51, 2.1].

Definition 47. A *modular pair* is a couple (δ,σ) when δ is a character and σ is a group-like element such that $\delta(\sigma)=1$.

Proof of Proposition 46. i) An element σ is a *group-like* element of K by definition if and only if $\Delta\sigma=\sigma\otimes\sigma$ and $\varepsilon(\sigma)=1$. This means that the linear map that we denoted again $\sigma:\mathbb{F}\to K$, mapping 1 to σ is a morphism of coalgebras. Therefore its dual $ev_\sigma=\sigma^\vee:K^\vee\to\mathbb{F}$ is a morphism of algebras, i.e. a *character* of $K^\vee$. Let $\delta:K\to\mathbb{F}$ be a character of K, i.e. a morphism of algebras. Its dual $\delta^\vee:\mathbb{F}\to K^\vee$, mapping 1 to δ, is a morphism of coalgebras, i.e δ is a group-like element of $K^\vee$. By definition, $ev_\sigma(\delta)=\delta(\sigma)$. Therefore (δ,σ) is a modular pair on K if and only if (ev_σ,δ) is a modular pair in $K^\vee$.

Let (δ,σ) be a modular pair on H. The *twisted antipode* $\widetilde{S}$ associated to (δ,σ) (in the sense of Connes-Moscovici) is by definition the convolution product $(\eta\circ\delta)\star S$ in $\mathrm{Hom}(H,H)$. Explicitly, for $h\in H$, $\widetilde{S}(h)=\delta(h^1)S(h^2)$, where $\Delta h=h^1\otimes h^2$. Consider the map $\tau_n:H^{\otimes n}\to H^{\otimes n}$ defined by

$$\tau_n(h_1\otimes\cdots\otimes h_n):=\mu_{H^{\otimes n}}\left(\Delta^{n-1}\widetilde{S}(h_1)\otimes(h_2\otimes\cdots\otimes h_n\otimes\sigma)\right)$$

Here $\mu_{H^{\otimes n}}:H^{\otimes n}\otimes H^{\otimes n}\to H^{\otimes n}$ is the product in $H^{\otimes n}$ and $\Delta^{n-1}:H\to H^{\otimes n}$ is the iterated diagonal on H. In particular, $\tau_1(h)=\widetilde{S}(h)\sigma$.

Definition 48. A modular pair (δ,σ) (Definition 47) is *in involution* in the sense of Connes-Moscovici if and only if $\tau_1^2=id_H$, i.e. $\forall h\in H$, $\widetilde{S}^2(h)=\sigma h\sigma^{-1}$.

Let $(\Omega H)_{(\delta,\sigma)}$ be the usual cosimplicial module defining the Cobar construction, except that $\delta_{n+1}:H^{\otimes n}\to H^{\otimes n+1}$ is given by [6, (2.9)] $\delta_{n+1}(h_1\otimes\cdots\otimes h_n)=h_1\otimes\cdots\otimes h_n\otimes\sigma$. Connes and Moscovici have shown that if $\tau_1^2=id_H$, then $(\Omega H)_{(\delta,\sigma)}$ (equipped with the τ_n) is a cocyclic module.

Let (δ,σ) be a modular pair on K. Let $t_n : K^{\otimes n}\to K^{\otimes n}$ defined by ([29, Theorem 2.1] or [51, 2.1] which generalizes [34, (7.3.3.1)])

$$t_n(k_1\otimes\cdots\otimes k_n)=\sigma S(k_1^{(1)}\dots k_n^{(1)})\otimes k_1^{(2)}\otimes\cdots\otimes k_{n-1}^{(2)}\delta(k_n^{(2)})$$

where $\Delta(k_i)=k_i^{(1)}\otimes k_i^{(2)}$. In particular, t_1 is equal to $\sigma(S\star\eta\circ\delta)$, the left multiplication by σ of the convolution product $\star$ of S and $\eta\circ\delta$. By definition, the couple (δ,σ) is a *modular pair in involution* in the sense of Khalkhali-Rangipour [29, (1)] if and only if $t_1^2=id_K$.

Therefore to prove part i) of this Proposition, it suffices to show that $\tau_1=t_1^\vee$. This will be proved in the proof of ii) below.

(Denote by $K^{op,cop}$ the Hopf algebra with the opposite multiplication, the opposite diagonal and the same antipode [8, Remark 4.2.10], since the convolution product $\star$ on $\mathrm{Hom}(K^{op,cop},K^{op,cop})$ is the opposite of the convolution product $\star$ on $\mathrm{Hom}(K,K)$, note that a modular pair in involution for K in the sense of Khalkhali-Rangipour is the same as a modular pair in involution for $K^{op,cop}$ in the sense of Connes-Moscovici.)

ii) Let $B(K)^{(\delta,\sigma)}$ be the usual simplicial module defining the Bar construction except that $d_{n+1}:K^{\otimes n+1}\to K^{\otimes n}$ is given by ([29, Theorem 2.1] or [51, 2.1]) $d_{n+1}(k_1\otimes\cdots\otimes k_{n+1})=k_1\otimes\cdots\otimes k_n\delta(k_{n+1})$. It is well known [27, Lemma XVIII.7.3] that ψ_* is an isomorphism of cosimplicial modules from the usual Cobar construction on $K^\vee$, $\Omega(K^\vee)_{(ev_\sigma,\varepsilon)}$, to the dual of the usual Bar construction on K, $(B(K)^{(\varepsilon,\sigma)})^\vee$. Obviously, $\psi_{n+1}\circ\delta_{n+1}=d_{n+1}^\vee\circ\psi_n$. Therefore $\psi_*:\Omega(K^\vee)_{(ev_\sigma,\delta)}\overset{\cong}{\to}(B(K)^{(\delta,\sigma)})^\vee$ is an isomorphism of cosimplicial modules even if $\delta\neq\varepsilon$.

Denote by $\sigma S:K\to K$ the linear map defined by $(\sigma S)(k)=\sigma S(k)$, $k\in K$. The transposition map $\mathrm{Hom}(K,K)\to\mathrm{Hom}(K^\vee,K^\vee)$, mapping a linear map $f:K\to K$ to its dual $f^\vee:K^\vee\to K^\vee$ is a morphism of algebras with respect to the convolution products $\star$. Since σS can be written as the convolution product $(\sigma\circ\varepsilon)\star S$ of the composite $K\overset{\varepsilon}{\to}\mathbb{F}\overset{\sigma}{\to}K$ and of the antipode S, its dual $(\sigma S)^\vee$ is equal to $(\varepsilon^\vee\circ\sigma^\vee)\star S^\vee=(\varepsilon\circ ev_\sigma)\star S^\vee$ which is the twisted antipode $\widetilde{S}$ on $K^\vee$ associated to the modular pair (ev_σ,δ).

The cyclic operator $t_n:K^{\otimes n}\to K^{\otimes n}$ can be written as the composite

$$K^{\otimes n}\overset{\Delta_{K^{\otimes n}}}{\to}K^{\otimes n}\otimes K^{\otimes n}\overset{\mu^{(n-1)}\otimes K^{\otimes n}}{\to}K\otimes K^{\otimes n}\overset{\sigma S\otimes K^{\otimes n-1}\otimes\delta}{\to}K\otimes K^{\otimes n-1}\otimes\mathbb{F}.$$

Here $\mu^{(n-1)}:K^{\otimes n}\to K$ is the iterated product on K and $\Delta_{K^{\otimes n}}$ is the diagonal of $K^{\otimes n}$. The cocyclic operator $\tau_n:H^{\otimes n}\to H^{\otimes n}$ can be written as the composite

$$H\otimes H^{\otimes n-1}\otimes\mathbb{F}\overset{\widetilde{S}\otimes H^{\otimes n-1}\otimes\sigma}{\to}H\otimes H^{\otimes n}\overset{\Delta^{(n-1)}\otimes H^{\otimes n}}{\to}H^{\otimes n}\otimes H^{\otimes n}\overset{\mu_{H^{\otimes n}}}{\to}H^{\otimes n}$$

Here $\Delta^{(n-1)} : H \to H^{\otimes n}$ is the iterated diagonal on H and $\mu_{H^{\otimes n}}$ is the multiplication of $H^{\otimes n}$. We saw that the twisted antipode $\widetilde{S}$ on $K^\vee$ associated to (ev_σ, δ) was $(\sigma S)^\vee$, the dual of σS. Therefore $\psi_n \circ \tau_n = t_n^\vee \circ \phi_n$. In particular when $n = 1$, since ψ_1 is the identity, $\tau_1 = t_1^\vee$. So finally, $\psi_* : \Omega(K^\vee)_{(ev_\sigma,\delta)} \overset{\cong}{\to} (B(K)^{(\delta,\sigma)})^\vee$ is an isomorphism of cocyclic modules. □

Corollary 49. *Let K be a finite dimensional Hopf algebra equipped with a group-like element σ such that $\forall k \in K$, $S \circ S(k) = \sigma^{-1} k \sigma$. Then $\psi_* : \Omega(K^\vee)_{(ev_\sigma,\varepsilon)} \overset{\cong}{\to} (B(K)^{(\varepsilon,\sigma)})^\vee$ is an isomorphism of cyclic operads with multiplication. In particular, $H^*(\psi_*) : Cotor^*_{K^\vee}(\mathbb{F}, \mathbb{F}) \overset{\cong}{\to} Ext^*_K(\mathbb{F}, \mathbb{F})$ is an isomorphism of Batalin-Vilkovisky algebras and ψ_* induces an isomorphism of graded Lie algebras $HC^*_{(ev_\sigma,\varepsilon)}(K^\vee) \overset{\cong}{\to} \widetilde{HC}^*_{(\varepsilon,\sigma)}(K)$.*

Proof. The canonical injection of K into its bidual $K^{\vee\vee}$, $\nu : K \hookrightarrow K^{\vee\vee}$, is an isomorphism of bialgebras. Let $C := K^\vee$ be the dual bialgebra. In the proof of Proposition 22, we saw that $\phi : \Omega C \overset{\cong}{\to} B(C^\vee)^\vee$ is an isomorphism of linear operads with multiplication. Therefore the composite $\Omega(K^\vee) \overset{\phi}{\to} B(K^{\vee\vee})^\vee \overset{B(\nu)^\vee}{\to} B(K)^\vee$ is also an isomorphism of linear operads with multiplication. But this composite coincides with the isomorphism of cocyclic modules $\psi_* : \Omega(K^\vee)_{(ev_\sigma,\varepsilon)} \overset{\cong}{\to} (B(K)^{(\varepsilon,\sigma)})^\vee$ given by part ii) of Proposition 46. □

Theorem 50. *Let K be a Hopf algebra equipped with a group-like element σ such that for all $k \in K$, $S^2(k) = \sigma^{-1} k \sigma$. Let $t_n : K^{\otimes n} \to K^{\otimes n}$ be the linear map defined by*

$$t_n(k_1 \otimes \cdots \otimes k_n) = \sigma S(k_1^{(1)} \dots k_{n-1}^{(1)} k_n) \otimes k_1^{(2)} \otimes \cdots \otimes k_{n-1}^{(2)}.$$

The dual of the Bar construction on K, $B(K)^\vee$ is a cyclic operad with multiplication. In particular, the Gerstenhaber algebra given by Theorem 16, $Ext^*_K(\Bbbk, \Bbbk)$, is in fact a Batalin-Vilkovisky algebra and the cyclic cohomology of K, $\widetilde{HC}^*_{(\varepsilon,\sigma)}(K)$ has a Lie bracket of degree -1.

Proof. Corollary 49 explains in details that this Theorem is the dual of Corollaries 44 and 45. Therefore, the computations dual to [41, Proof of Theorem 1.1] show that the operad with multiplication $B(K)^\vee$ given in the proof of Theorem 16 together with the cyclic operators t_n defines a cyclic operad with multiplication. Using Theorems 36 and 37, we conclude. □

7. Characteristic maps

Lemma 51. *Let H be a bialgebra. Let A be a left module algebra over H (in the sense of* [27, Definition V.6.1]*). Then the application $\Phi : H^{\otimes n} \to Hom_{\Bbbk-mod}(A^{\otimes n}, A)$ mapping $h_1 \otimes \cdots \otimes h_n$ to $f : A^{\otimes n} \to A$ defined by $f(a_1 \otimes \cdots \otimes a_n) = (h_1.a_1) \dots (h_n.a_n)$ defines a morphism of linear operads with multiplication from the coendomorphism operad of H, $\mathcal{C}o\mathcal{E}nd_{H-mod}(H)$, to the endomorphism operad of A, $\mathcal{E}nd_{\Bbbk-mod}(A)$. In particular, Φ induces a morphism of Gerstenhaber algebras $H^*(\Phi) : Cotor^*_H(\Bbbk, \Bbbk) \to HH^*(A, A)$.*

Proof. Since $1_H.a_1 = a_1$, $\Phi(1_H) = id_A$. Let $h_1 \otimes \cdots \otimes h_m \in H^{\otimes m}$, $k_1 \otimes \cdots \otimes k_m \in H^{\otimes n}$ and $a_1 \otimes \cdots \otimes a_{m+n-1} \in A^{\otimes m+n-1}$. Using (18), we have that $\Phi[(h_1 \otimes \cdots \otimes h_m) \circ_i (k_1 \otimes \cdots \otimes k_n)]$ evaluated on $a_1 \otimes \cdots \otimes a_{m+n-1}$ is equal to the product

$$(h_1.a_1) \dots (h_{i-1}.a_{i-1})(h_i^{(1)} k_1.a_i) \dots (h_i^{(n)} k_n.a_{i+n-1})(h_{i+1}.a_{i+n}) \dots (h_m.a_{m+n-1}).$$

On the other hand, using example 2, $\Phi(h_1 \otimes \cdots \otimes h_m) \circ_i \Phi(k_1 \otimes \cdots \otimes k_n)$ evaluated on $a_1 \otimes \cdots \otimes a_{m+n-1}$ is equal to the product

$$(h_1.a_1) \dots (h_{i-1}.a_{i-1}) \left(h_i.[(k_1.a_i) \dots (k_n.a_{i+n-1})]\right) (h_{i+1}.a_{i+n}) \dots (h_m.a_{m+n-1}).$$

Since for any $h \in H$, a and $b \in A$, $h.(ab) = (h^{(1)}.a)(h^{(2)}.a)$, the previous two products are equal. So Φ is a morphism of operads. Now Φ is a morphism of operads with multiplication, since $\Phi(1_{\Bbbk})$ is the unit map $\eta : \Bbbk \to A$ and since $\Phi(1_H \otimes 1_H)$ is the multiplication $\mu : A \otimes A \to A$ of A. ☐

The following Lemma is a variant of the previous lemma if H is finite dimensional, since in this case, A is a left module algebra over H if and only if A be a right comodule algebra over the dual of H, $H^\vee$.

Lemma 52. *Let H be a bialgebra. Let A be a right comodule algebra over H (in the sense of* [27, Definition III.7.1]*). Then the application $\Phi : (H^{\otimes n})^\vee \to Hom_{\Bbbk-mod}(A^{\otimes n}, A)$ mapping $f : H^{\otimes n} \to \Bbbk$ to $F : A^{\otimes n} \to A$ defined by $F(a_1 \otimes \cdots \otimes a_n) = a_1^{(1)} \dots a_n^{(1)} f(a_1^{(2)} \otimes \cdots \otimes a_n^{(2)})$. defines a morphism of linear operads with multiplication from the endomorphism operad of H, $\mathcal{E}nd_{H-comod}(H)$, to the endomorphism operad of A, $\mathcal{E}nd_{\Bbbk-mod}(A)$. In particular, Φ induces a morphism of Gerstenhaber algebras $H^*(\Phi) : Ext^*_H(\Bbbk, \Bbbk) \to HH^*(A, A)$.*

Note that in the case $A = H$, $H^*(\Phi)$ coincides with the inclusion of Gerstenhaber algebras given by Theorem 16. The proof of Lemma 52 is a computation similar to the proof of Lemma 51.

Theorem 53. *Let H be a Hopf algebra equipped with a group-like element $\sigma \in H$ such that $\forall h \in H$, $S^2(h) = \sigma^{-1}h\sigma$. Let A be a right comodule algebra over H. Let $\tau : A \to \mathbb{k}$ be a non-degenerate 1-trace, i.e. the morphism of left A-modules $\Theta : A \overset{\cong}{\to} A^\vee$, mapping $b \in A$ to $\varphi : A \to \mathbb{k}$ given by $\varphi(a) = \tau(ab)$, is an isomorphism of A-bimodules. Suppose that τ is σ-invariant in the sense of* [29, Definition 3.1]*: $\forall a, b \in A$, $\tau(a^{(1)})a^{(2)} = \tau(a)\sigma$. Then*

1) the morphism $H^*(\Phi) : Ext^*_H(\mathbb{k}, \mathbb{k}) \to HH^*(A, A)$ given by Lemma 51, is a morphism of Batalin-Vilkovisky algebras,

2) the characteristic map defined by Khalkhali-Rangipour [29, (10)] *$\gamma^* : \widetilde{HC}^*_{(\varepsilon,\sigma)}(H) \to HC^*_\lambda(A)$ is a morphism of graded Lie algebras.*

Proof. Recall from the proof of Corollary 42 that

$$\mathcal{C}^*(A, A) \underset{\cong}{\overset{\mathcal{C}^*(A,\Theta)}{\to}} \mathcal{C}^*(A, A^\vee) \underset{\cong}{\overset{Ad}{\to}} \mathcal{C}_*(A, A)^\vee$$

is a cyclic operad with multiplication. By Lemma 52, $\Phi : B(H)^\vee \to \mathcal{C}^*(A, A)$ is a morphism of linear operads with multiplication. Let $\gamma : \mathcal{C}_*(A, A) \to B(H)$ be the morphism of cyclic modules defined by [29, Proposition 3.1]

$$\gamma(a_0 \otimes a_1 \otimes \cdots \otimes a_n) = \tau(a_0 a_1^{(1)} \ldots a_n^{(1)})(a_1^{(2)} \otimes \cdots \otimes a_n^{(2)}).$$

Here the coaction of a_i, $\Delta a_i = a_i^{(1)} \otimes a_i^{(2)}$. A straightforward calculation shows that the composite

$$B(H)^\vee \overset{\Phi}{\to} \mathcal{C}^*(A, A) \underset{\cong}{\overset{\mathcal{C}^*(A,\Theta)}{\to}} \mathcal{C}^*(A, A^\vee) \underset{\cong}{\overset{Ad}{\to}} \mathcal{C}_*(A, A)^\vee$$

is the dual of γ, $\gamma^\vee : \mathcal{C}_*(A, A)^\vee \to B(H)^\vee$. Since $\gamma^\vee$ is a morphism of cocyclic modules, $\Phi : B(H)^\vee \to \mathcal{C}^*(A, A)$ is a morphism of linear cyclic operads with multiplication. By applying Theorem 36, $H(\Phi)$ is a morphism of Batalin-Vilkovisky algebras between the Batalin-Vilkovisky algebras given by Theorem 50 and Corollary 42. This is 1). By applying Theorem 37, we obtain 2). ☐

Using this time, Lemma 51 and the cocyclic map γ defined by Connes and Moscovici [6, Theorem 6], we obtain easily the following variant of the previous Theorem.

Theorem 54. *Let H be a Hopf algebra endowed with a modular pair in involution $(\delta, 1)$. Let A be a module algebra over H. Let $\tau : A \to \mathbb{k}$ be a non-degenerate 1-trace, i.e. the morphism of left A-modules $\Theta : A \overset{\cong}{\to} A^\vee$, mapping $b \in A$ to $\varphi : A \to \mathbb{k}$ given by $\varphi(a) = \tau(ab)$, is an isomorphism of A-bimodules. Suppose that τ is δ-invariant. Then*

1) the morphism $H^*(\Phi) : Cotor^*_H(\mathbb{k},\mathbb{k}) \to HH^*(A,A)$ given by Lemma 51, is a morphism of Batalin-Vilkovisky algebras,

2) the characteristic map defined by Connes and Moscovici $\chi_\tau : HC^*_{(\delta,1)}(H) \to HC^*_\lambda(A)$ is a morphism of graded Lie algebras.

In [6, Theorem 6] or[49, Section 4.4], Connes and Moscovici have defined more generally a characteristic map $\chi_\tau : HC^*_{(\delta,\sigma)}(H) \to HC^*_\lambda(A)$ without assuming that

i) the group-like element σ is the unit 1 of H,

and without assuming that

ii) the σ-trace τ is non-degenerated.

But we need i) to have a Lie bracket on $HC^*_{(\delta,\sigma)}(H)$ (Corollary 45) and we need i) and ii) to have a Lie bracket on $HC^*_\lambda(A)$ (Corollary 43). However, note that in their first construction of the characteristic map in [5], Connes and Moscovici were assuming i) like us. We believe that ii) can be weakened, since the Batalin-Vilkovisky algebra on $HH^*(A,A^\vee)$ can be defined for non-counital symmetric Frobenius algebras, i. e "unital associative algebras with an invariant co-inner product" [57, p. 61-2]. In particular, as Tradler explained us, A does not need to be finite dimensional.

8. Hopf algebras that are symmetric Frobenius

In this section, we work over an arbitrary field $\mathbb{F}$. We want to consider in Theorem 53, the case where the comodule algebra A over H is the Hopf algebra H itself. We remark that for a finite dimensional Hopf algebra H, there is a close relationship between being a symmetric Frobenius algebra and being equipped with a modular pair in involution of the form (ε, u) (Theorem 61). Therefore (Theorem 63), for Hopf algebras which are symmetric Frobenius algebras, often we have an inclusion of Batalin-Vilkovisky algebras $\mathrm{Ext}^*_H(\mathbb{F},\mathbb{F}) \hookrightarrow HH^*(H,H)$ and in some cases, the characteristic map $\widetilde{HC}^*_{(\varepsilon,\sigma)}(H) \to HC^*_\lambda(H)$ is injective.

First, we recall the notion of (symmetric) Frobenius algebra and that the Nakayama automorphisms of a symmetric Frobenius algebra are all inner automorphisms. Then we recall that an augmented symmetric Frobenius algebra is always unimodular. Specializing to Hopf algebras, we recall that finite dimensional Hopf algebras are always Frobenius algebras and that the square $S \circ S$ of the antipode of an unimodular Hopf algebra is a particular Nakayama automorphism. Finally, we can recall Theorem 61 due to Oberst and Schneider [44], which explains when a Hopf algebra is a symmetric Frobenius algebra. In the proof of Theorem 61, we recall the construction of a non-degenerated trace τ on H. Checking that the diagonal of H is compatible with this trace τ, we obtain Theorem 63.

8.1. **Frobenius algebras.** Let A be an algebra. The morphism of right A-modules $\Theta : A \to A^\vee$, mapping 1 to the form ϕ, is an isomorphism (of A-bimodules) if and only if A is finite dimensional and the bilinear form $< \ , \ >: A \otimes A \to \mathbb{F}$ defined by $< a, b >:= \phi(ab)$ is non-degenerate (and symmetric).

Definition 55. An algebra A is a *(symmetric) Frobenius algebra* if there exists an isomorphism $\Theta : A \overset{\cong}{\to} A^\vee$ of right A-modules (respectively of A-bimodules). We call $\phi := \Theta(1)$ a *Frobenius form.*

Example 56. Let G be a finite group then its group algebra $\mathbb{F}[G]$ is a non-commutative symmetric Frobenius algebra. By definition, the group ring $\mathbb{F}[G]$ admits the set $\{g \in G\}$ as a basis. Denote by δ_g the dual basis in $\mathbb{F}[G]^\vee$. The linear isomorphism $\Theta : \mathbb{F}[G] \to \mathbb{F}[G]^\vee$, sending g to $\delta_{g^{-1}}$ is an isomorphism of $\mathbb{F}[G]$-bimodules.

Let A be a Frobenius algebra with Frobenius form ϕ. By definition [32, (16.42)], the *Nakayama automorphism* of ϕ is the unique automorphism of algebras $\sigma : A \overset{\cong}{\to} A$ such that for all a and $b \in A$, $\phi(ab) = \phi(\sigma(b)a)$. Let σ and σ' be two Nakayama automorphisms of a Frobenius algebra A. Then, by [32, (16.43)], there exists an invertible element $u \in A$ such that for all $x \in A$, $\sigma'(x) = u\sigma(x)u^{-1}$. In particular, if A is a symmetric Frobenius algebra, the identity map of A, $id_A : A \to A$ is a particular Nakayama automorphism of A. And all the other Nakayama automorphisms are inner automorphisms [35, p. 483 Lemma (b)].

Definition 57. Let $(A, \mu, \eta, \varepsilon)$ be an augmented algebra. A left (respectively right) *integral* in A is an element l of A such that $\forall h \in A$, $h \times l = \varepsilon(h)l$ (respectively $l \times h = \varepsilon(h)l$). The augmented algebra A is *unimodular* if the set of left integrals in A coincides with the set of right integrals in A.

Remark 58. An element l of A is a right integral in A such that $\varepsilon(l) = 1$ if and only if $1_A - l$ is a left unit in $\overline{A}$, the augmentation ideal of A. Suppose that there exists a right integral l in A such that $\varepsilon(l) = 1$. Then l defines a morphism of right A-modules $s : \mathbb{F} \to A$ such that $\varepsilon \circ s = id_{\mathbb{F}}$. Therefore $\mathbb{F}$ is a right projective A-module and $\mathrm{Ext}^*_A(\mathbb{F}, \mathbb{F})$ is concentrated in degree 0 (Compare with [7, Proof of Proposition 5.4]).

The set of right (respectively left) integrals in an augmented Frobenius algebra is an $\mathbb{F}$-vector space of dimension 1 [26, Proposition 6.1]. Let A be an augmented algebra and let $\Theta : A \overset{\cong}{\to} A^\vee$ be an isomorphism of left (respectively right) A-modules. Then $\Theta^{-1}(\varepsilon)$ is non-zero left (respectively right) integral in A [26, just above Proposition 6.1]. In particular, if $\Theta : A \overset{\cong}{\to} A^\vee$ is an isomorphism of A-bimodules, $\Theta^{-1}(\varepsilon)$ is both a non-zero left et right integral in A. Therefore a symmetric Frobenius algebra with an augmentation is always unimodular.

Let A be a Frobenius algebra with an augmentation. Let t be any non-zero left integral in A. The *distinguished group-like element* or left modular function [26, (6.2)] in $A^\vee$ is the unique morphism of algebras $\alpha : A \to \mathbb{F}$ such that for all $h \in A$, $t \times h = \alpha(h)t$ ([43, 2.2.3] or [45, p. 590]). Note that A is unimodular if and only if the distinguished group-like element in $A^\vee$ is ε the augmentation of A.

8.2. **Hopf algebras.** Let $(H, \mu, \eta, \Delta, \varepsilon, S)$ be a finite dimensional Hopf algebra. Its dual is also a Hopf algebra $(H^\vee, \Delta^\vee, \varepsilon^\vee, \mu^\vee, \eta^\vee, S^\vee)$. In particular, a form λ on H is a left (respectively right) integral in $H^\vee$ if and only if for every $\varphi \in H^\vee$ and $k \in H$, $\sum \varphi(k^{(1)})\lambda(k^{(2)}) = \varphi(1_H)\lambda(k)$ (respectively $\sum \lambda(k^{(1)})\varphi(k^{(2)}) = \varphi(1_H)\lambda(k)$). Here $\Delta k = \sum k^{(1)} \otimes k^{(2)}$.

Example 59. [43, 2.1.2] If G is a finite group, $\sum_{g\in G} g$ is both a left and right integral in the group algebra $\mathbb{F}[G]$. And δ_1, the form mapping $g \in G$ to 1 if $g = 1$ and 0 otherwise, is both a left and right integral in $\mathbb{F}[G]^\vee$. Since $\delta_1(1) = 1$, by Remark 58, $\mathrm{Cotor}^*_{\mathbb{F}[G]}(\mathbb{F}, \mathbb{F})$ and $\mathrm{Ext}^*_{\mathbb{F}[G]^\vee}(\mathbb{F}, \mathbb{F})$ are both concentrated in degree 0 (Note that here the product on G is not used and that G does not need to be finite [28, 4. p. 97]).

The set of left (respectively right) integrals in the dual Hopf algebra $H^\vee$ is an $\mathbb{F}$-vector space of dimension 1 [50, Corollary 5.1.6 2)]. So let λ be any non-zero left (respectively right) integral in $H^\vee$. The morphism of left (respectively right) H-modules, $H \overset{\cong}{\to} H^\vee$, sending g to the form, denoted [50, p. 95] $g \rightharpoonup \lambda$, mapping h to $\lambda(hg)$ (respectively to the form mapping h to $\lambda(gh)$), is an isomorphism [50, Proof of Corollary 5.1.6 2)]. So a finite dimensional Hopf algebra is always a Frobenius algebra, but not always a symmetric Frobenius algebra as Theorem 61 will show.

Lemma 60. *(*[45, Theorem 3(a)] *or* [26, (6.8)]*) Let H be a finite dimensional Hopf algebra. Let λ be a non-zero right integral in $H^\vee$. Let α be the distinguished group-like element in $H^\vee$. Then for all a and $b \in H$,*

i) $\lambda(ab)=\lambda(S^2(b\leftharpoonup\alpha)a)$ *where* $b\leftharpoonup\alpha=\sum\alpha(b^{(1)})b^{(2)}$ *(*[50, p. 95]*,* [45, p. 585] *or* [26, p. 55]*),*
ii) In the case H *is unimodular,* $\lambda(ab)=\lambda(S^2(b)a)$ [33, Proposition 8]*.*

We have seen that if H is unimodular, then $\alpha=\varepsilon$. Therefore ii) follows from i). Note that Kadison's distinguished group-like element [26, (6.2) or p. 57] m in $H^\vee$ is $\alpha^{-1}=\alpha\circ S$, the inverse of ours ([43, 2.2.3] or[26, p. 57]), since he uses right integrals to define it and we use left integrals. Lemma 60 means that the Nakayama automorphism σ of any non-zero right integral λ in $H^\vee$ is given by $\sigma(b)=S^2(b\leftharpoonup\alpha)$ for any $b\in H$.

Theorem 61. [44, 10, 35, 24] *A finite dimensional Hopf algebra* H *is a symmetric Frobenius algebra if and only if* H *is unimodular and its antipode* S *satisfies* S^2 *is an inner automorphism of* H*.*

Proof. Suppose that H is a symmetric Frobenius algebra. Then we saw that H is unimodular and that all its Nakayama automorphisms are inner automorphisms. By ii) of Lemma 60, S^2 is a Nakayama automorphism of H.

Conversely, assume that H is unimodular and that S^2 is an inner automorphism of H. Let u be an invertible element of H such that $\forall h\in H$, $S^2(h)=uhu^{-1}$. Let λ be any non-zero right integral in $H^\vee$. We saw above that $\lambda(ab)$ is a non-degenerate bilinear form on H. By ii) of Lemma 60, $\lambda(ab)=\lambda(S^2(b)a)=\lambda(ubu^{-1}a)$. Therefore $\beta(h,k):=\lambda(uhk)$ is a non-degenerate symmetric bilinear form [35, p. 487 proof of Proposition]. □

Example 62. Let G be a finite group. Since $S^2=Id$ and since δ_1 is a right integral for $\mathbb{F}[G]^\vee$, we recover that the the linear isomorphism $\mathbb{F}[G]\to\mathbb{F}[G]^\vee$, sending g to $\delta_1(g-)=\delta_{g^{-1}}$ is an isomorphism of $\mathbb{F}[G]$-bimodules.

The Sweedler algebra is an example of non-unimodular Hopf algebra over any field of characteristic different from 2 [43, 2.1.2]. Notice that a cocommutative Hopf algebra over a field of characteristic different from 0 can be non unimodular [35, p. 487-8, Remark and Examples (1) and (4)].

The square of the antipode of every quasi-cocommutative Hopf algebra with bijective antipode is an inner automorphism ([27, Proposition VIII.4.1] or [43, 10.1.4]). Therefore by Theorem 61, every braided (also called quasitriangular) unimodular finite dimensional Hopf algebra is a symmetric Frobenius algebra. In particular, the Drinfeld double $D(H)$ of any finite dimensional Hopf algebra is a symmetric Frobenius algebra [26, Theorem 6.10].

Theorem 63. *Let H be a finite dimensional unimodular (Definition 57) Hopf algebra equipped with a group-like element σ such that $\forall h \in H$, $S^2(h) = \sigma^{-1}h\sigma$. Then 1) $H^*(\Phi) : Ext^*_H(\mathbb{F},\mathbb{F}) \hookrightarrow HH^*(H,H)$ is an inclusion of Batalin-Vilkovisky algebras.*

2) Suppose moreover that H is cosemisimple. Then $\gamma^* : \widetilde{HC}^*_{(\varepsilon,\sigma)}(H) \to HC^*_\lambda(H)$ is an inclusion of graded Lie algebras.

Remark that by [8, Exercise 5.5.10], a finite dimensional cosemisimple Hopf algebra is always unimodular.

Proof. By Theorem 16, $H^*(\Phi) : \mathrm{Ext}^*_H(\mathbb{F},\mathbb{F}) \hookrightarrow HH^*(H,H)$ is an inclusion of Gerstenhaber algebras. By Theorem 50 (or Corollary 49), $\mathrm{Ext}^*_H(\mathbb{F},\mathbb{F})$ is a Batalin-Vilkovisky algebra and $\widetilde{HC}^*_{(\varepsilon,\sigma)}(H)$ has a Lie bracket of degree -1. By Theorem 61, H is a symmetric Frobenius algebra. Therefore by Corollary 42, $HH^*(H,H)$ is a Batalin-Vilkovisky algebra. And by Corollary 43, $HC^*_\lambda(H)$ has a Lie bracket of degree -1.

More precisely, let λ be any non-zero right integral in $H^\vee$. Let $\tau : H \to \mathbb{F}$ given by $\tau(a) = \lambda(\sigma^{-1}a)$ for all $a \in H$. In the proof of Theorem 61, we saw that τ is a non-degenerate 1-trace. Since λ is right integral in $H^\vee$, using the canonical injection of H into its bidual, for every $k \in H$, $\lambda(k^{(1)})k^{(2)} = \lambda(k)1_H$. Here $\Delta k = k^{(1)} \otimes k^{(2)}$. By taking $k = \sigma^{-1}a$, since σ^{-1} is a group-like element, for all $a \in H$, $\tau(a^{(1)})\sigma^{-1}a^{(2)} = \lambda(\sigma^{-1}a^{(1)})\sigma^{-1}a^{(2)} = \lambda(\sigma^{-1}a)1_H = \tau(a)1_H$. This means that τ is σ-invariant in the sense of [29, Definition 3.1]. Therefore by applying part 1) of Theorem 53 in the case $A = H$, we obtain that $H^*(\Phi) : \mathrm{Ext}_H(\mathbb{F},\mathbb{F}) \hookrightarrow HH^*(H,H)$ is a morphism of Batalin-Vilkovisky algebras. This is 1).

By [43, 2.4.6] or [8, Exercise 5.5.9], H is cosemisimple means that there exists a right integral t in $H^\vee$ such that $t(1) = 1$. Since the set of right integrals in $H^\vee$ is an $\mathbb{F}$-vector space of dimension 1, any non-zero right integral λ in $H^\vee$ satisfies $\lambda(1) \neq 0$. Since $\tau(\sigma) = \lambda(\sigma^{-1}\sigma) = \lambda(1)$ is different from zero, by [29, Theorem 3.1], the morphism of graded Lie algebras given by part 2) of Theorem 53, $\gamma^* : \widetilde{HC}^*_{(\varepsilon,\sigma)}(H) \to HC^*_\lambda(H)$ is injective. So 2) is proved. □

Note that by Theorem 61, any Hopf algebra satisfying the hypotheses of Theorem 63 is a symmetric Frobenius algebra. On the contrary, any Hopf algebra which is also a symmetric Frobenius algebra does not necessarily satisfies the hypotheses of Theorem 63. Indeed, in a symmetric Frobenius Hopf algebra, S^2 is an inner automorphism, not necessarily given by a group-like element σ. But in order, to apply Connes-Moscovici (or more precisely its dual Khalkhali-Rangipour-Taillefer)

Hopf cyclic cohomology, we have to suppose that σ is a group-like element.

9. The Batalin-Vilkovisky algebra $\mathrm{Cotor}^*_{UL}(\mathbb{Q},\mathbb{Q})$

Let A be a Gerstenhaber algebra. Then A^1 is a Lie algebra. This forgetful functor from Gerstenhaber algebras to Lie algebras, has a left adjoint ([16, Theorem 5 p. 67] or [15, beginning of Section 4]), namely $L \mapsto \Lambda^* L$ where $\Lambda^* L$ is the exterior algebra on the Lie algebra L equipped with the *Schouten bracket*: for $x_1 \wedge \cdots \wedge x_p \in \Lambda^p L$ and $y_1 \wedge \cdots \wedge y_q \in \Lambda^q L$,

$$\{x_1 \wedge \cdots \wedge x_p, y_1 \wedge \cdots \wedge y_q\} = \sum_{1\leq i\leq p, 1\leq j\leq q} \pm\{x_i, y_j\} \wedge x_1 \wedge \cdots \wedge \widehat{x_i} \wedge \cdots \wedge x_p \wedge y_1 \wedge \cdots \wedge \widehat{y_j} \wedge \cdots \wedge y_q. \tag{64}$$

Here the symbol $\widehat{}$ denotes omission and $\pm$ is the sign $(-1)^{i+j+(p+1)(q-1)}$. A tedious calculation shows that more generally in any Gerstenhaber algebra A, for $x_1, \dots, x_p, y_1, \dots, y_q \in A$,

$$\{x_1 \dots x_p, y_1 \dots y_q\} = \sum_{1\leq i\leq p, 1\leq j\leq q} \pm\{x_i, y_j\} x_1 \dots \widehat{x_i} \dots x_p y_1 \dots \widehat{y_j} \dots y_q.$$

where here $\pm$ is the sign $(-1)^{|x_i||x_1 \dots x_{i-1}| + |y_j||y_1 \dots y_{j-1}| + (|x_1 \dots x_p|+1)|y_1 \dots \widehat{y_j} \dots y_q|}$.

In particular for any bialgebra C, the inclusion of Lie algebras $P(C) \hookrightarrow \mathrm{Cotor}^*_C(\mathbb{k},\mathbb{k})$ given by Property 17 induces a unique morphism of Gerstenhaber algebras $\varphi_C : \Lambda^* P(C) \to \mathrm{Cotor}^*_C(\mathbb{k},\mathbb{k})$.

Proposition 65. [16, Theorem 8 p. 70] *Let L be a Lie algebra over the rationals $\mathbb{Q}$. Consider the universal enveloping algebra UL with its canonical bialgebra structure. Then the morphism of Gerstenhaber algebras $\varphi_{UL} : \Lambda^* L \to \mathrm{Cotor}^*_{UL}(\mathbb{Q},\mathbb{Q})$ is an isomorphism.*

Since we have not be able to fully understand the proof of Gerstenhaber and Schack, we give our own detailed proof of this proposition.

Proof. Let V be a graded $\mathbb{Q}$-vector space. Let ΛV be the free graded commutative algebra on V. By [13, Proposition 22.7] applied to V considered as a differential graded abelian Lie algebra, the linear map $\lambda_V : \Lambda(sV), 0 \overset{\simeq}{\hookrightarrow} \overline{B}(\Lambda V)$, mapping $v_1 \wedge \cdots \wedge v_n$ to the shuffle product $[v_1] \star \cdots \star [v_n]$, is an injective quasi-isomorphism of differential graded coaugmented coalgebras (This is a consequence of the (graded) Koszul resolution).

Let A be an augmented algebra. Denote by $\overline{B}A$ the normalized reduced Bar construction. Let $\tau_A : s^{-1}\overline{B}A \to A$ be the linear map of degree 0, mapping $s^{-1}[sa_1|\dots|sa_n]$ to a_1 if $n = 1$ and to 0 otherwise. Then the unique morphism of graded algebras $\Omega\overline{B}A \overset{\simeq}{\to} A$, extending τ_A is a quasi-isomorphism [11, Proposition 2.14]. Suppose that V is concentrated in negative (lower) degre, the both ΛsV and $\overline{B}(\Lambda V)$ are concentrated in non-positive degree. Therefore, by [11, Remark 2.3], the morphism of differential graded algebras

$$\Omega\lambda_V : \Omega(\Lambda(sV), 0) \overset{\simeq}{\to} \Omega\overline{B}(\Lambda V)$$

is a quasi-isomorphism. Therefore, by composing, we obtain a quasi-isomorphism of differential graded algebras

$$\nu_{sV} : \Omega\Lambda sV \overset{\simeq}{\to} \Omega\overline{B}(\Lambda V) \overset{\simeq}{\to} (\Lambda V, 0).$$

(This is a particular case of [40, line above Lemma 8.3].) Note that if V is finite dimensional, ν_V is the dual of $\lambda sV^\vee$.

Since L is ungraded, $V = s^{-1}L$ is concentrated in degree -1 and therefore, we have the quasi-isomorphism of differential graded algebras $\nu_L : \Omega\Lambda L \overset{\simeq}{\to} (\Lambda s^{-1}L, 0)$. Poincaré-Birkoff-Witt gives an isomorphism of coalgebras $PBW : UL \overset{\cong}{\to} \Lambda L$ which restricts to the identity map on the primitives. By definition, $\nu_L : \Omega\Lambda L \to \Lambda(s^{-1}L)$ extends the composite

$$s^{-1}\Lambda L \overset{s^{-1}\lambda_{s^{-1}L}}{\longrightarrow} s^{-1}\overline{B}\Lambda s^{-1}L \overset{\tau_{\Lambda s^{-1}L}}{\longrightarrow} \Lambda s^{-1}L$$

which maps $s^{-1}(l_1 \wedge \dots \wedge l_n)$ to $s^{-1}l_1$ if $n = 1$ and to 0 otherwise. Therefore, we have the commutative diagram

$$\begin{tikzcd}
\Omega UL \arrow[r, "\Omega(PBW)", "\cong"'] & \Omega\Lambda L \arrow[r, "\nu_L", "\simeq"'] & \Lambda s^{-1}L \\
s^{-1}PUL \arrow[r] \arrow[u] & s^{-1}P\Lambda L \arrow[r] \arrow[u] & s^{-1}L \arrow[u]
\end{tikzcd}$$

where the vertical arrows are the canonical inclusions and the bottom horizontal maps are the identity maps. Therefore the inverse of the algebra isomorphism

$$H_*(\Omega UL) \overset{H_*(\Omega PBW)}{\longrightarrow} H_*(\Omega\Lambda L) \overset{H_*(\nu_L)}{\longrightarrow} \Lambda s^{-1}L$$

coincides with φ_{UL}. $\square$

Let L be a Lie algebra. A *character* of L is by definition [7, Example 5.5] a morphism of Lie algebras $\delta : L \to \mathbb{k}$. Let A be a connected Batalin-Vilkovisky algebra. Then $B = A^1 \to A^0 = \mathbb{k}$ is a character for the Lie algebra A^1. Indeed for a, $b \in A^1$, since $\{1, b\} = 0$ and $\{a, 1\} = 0$, by [18, Proposition 1.2],

$$B\{a, b\} = \{Ba, b\} \pm \{a, Bb\} = 0 \pm 0 = 0.$$

This forgetful functor from connected Batalin-Vilkovisky algebras to Lie algebras equipped with a character has a left adjoint (Compare with [15, Freely generated BV_n algebras in Section 4]):

Let L be a Lie algebra equipped with a character $\delta : L \to \mathbb{k}$. Then $\lambda.x := \delta(x)\lambda$ for $x \in L$ and $\lambda \in \mathbb{k}$ defines a right (and also left) Lie action of L on $\mathbb{k}$. The differential $d : \Lambda^n L \to \Lambda^{n+1} L$ of the Chevalley-Eilenberg complex $C_*(L, \mathbb{k})$ is given by ([34, (10.1.3.1)] with the opposite differential or [58, 7.7.1] for exactly the same differential)

$$\begin{aligned} d(x_1 \wedge \cdots \wedge x_n) &= \sum_{i=1}^{n} (-1)^{i-1} \delta(x_i) x_1 \wedge \cdots \wedge \widehat{x_i} \wedge \cdots \wedge x_n \\ &\quad + \sum_{1 \leq i < j \leq n} (-1)^{i+j} \{x_i, x_j\} \wedge x_1 \wedge \cdots \wedge \widehat{x_i} \wedge \cdots \wedge \widehat{x_j} \wedge \cdots \wedge x_n. \end{aligned}$$

Here the symbol $\widehat{\ }$ denotes omission. A direct calculation shows that

$$\begin{aligned} (-1)^p (d(x_1 \wedge \cdots \wedge x_p \wedge y_1 \wedge \cdots \wedge y_q) \\ - d(x_1 \wedge \cdots \wedge x_p) y_1 \wedge \cdots \wedge y_q \\ (-1)^p x_1 \wedge \cdots \wedge x_p d(y_1 \wedge \cdots \wedge y_q)) \end{aligned}$$

is the Schouten bracket on the Gerstenhaber algebra $\Lambda^* L$ defined by equation (64). Therefore the Gerstenhaber algebra $\Lambda^* L$ equipped with the operator d is a Batalin-Vilkovisky algebra. By induction, one can check that in any Batalin-Vilkovisky algebra A, for $x_1, \ldots, x_n \in A$,

$$\begin{aligned} B(x_1 \ldots x_n) &= \sum_{i=1}^{n} (-1)^{|x_1| + \cdots + |x_{i-1}|} x_1 \ldots B(x_i) \ldots x_n \\ &\quad + \sum_{1 \leq i < j \leq n} (-1)^{|x_1| + \cdots + |x_{j-1}| + |x_1 \ldots x_{i-1}||x_i|} \{x_i, x_j\} x_1 \ldots \widehat{x_i} \ldots \widehat{x_j} \ldots x_n. \end{aligned}$$

It follows easily that the inclusion of Lie algebras with character

$$(L, \delta) \to (\Lambda^1 L, d : \Lambda^1 L \to \Lambda^0 L = \mathbb{k})$$

is universal.

Let C be a Hopf algebra endowed with a modular pair in involution of the form $(\delta, 1)$. By [6, (2.19) or (2.22)], the operator

$$B : \mathrm{Cotor}^1_C(\mathbb{k}, \mathbb{k}) = P(C) \to \mathrm{Cotor}^0_C(\mathbb{k}, \mathbb{k}) = \mathbb{k}$$

coincides with the restriction of δ, $\delta_{|P(C)} : P(C) \to \mathbb{k}$. Since δ is a character for the associative algebra C, $\delta_{|P(C)}$ is a character for the Lie algebra of primitive elements $P(C)$. By universal property, the morphism of Gerstenhaber algebra $\varphi_C : (\Lambda^* P(C), d) \to \mathrm{Cotor}^*_C(\mathbb{k}, \mathbb{k})$ is a morphism of Batalin-Vilkovisky algebras between the free Batalin-Vilkovisky algebra generated by the Lie algebra with character $P(C)$ and the Batalin-Vilkovisky algebra recalled in Corollary 44. As an immediate consequence of Proposition 65, we obtain the following theorem:

Theorem 66. *Let L be a Lie algebra over the rationals $\mathbb{Q}$. Let $\delta : L \to \mathbb{Q}$ be a character for L. Extend δ to a character $\delta : UL \to \mathbb{Q}$ for UL. The morphism of Batalin-Vilkovisky algebras*

$$\varphi_{UL} : (\Lambda^* L, d) \to \mathit{Cotor}^*_{UL}(\mathbb{Q}, \mathbb{Q})$$

is an isomorphism.

In [15, Theorem 4.4], Gérald Gaudens and the author showed that the rational homology $H_*(\Omega^2 X; \mathbb{Q})$ of the double loop space of a 2-connected space X is isomorphic as Batalin-Vilkovisky algebras to $(\Lambda(\pi_*(\Omega X) \otimes \mathbb{Q}), d)$, the free Batalin-Vilkovisky algebra generated by the (graded) Lie algebra $\pi_*(\Omega X) \otimes \mathbb{Q}$ equipped with the trivial character. The graded version of Theorem 66 shows that $(\Lambda(\pi_*(\Omega X) \otimes \mathbb{Q}), d)$ is isomorphic as Batalin-Vilkovisky algebras to $\mathrm{Cotor}^*_{H_*(\Omega X;\mathbb{Q})}(\mathbb{Q}, \mathbb{Q})$. So finally, we obtain an isomorphism of Batalin-Vilkovisky algebras

$$H_*(\Omega^2 X; \mathbb{Q}) \cong \mathrm{Cotor}^*_{H_*(\Omega X;\mathbb{Q})}(\mathbb{Q}, \mathbb{Q}).$$

Of course, such isomorphism must be compared with our Conjecture 27.

It would be interesting to compute the Batalin-Vilkovisky algebra $\mathrm{Cotor}^*_H(\mathbb{k}, \mathbb{k})$ and the Lie bracket on $HC^*_{(\delta,1)}(H)$ when $H = \mathcal{H}_n$, the Connes-Moscovici Hopf algebra. In [5, Theorem 11], Connes and Moscovici start the computation of $HC^*_{(\delta,1)}(\mathcal{H}_n)$ by first computing $HC^*_{(\delta,1)}(UL)$ [5, 1) of Proposition 7]. This gives a second reason why in this paper, we have chosen to compute the Batalin-Vilkovisky algebra $\mathrm{Cotor}^*_{UL}(\mathbb{k}, \mathbb{k})$ when $\mathbb{k}$ is a field of characteristic 0 (Theorem 66).

10. Future extension

Let H be a Hopf algebra endowed with a modular pair in involution (δ, σ) where σ is different from 1. Then the cocyclic module of Connes and Moscovici $(\Omega H)_{(\delta,\sigma)}$ (recalled just after definition 48) is not the usual Cobar construction and therefore has no obvious cup product. Therefore we believe that its cohomology $\mathrm{Cotor}^*_H(\mathbb{k}, \mathbb{k}\sigma)$ has no algebra structure and that its cyclic cohomology $HC^*_{(\delta,\sigma)}(H)$ has no Lie bracket: we believe that Corollaries 44, 45 and Theorem 50 cannot be generalized to any Hopf algebra admitting "full" modular pair in involution.

In [21, 20], Connes-Moscovici Hopf cyclic cohomology was extended to Hopf algebra with right-left stable-anti-Yetter-Drinfeld modules (SAYD). By [31, Example 4.2], Hopf cyclic cohomology $HC^{(*,H)}(B,M)$ for Hopf algebras H with a right-right SAYD H-module M gives, when $B=H$ and $M={}^{\sigma}\,\Bbbk_{\delta}$, the dual of Khalkhali-Rangipour-Taillefer cyclic homology, $\widetilde{HC}^{*}_{(\varepsilon,\sigma)}(H)$, and when $H=\Bbbk$ and $M=\Bbbk$ the cyclic cohomology of the algebra B, $HC^{*}_{\lambda}(B)$. Therefore, we hope that the Hopf cyclic cohomology $HC^{(*,H)}(B,M)$ has a Lie bracket of degre -1 if (H,B,M) satisfies some hypotheses generazing the hypotheses of both corollaries 43 and 45.

The characteristic map of Connes and Moscovici has been generalized for invariant higher trace in [7, 19]. For Hopf algebras with SAYD modules, the characteristic map of Connes and Moscovici is generalized by a cup product in Hopf cyclic cohomology [30, 47, 46]. It would be interesting to generalize Theorem 54 by looking at what kind of algebraic structures those cup products preserves.

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